ON PROPERTIES OF THE APPROXIMATE PEANO DERIVATIVES⁽¹⁾

BY

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ABSTRACT. The notion of k th approximate Peano differentiation not only generalizes k th ordinary differentiation but also k th Peano differentiation and k th L_p differentiation. Recently, M. Evans has shown that a k th approximate Peano derivative at least shares with these other derivatives the property of belonging to Baire class one. In this paper the author extends the properties possessed by a k th approximate Peano derivative by showing that it is like the above derivatives in that it also possesses the following properties: Darboux, Denjoy, Zahorski, and a new property stronger than the Zahorski property, Property Z.

1. Introduction. Let k be a positive integer. Let f be a real-valued, measurable function defined on the closed interval $I = [a, b]$ and let $x \in I$. If there are numbers $f_{(1)}(x), f_{(2)}(x), \dots, f_{(k)}(x)$ and a measurable set E having 0 as a point of density so that

$$f(x+h) = f(x) + hf_{(1)}(x) + \dots + (h^k/k!)f_{(k)}(x) + o(h^k)$$

as $h \rightarrow 0, h \in E$ and $x+h \in I$, then the number $f_{(k)}(x)$ is called the k th approximate Peano derivative of f at x . We will find it convenient to write $f_{(0)}(x) = f(x)$. It is easily seen from the definition that if $f_{(k)}(x)$ exists then so does $f_{(n)}(x)$ for $0 \leq n < k$. Also, $f_{(1)}(x) = f'_{ap}(x)$, the approximate derivative. The notion of k th approximate Peano differentiation not only generalizes k th ordinary differentiation but also k th Peano differentiation and k th L_p differentiation. For definitions of the latter two types of derivatives see [2].

Recently, Evans [2] proved that if $f_{(k)}$ is defined on I then $f_{(k)}$ is in the first class of Baire (a pointwise limit of continuous functions). The purpose of this paper is to show that $f_{(k)}$ also has the following properties: (1) Darboux,

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(2) Denjoy, (3) Zahorski, and (4) Property Z. That an ordinary derivative, approximate derivative, k th Peano derivative and k th L_p derivative are in the first class of Baire and possess the above four properties, we refer the reader to [1]–[4], [6]–[11].

We begin in §2 by defining the four properties stated above and by giving notation and terminology which will be used throughout this paper. In §3, a density lemma is proved which plays a key role in §4 where the following major result is proved. If $f_{(k)}$ is defined on I and if $f_{(k)}$ is bounded above or below on I then $f^{(k)}$, the ordinary k th derivative of f , exists and $f^{(k)} = f_{(k)}$ on I . Properties 1 and 2 are shown to hold for $f_{(k)}$ in §5 by using known theorems together with the major result. In §6, the final section, a lemma is proved from which property 4 is shown to hold for $f_{(k)}$. Again from a known theorem, property 3 is shown to follow from property 4 for $f_{(k)}$.

2. Notation, terminology and definitions. All of the functions in this paper are assumed to be real-valued, measurable functions defined on the closed interval $I = [a, b]$ unless specified otherwise. R will denote the real numbers and if $E \subset R$ is a measurable set then we denote the measure of E by either $m(E)$ or $|E|$. The notation $E\text{-}\lim_{y \rightarrow x}$ denotes $\lim_{y \rightarrow x, y \in E}$. For convenience we now define the four properties stated in the introduction.

Let g be a function defined on I .

1. g possesses the Darboux property if g maps connected sets of I into connected sets.
2. g satisfies the Denjoy property if, for every open interval (c, d) , $g^{-1}((c, d))$ either is empty or has positive measure.
3. g has the Zahorski property if the following condition is fulfilled: If $c < d$, $x \in g^{-1}((c, d))$, and if $\{I_n\}$ is a sequence of closed intervals of I not containing x so that $I_n \rightarrow x$ and $m(I_n \cap g^{-1}((c, d))) = 0$ for all n , then

$$\lim_{n \rightarrow \infty} \frac{m(I_n)}{\text{dist}(x, I_n)} = 0.$$

The notation $I_n \rightarrow x$ means that every neighborhood of x contains all but finitely many of the I_n 's.

4. g is said to have Property Z if the following condition is satisfied: If for each $\epsilon > 0$ and each sequence $\{I_n\}$ of closed intervals of I such that $I_n \rightarrow x$ and $g(y) \geq g(x)$ on I_n or $g(y) \leq g(x)$ on I_n for each n , then

$$\lim_{n \rightarrow \infty} \frac{m\{y \in I_n : |g(y) - g(x)| \geq \epsilon\}}{m(I_n) + \text{dist}(x, I_n)} = 0.$$

Let $E \subset R$ be a measurable set and let $x \in R$. Define

$$d(x, E) = \lim_{h \rightarrow 0} \frac{m(E \cap [x - h, x + h])}{2h},$$

$$d_+(x, E) = \lim_{h \rightarrow 0+} \frac{m(E \cap [x, x + h])}{h},$$

and $d_-(x, E)$ in the obvious fashion. If $d(x, E) = 1$ (0) then x is called a point of density (dispersion) of E . If $d_+(x, E) = 1$ (0) then x is called a point of right-hand density (dispersion) of E ; if $d_-(x, E) = 1$ (0) then x is said to be a point of left-hand density (dispersion) of E .

The following simple observations, which will be used later, are now noted. If $d_+(0, E) = d_+(0, F) = 0$ (1) then $d_+(0, E \cap F) = 0$ (1) and $d_+(0, E^c) = 1$ (0), where E^c is the complement of E .

REMARK. If $x = a$ ($x = b$) in the definition of a k th approximate Peano derivative then the expression "there exists a measurable set E having 0 as a point of density" is understood to mean that $E \subset [0, \infty)$ ($E \subset (-\infty, 0]$) and 0 is a point of right-hand (left-hand) density of E .

3. A density lemma. If $E \subset R$ and $\lambda \in R$ then we define $\lambda E = \{\lambda e: e \in E\}$. Before proceeding to the density lemma we need

LEMMA 3.1. Let $E \subset R$ be a set of finite measure and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|\lambda E - E| < \epsilon$, whenever $|1 - \lambda| < \delta$.

PROOF. If $|E| = 0$ or if E is a finite union of intervals then the lemma is easily verified. Thus, assume E is a set of finite measure and let G be an open set such that $E \subset G$ and $|G - E| < \epsilon/3$. Without loss of generality we may assume $G = \bigcup_{n=1}^{\infty} I_n$, a pairwise disjoint union of open intervals. Choose $N > 0$ so that if $H = \bigcup_{n=N+1}^{\infty} I_n$, then $|H| < \epsilon/6$. Choose $0 < \delta < 1$ so that $|\lambda F - F| < \epsilon/3$, whenever $|1 - \lambda| < \delta$, where $F = \bigcup_{n=1}^N I_n$. If $|1 - \lambda| < \delta$, then

$$\begin{aligned} |\lambda E - E| &\leq |\lambda G - E| \leq |\lambda G - G| + |G - E| \leq |\lambda G - F| + |G - E| \\ &\leq |\lambda F \cup \lambda H - F| + |G - E| \leq |\lambda F - F| + \lambda |H| + |G - E| \\ &\leq \epsilon/3 + 2 \cdot \epsilon/6 + \epsilon/3 = \epsilon. \end{aligned}$$

DENSITY LEMMA 3.2. Let $d_+(0, E) = 1$. Then there are numbers α_n, β_n such that $0 < \alpha_n < \alpha_{n+1} < 1 < \beta_{n+1} < \beta_n$ ($n = 1, 2, \dots$), $\alpha_n \rightarrow 1, \beta_n \rightarrow 1$ and

$$d_+\left(0, \bigcap_{n=1}^{\infty} \alpha_n E\right) = d_+\left(0, \bigcap_{n=1}^{\infty} \beta_n E\right) = 1.$$

PROOF. Set $H = E^c$, then $d_+(0, H) = 0$. There are numbers δ_k such that $1/k > \delta_k > \delta_{k+1} > 0$ and $|H \cap (0, t)| < t/k^2$ for each $t \in (0, 2\delta_k)$ ($k = 1, 2, \dots$). Set $F = H \cap (0, 2)$. By Lemma 3.1 there are numbers α_n, β_n such that

$$n/(n+1) < \alpha_n < \alpha_{n+1} < 1 < \beta_{n+1} < \beta_n < (n+1)/n$$

and

$$|\alpha_n F - F| < \delta_n/2^n, \quad |\beta_n F - F| < \delta_n/2^n$$

for $n = 1, 2, \dots$. Let $\epsilon > 0$; let k be an integer greater than $3/\epsilon$ and let $0 < h < \delta_k$. Choose $j \geq k$ so that $\delta_{j+1} \leq h < \delta_j$. Since $\alpha_n H \cap (0, h) = \alpha_n (H \cap (0, h/\alpha_n))$ and since $h/\alpha_n < 2h < 2\delta_j < 2$, we have

$$|\alpha_n H \cap (0, h)| = \alpha_n |H \cap (0, h/\alpha_n)| < h/j^2$$

and

$$\alpha_n H \cap (0, h) \subset \alpha_n F$$

for each n . As

$$\bigcup_{n=1}^{\infty} \alpha_n H \subset \left(\bigcup_{n=1}^j \alpha_n H \right) \cup \left(\bigcup_{n=j+1}^{\infty} (\alpha_n H - H) \right) \cup H,$$

and $(\alpha_n H - H) \cap (0, h) \subset \alpha_n F - F$, we get

$$\begin{aligned} \left| \left(\bigcup_{n=1}^{\infty} \alpha_n H \right) \cap (0, h) \right| &\leq \sum_{n=1}^j |\alpha_n H \cap (0, h)| \\ &\quad + \sum_{n=j+1}^{\infty} |\alpha_n F - F| + |H \cap (0, h)| \\ &< j \cdot h/j^2 + \sum_{n=j+1}^{\infty} \delta_n/2^n + h/j^2 \\ &< h/j + h/2^j + h/j^2 \\ &< 3h/j \leq 3h/k < \epsilon h. \end{aligned}$$

Thus $d_+(0, \bigcup_{n=1}^{\infty} \alpha_n H) = 0$. Since

$$\left(\bigcup_{n=1}^{\infty} \alpha_n H \right)^c = \bigcap_{n=1}^{\infty} (\alpha_n H)^c = \bigcap_{n=1}^{\infty} \alpha_n H^c = \bigcap_{n=1}^{\infty} \alpha_n E,$$

$d_+(0, \bigcap_{n=1}^{\infty} \alpha_n E) = 1$. Similarly it can be proved that $d_+(0, \bigcap_{n=1}^{\infty} \beta_n E) = 1$.

4. **The major theorem.** In this section we deduce the fundamental result stated in

THEOREM 4.1. *Let $f_{(k)}$ be defined on I .*

- (i) *If $f_{(k)} > 0$ on I , then $f_{(k-1)}$ is continuous and nondecreasing on I .*
- (ii) *If $f_{(k)}$ is bounded above or below on I , then $f_{(k)} = f^{(k)}$ on I .*

The proof of this theorem will require some additional lemmas.

LEMMA 4.2. *Let $f_{(k)}$ be defined on $I = [a, b]$. Assume $f_{(1)}$ is nondecreasing on I , and if $k \geq 2$ furthermore assume $f_{(2)}(a) = f_{(3)}(a) = \dots = f_{(k-1)}(a) = 0$. Then $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$.*

PROOF. By subtracting from f a multiple of x , we may assume that $f_{(1)}(a) = 0$. By hypothesis there exists a measurable set F such that $d_+(0, F) = 1$ and

$$(4.3) \quad F\text{-}\lim_{h \rightarrow 0} \frac{1}{h^k} \{f(a+h) - f(a) - Ah^k\} = 0,$$

where $A = f_{(k)}(a)/k!$.

By the Density Lemma there exist two sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_m\}$ such that $\alpha_n \rightarrow 0$, $\beta_m \rightarrow 0$ and

$$d_+\left(0, \bigcap_{n=1}^{\infty} (1 - \alpha_n)F\right) = d_+\left(0, \bigcap_{m=1}^{\infty} (1 + \beta_m)F\right) = 1.$$

If we set

$$E = F \cap \left[\bigcap_{n=1}^{\infty} (1 - \alpha_n)F \right] \cap \left[\bigcap_{m=1}^{\infty} (1 + \beta_m)F \right],$$

then $d_+(0, E) = 1$. To complete the proof of the lemma we need only show

$$E\text{-}\lim_{h \rightarrow 0} \frac{f_{(1)}(a+h)}{h^{k-1}} = Ak.$$

Let $\epsilon > 0$ be given; choose n, m so that if $\alpha = \alpha_n/(1 - \alpha_n)$ and $\beta = \beta_m/(1 + \beta_m)$ then

$$A \cdot \frac{(1 + \alpha)^k - 1}{\alpha} < Ak + \frac{\epsilon}{2} \quad \text{and} \quad A \cdot \frac{1 - (1 - \beta)^k}{\beta} > Ak - \frac{\epsilon}{2}.$$

Set

$$\epsilon' = \min \left(\frac{\epsilon\alpha}{2[(1+\alpha)^k + 1]}, \frac{\epsilon\beta}{2[(1+\beta)^k + 1]} \right).$$

By (4.3) there exists a $\delta' > 0$ such that $|f(a+h) - f(a) - Ah^k| < \epsilon'h^k$ whenever $0 < h < \delta'$, $h \in F$. If $0 < u < v < \delta'$ and $u, v \in F$ then

$$|[f(a+v) - f(a+u)] - A(v^k - u^k)| < \epsilon'(v^k + u^k).$$

Hence,

$$A \frac{(v^k - u^k)}{v - u} - \epsilon' \frac{(v^k + u^k)}{v - u} < \frac{f(a+v) - f(a+u)}{v - u} < A \frac{(v^k - u^k)}{v - u} + \epsilon' \frac{(v^k + u^k)}{v - u}.$$

Since $f_{(1)}$ is nondecreasing on $[a, b]$ and $f_{(1)} = f'_{\text{ap}}$ we have $f_{(1)} = f'$ on $[a, b]$ (see [3]) and hence

$$f_{(1)}(a+u) \leq \frac{f(a+v) - f(a+u)}{v - u} \leq f_{(1)}(a+v).$$

Thus, whenever $0 < u < v < \delta'$ and $u, v \in F$,

$$(4.4) \quad f_{(1)}(a+u) < A \frac{(v^k - u^k)}{v - u} + \epsilon' \frac{(v^k + u^k)}{v - u}$$

and

$$(4.5) \quad f_{(1)}(a+v) > A \frac{(v^k - u^k)}{v - u} - \epsilon' \frac{(v^k + u^k)}{v - u}.$$

Set $\delta = \min\{\delta'/(1+\alpha), \delta'(1-\beta)\}$ and let $h \in E$ such that $0 < h < \delta$. Since $h \in (1-\alpha_n)F$, there exists a $v \in F$ such that $h = (1-\alpha_n)v$. Hence, $v = \{1 + [\alpha_n/(1-\alpha_n)]\}h = (1+\alpha)h$ and $h < v < \delta'$. Thus from (4.4) we have

$$\begin{aligned} \frac{f_{(1)}(a+h)}{h^{k-1}} &< A \frac{[h^k(1+\alpha)^k - h^k]}{\alpha h^k} + \epsilon' \frac{[h^k(1+\alpha)^k + h^k]}{\alpha h^k} \\ (4.6) \quad &< A[(1+\alpha)^k - 1]/\alpha + \epsilon'[(1+\alpha)^k + 1]/\alpha \\ &< Ak + \epsilon/2 + \epsilon/2 < Ak + \epsilon. \end{aligned}$$

Moreover, since $h \in (1+\beta_m)F$ there exists a $u \in F$ such that $h = (1+\beta_m)u$. Hence, $u = \{1 - [\beta_m/(1+\beta_m)]\}h = (1-\beta)h$ and $u < h < \delta'$. Thus from (4.5) it follows that

$$(4.7) \quad f_{(1)}(a+h)/h^{k-1} > Ak - \epsilon.$$

Thus from (4.6) and (4.7) we have

$$|f_{(1)}(a+h)/h^{k-1} - Ak| < \epsilon$$

whenever $0 < h < \delta$ and $h \in E$.

COROLLARY 4.8. *Let $f_{(k)}$ be defined on $I = [a, b]$. Assume $f_{(1)}$ is non-decreasing on I , and if $k \geq 2$ furthermore assume $f_{(2)}(b) = f_{(3)}(b) = \dots = f_{(k-1)}(b) = 0$. Then $(f_{(1)})_{(k-1)}(b) = f_{(k)}(b)$.*

PROOF. Define a function g on $[-b, -a]$ as follows:

$$g(x) = f(-x) \quad \text{for each } x \in [-b, -a].$$

Then $g_{(k)}(x)$ exists for each $x \in [-b, -a]$ and $g_{(n)}(x) = (-1)^n f_{(n)}(-x)$ for $n = 0, 1, \dots, k$. It is easy to verify that $g_{(1)}$ is nondecreasing on $[-b, -a]$, and that if $k \geq 2$, $g_{(2)}(-b) = g_{(3)}(-b) = \dots = g_{(k-1)}(-b) = 0$. The proof of the corollary is now easily completed by applying Lemma 4.2.

COROLLARY 4.9. *Let $f_{(2)}$ be defined on I . If $f_{(1)}$ is nondecreasing on I , then $(f_{(1)})_{(1)} = f_{(2)}$ on I .*

LEMMA 4.10. *Suppose f has $(k-1)$ derivatives at the point x , then for each sufficiently small nonzero h , there is a $0 < \theta < 1$ such that*

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \left\{ f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) \right\} \\ = f^{(k-2)}(x + \theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x) \end{aligned}$$

where $f^{(0)}(x) = f(x)$.

PROOF. Let

$$(4.11) \quad g(t) = f(x+t) - \sum_{n=0}^{k-1} \frac{t^n}{n!} f^{(n)}(x).$$

Then g is $(k-2)$ times differentiable around 0 and

$$(4.12) \quad g^{(j)}(t) = f^{(j)}(x+t) - \sum_{n=0}^{k-j-1} \frac{t^n}{n!} f^{(n+j)}(x)$$

for $j = 0, 1, \dots, (k-2)$. By the extended mean value theorem for each sufficiently small h there exists a $0 < \theta < 1$ so that

$$g(h) = \sum_{n=0}^{k-3} \frac{h^n}{n!} g^{(n)}(0) + \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h)$$

where $g^{(0)}(0) = g(0)$. By (4.12) it follows that $g^{(j)}(0) = 0$ for $j = 0, 1, \dots, (k-3)$; hence

$$(4.13) \quad g(h) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h).$$

Thus, by replacing the left-hand side of (4.11) by (4.13) we have

$$f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h).$$

If $h \neq 0$ then this last equation together with (4.12) yields

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \left\{ f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) \right\} &= g^{(k-2)}(\theta h) \\ &= f^{(k-2)}(x + \theta h) - \sum_{n=0}^1 \frac{(\theta h)^n}{n!} f^{(k+n-2)}(x) \\ &= f^{(k-2)}(x + \theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x). \end{aligned}$$

Before stating the next two lemmas, proofs of which can be found in the papers of Verblunsky [8] and Zygmund [12] respectively, we need the following definitions.

DEFINITION 4.14. A function f defined on an interval is said to be convex if for every pair of points P_1, P_2 on the curve $y = f(x)$ the points of the arc P_1P_2 are below, or on, the chord P_1P_2 .

DEFINITION 4.15. Let f be a function defined in a neighborhood of x . Then define

$$\bar{D}_2 f(x) = \limsup_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

$\bar{D}_2 f(x)$ is called the upper symmetric second derivative of f at x .

REMARK. It can easily be shown that if $f''(x)$ exists at x then $\bar{D}_2 f(x) = f''(x)$. However, the upper symmetric second derivative may exist at a point without the second derivative existing.

LEMMA 4.16. Let f have a finite derivative at each point of (a, b) . Suppose that for each $x_0 \in (a, b)$ there are, in every neighborhood of $(x_0, f(x_0))$, points of the graph of f above the line $y = f(x_0) + f'(x_0)(x - x_0)$. Then f is convex on (a, b) .

LEMMA 4.17. *A necessary and sufficient condition for a continuous function f to be convex on (a, b) is that $\overline{D}_2 f(x) \geq 0$ for each x in (a, b) .*

The following lemma is a special case of Lemma 2 in [2].

LEMMA 4.18. *Suppose $f_{(2)}$ exists at a point $x \in (a, b)$. Then there exists a measurable set E so that $d(0, E) = 1$ and*

$$E\text{-}\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f_{(2)}(x).$$

COROLLARY 4.19. *Suppose $f_{(2)}$ exists on (a, b) and $f_{(2)} \geq 0$ on (a, b) . Then $\overline{D}_2 f \geq 0$ on (a, b) .*

In what follows we shall use without specific reference several well-known results. We list these results here without proof.

Let g be a function defined on an interval J and let g have an ordinary derivative g' on J . If g is convex on J then g' is nondecreasing on J .

Let g be a function defined on $[a, b]$. If g is nondecreasing on (a, b) and has the Darboux property on $[a, b]$ then g is nondecreasing on $[a, b]$.

Let g be a function defined on an interval J . If g is nondecreasing on J and has the Darboux property on J , then g is continuous on J .

Let g be a function of Baire class one on $[a, b]$. Then every nonempty closed set F , contained in $[a, b]$, contains points of continuity of g relative to F .

Let g be a function defined on an interval J and assume g'_{ap} exists at each point in J . Then the following are true (see [3]):

- (1) g'_{ap} is a function of Baire class one on J ,
- (2) g'_{ap} has the Darboux property on J ,
- (3) if g'_{ap} is bounded above or below on J then $g'_{\text{ap}} = g'$ on J .

LEMMA 4.20. *Let f be a function satisfying the following two conditions on $[a, b]$:*

- (i) f'_{ap} exists for each x in $[a, b]$;
- (ii) $\overline{D}_2 f \geq 0$ on (a, b) .

Then f'_{ap} is continuous and nondecreasing on $[a, b]$.

PROOF. Let G be the set of all points x in $[a, b]$ with the property that there is a neighborhood of x on which f'_{ap} is bounded. Then G is an open set. Let $(c, d) \subset G$; then a simple compactness argument shows f'_{ap} is bounded on $[c', d']$, where $c < c' < d' < d$. Hence $f'_{\text{ap}} = f'$ on $[c', d']$. Therefore it follows that $f'_{\text{ap}} = f'$ on (c, d) . Since f is continuous on (c, d) and $\overline{D}_2 f \geq 0$ on (c, d) , f is convex on (c, d) by Lemma 4.17. Hence f'_{ap} is nondecreasing on

(c, d) . Moreover, since f'_{ap} has the Darboux property on $[c, d]$ it follows that f'_{ap} is continuous and nondecreasing on $[c, d]$. In particular, f'_{ap} is continuous and nondecreasing in the closure of each component of G .

To complete the proof of the lemma we show $G = [a, b]$. Let $H = [a, b] - G$; then H is a closed set having no isolated points. Suppose H is nonempty. Then H is a perfect set. Since f'_{ap} is a function of Baire class one on $[a, b]$ there exists an $x_0 \in H$ so that f'_{ap} is continuous at x_0 relative to H . Hence there is an $M \geq 0$ and a $\delta > 0$ such that $|f'_{ap}(x)| \leq M$ for each $x \in [x_0 - \delta, x_0 + \delta] \cap H$. Set

$$c = \min\{x: x \in [x_0 - \delta, x_0] \cap H\}, \quad d = \max\{x: x \in [x_0, x_0 + \delta] \cap H\}.$$

Notice that since H is perfect, $c, d \in H$ and $c < d$. If $x \in (c, d) - H$ then there exists a component of G , say (α, β) , where $\alpha, \beta \in H$ such that $x \in (\alpha, \beta) \subset (c, d)$. From the first part of the proof f'_{ap} is nondecreasing on $[\alpha, \beta]$; hence

$$-M \leq f'_{ap}(\alpha) \leq f'_{ap}(x) \leq f'_{ap}(\beta) \leq M.$$

Thus, for each $x \in (c, d)$, $|f'_{ap}(x)| \leq M$ and so $(c, d) \subset G$.

Since $x_0 \in H$, $x_0 \notin (c, d)$; so either $x_0 = c$ or $x_0 = d$. But if $x_0 = c$ then $(x_0 - \delta, x_0) \subset G$ and there exists a number $M' \geq 0$, so that f'_{ap} is bounded by M' on $[x_0 - \delta, x_0]$. In the last paragraph it was shown that f'_{ap} was bounded by M on (x_0, d) . So f'_{ap} is bounded by $\max\{M, M'\}$ on $(x_0 - \delta, d)$ and $x_0 \in G$. Similarly, it can be shown that if $x_0 = d$ then $x_0 \in G$. Thus, the assumption that $H \neq \emptyset$ is false. Therefore, $H = \emptyset$ and $G = [a, b]$.

PROOF OF THEOREM 4.1. Consider first the case $k = 1$. If $f_{(1)} > 0$ on $[a, b]$ then $f_{(1)} = f'$ on $[a, b]$. Thus, $f_{(0)} = f$ is continuous and nondecreasing on $[a, b]$. Moreover, if $f_{(1)}$ is bounded either above or below on $[a, b]$ then again $f_{(1)} = f'$ on $[a, b]$. Thus, the theorem holds when $k = 1$.

Secondly, consider $k = 2$. By Corollary 4.19 and Lemma 4.20 the proof of (i) is immediate. Turning to case (ii), there is no loss of generality to assume $f_{(2)} > 0$ on $[a, b]$. From (i) it follows that $f_{(1)}$ is nondecreasing on $[a, b]$; hence $f_{(1)} = f'$ on $[a, b]$. By Corollary 4.9, $(f')_{(1)} = f_{(2)}$ on $[a, b]$. Moreover, by assumption $(f')_{(1)} > 0$ on $[a, b]$; hence $(f')_{(1)} = (f')' = f^{(2)}$. Thus, $f_{(2)} = f^{(2)}$ on $[a, b]$.

We may now assume that $k > 2$, and we can complete the proof by induction. We therefore assume the following:

If f possesses a $(k - 1)$ th approximate Peano derivative everywhere on an interval $[a, b]$, then for $1 \leq n \leq (k - 1)$:

(i) if $f_{(n)} > 0$ on $[a, b]$, then $f_{(n-1)}$ is continuous and nondecreasing on $[a, b]$;

(ii) if $f_{(n)}$ is bounded either above or below on $[a, b]$, then $f_{(n)} = f^{(n)}$ on $[a, b]$.

Let $k > 2$ and assume $f_{(k)} > 0$ at each point in $[a, b]$. Let G be the set of all points x of $[a, b]$ with the property that there is a neighborhood of x on which $f_{(k-1)}$ is bounded. Obviously G is open. Let $(c, d) \subset G$. If $c < \alpha < \beta < d$, then a simple compactness argument shows $f_{(k-1)}$ is bounded on $[\alpha, \beta]$. By (ii) of the induction hypothesis, $f_{(k-1)} = f^{(k-1)}$ on $[\alpha, \beta]$ and therefore $f_{(k-2)} = f^{(k-2)}$ on $[\alpha, \beta]$. Moreover, these relations hold on (c, d) . Thus, $f_{(k-1)} = f'_{(k-2)}$ on (c, d) and $f_{(k-2)}$ is continuous on (c, d) . If $x \in (c, d)$ then there exists a measurable set E such that 0 is a point of density of E and

$$f(x+h) = \sum_{n=0}^{k-1} \frac{h^n}{n!} f_{(n)}(x) + \frac{h^k}{k!} [f_{(k)}(x) + \epsilon(x, h)]$$

where $E\text{-}\lim_{h \rightarrow 0} \epsilon(x, h) = 0$. From Lemma 4.10 for each sufficiently small nonzero $h \in E$ there is a θ between 0 and 1 such that

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \left\{ f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f_{(n)}(x) \right\} \\ = f_{(k-2)}(x + \theta h) - f_{(k-2)}(x) - \theta h f_{(k-1)}(x). \end{aligned}$$

Hence

$$\begin{aligned} ((k-2)!/h^{k-2}) \{ (h^k/k!) [f_{(k)}(x) + \epsilon(x, h)] \} \\ = f_{(k-2)}(x + \theta h) - f_{(k-2)}(x) - \theta h f_{(k-1)}(x). \end{aligned}$$

Thus

$$\begin{aligned} f_{(k-2)}(x + \theta h) &= f_{(k-2)}(x) + \theta h f_{(k-1)}(x) + \frac{h^2}{k(k-1)} [f_{(k)}(x) + \epsilon(x, h)] \\ &> f_{(k-2)}(x) + \theta h f_{(k-1)}(x) \end{aligned}$$

for all sufficiently small nonzero $h \in E$. Thus, it follows by Lemma 4.16 that $f_{(k-2)}$ is convex on (c, d) ; hence $f_{(k-1)}$ is nondecreasing on (c, d) . Choose λ between c and d . Then $f_{(k-1)}$ is bounded below on $[\lambda, d]$. Applying (ii) of the induction hypothesis to the function $f_{(k-1)}$ on the interval $[\lambda, d]$, it follows that $f_{(k-1)} = f^{(k-1)}$ on $[\lambda, d]$. Now since $f_{(k-1)}$ is nondecreasing on (λ, d) and has the Darboux property on $[\lambda, d]$ we have that $f_{(k-1)}$ is continuous and nondecreasing on $[\lambda, d]$. Similarly, since $f_{(k-1)}$ is bounded above on $[c, \lambda]$, we deduce that $f_{(k-1)}$ is continuous and nondecreasing on $[c, \lambda]$. Thus,

it follows that $f_{(k-1)}$ is continuous and nondecreasing on $[c, d]$. In particular, $f_{(k-1)}$ is nondecreasing and continuous in the closure of each component of G .

To complete the proof of (i) we show $G = [a, b]$. Let $H = [a, b] - G$. From above H is a closed set having no isolated points. Since $f_{(k-1)}$ is a function of Baire class one on $[a, b]$ (see [2]), the same type of argument given in the proof of Lemma 4.20 shows H is empty. Hence $G = [a, b]$ and the proof of (i) is complete.

Consider, finally, (ii) for $k > 2$. It is no loss of generality to suppose that $f_{(k)} > 0$ on $[a, b]$. By (i), $f_{(k-1)}$ is nondecreasing on $[a, b]$ and by (ii) of the induction hypothesis $f_{(k-1)} = f^{(k-1)}$ on $[a, b]$. Thus, it follows that $f_{(1)} = f'$ on $[a, b]$. We shall prove that $(f_{(1)})_{(k-1)} = f_{(k)}$ on $[a, b]$. It will then follow by the induction hypothesis (ii) applied to $f_{(1)}$ that in $[a, b]$, $f_{(k)} = (f_{(1)})_{(k-1)} = (f_{(1)})^{(k-1)} = f^{(k)}$.

It suffices to prove that in $[a, b]$ the $(k-1)$ th approximate Peano derivative of $f_{(1)}$ on the right equals $f_{(k)}$. For, applying Corollary 4.8, it will follow that in (a, b) the $(k-1)$ th approximate Peano derivative of $f_{(1)}$ on the left equals $f_{(k)}$. Without altering $f_{(k)}$, by adding to f a suitable polynomial of degree less than k , we may assume that $f_{(j)}(a) = 0$ for $j = 2, 3, \dots, (k-1)$. Note, since $f^{(k-1)}(a) = 0$ and $f^{(k-1)}$ is nondecreasing on $[a, b]$, $f^{(k-1)} \geq 0$ on $[a, b]$. Now for each h , $0 < h < (b-a)$, there exists by the extended mean value theorem a number ξ , $a < \xi < a+h$ such that

$$f^{(2)}(a+h) = \frac{h^{k-3}}{(k-3)!} f^{(k-1)}(\xi).$$

Hence $f^{(2)} \geq 0$ in (a, b) . Thus, $f_{(1)}$ is nondecreasing on $[a, b]$. By Lemma 4.2, $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$. Since a may be replaced throughout by any $\alpha \in [a, b]$ the proof of the theorem is complete.

5. The Darboux and Denjoy properties. Neugebauer [5] proved that if g is a function of Baire class one on an interval J , then g has the Darboux property on J if and only if for each real number λ , the sets $E^\lambda = \{x \in J: g(x) \geq \lambda\}$ and $E_\lambda = \{x \in J: g(x) \leq \lambda\}$ have closed components relative to J . We thus have the following corollary to Theorem 4.1.

COROLLARY 5.1. *If $f_{(k)}$ is defined on $[a, b]$ then $f_{(k)}$ has the Darboux property on $[a, b]$.*

PROOF. Since $f_{(k)}$ is of Baire class one on $[a, b]$ (see [2]), in order to show $f_{(k)}$ has the Darboux property we need only show that the components of the sets $E^\lambda = \{x: f_{(k)}(x) \geq \lambda\}$ and $E_\lambda = \{x: f_{(k)}(x) \leq \lambda\}$ are closed for each real number λ . So suppose $f_{(k)}(x) \geq \lambda$ for all x in the interval (α, β) . We must

show that $f_{(k)}(\alpha) \geq \lambda$ and $f_{(k)}(\beta) \geq \lambda$. Since $f_{(k)}$ is bounded below on (α, β) , $f_{(k)}$ is bounded below on $[\alpha, \beta]$. Thus by Theorem 4.1, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Since $f^{(k)}$ has the Darboux property on $[\alpha, \beta]$, $f^{(k)}(\alpha) \geq \lambda$ and $f^{(k)}(\beta) \geq \lambda$. Hence, $f_{(k)}(\alpha) \geq \lambda$ and also $f_{(k)}(\beta) \geq \lambda$. Thus, the components of E^λ are closed. Similarly, the components of E_λ are closed. Hence, $f_{(k)}$ has the Darboux property on $[a, b]$.

In [9], Weil proved that a function g of Baire class one has the Denjoy property on an interval J if, for every subinterval L of J on which g is bounded either above or below, g restricted to L has the Denjoy property. Since an ordinary k th derivative has the Denjoy property, we also have the following corollary to Theorem 4.1.

COROLLARY 5.1. *If $f_{(k)}$ is defined on $[a, b]$, then $f_{(k)}$ has the Denjoy property on $[a, b]$.*

6. Property Z. To prove that $f_{(k)}$ has Property Z we first need a lemma which is a slight generalization of a lemma due to Weil [10].

LEMMA 6.1. *Suppose f is a function whose k th derivative exists and is nonnegative on the interval $[a, b]$, and let $A = \{x \in [a, b] : f^{(k)}(x) \geq \epsilon\}$ where ϵ is a fixed positive number. Then there exists a partition $\{a = t_0 < t_1 < \dots < t_l = b\}$ of the interval $[a, b]$ with $l \leq 2^k$ and such that for each $i = 1, 2, \dots, l$ with $x, y \in [t_{i-1}, t_i]$ and $x \leq y$*

$$|f(y) - f(x)| \geq (\epsilon/k!)(m(A \cap [x, y]))^k.$$

PROOF. It will be shown by induction that for each integer $j = 1, 2, \dots, k$, there is a partition of $[a, b]$,

$$\{a = t_{0,j} < t_{1,j} < \dots < t_{l(j),j} = b\},$$

with $l(j) \leq 2^j$ and such that for each $i = 1, 2, \dots, l(j)$ one of the following holds on $I_{i,j} = [t_{i-1,j}, t_{i,j}]$.

1(j): $f^{(k-j)} \geq 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leq y$,

$$f^{(k-j)}(y) - f^{(k-j)}(x) \geq (\epsilon/j!)(m(A \cap [x, y]))^j.$$

2(j): $f^{(k-j)} \leq 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leq y$,

$$f^{(k-j)}(y) - f^{(k-j)}(x) \geq (\epsilon/j!)(m(A \cap [x, y]))^j.$$

3(j): $f^{(k-j)} \leq 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leq y$,

$$f^{(k-j)}(x) - f^{(k-j)}(y) \geq (\epsilon/j!)(m(A \cap [x, y]))^j.$$

4(j): $f^{(k-j)} \geq 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leq y$

$$f^{(k-j)}(x) - f^{(k-j)}(y) \geq (\epsilon/j!)(m(A \cap [x, y]))^j.$$

The desired partition is then the one corresponding to $j = k$ and the desired inequality is obtained by taking absolute values, where, of course $f^{(0)} = f$.

If the conditions 1(j)–4(j) above are used in place of Weil's conditions 1(j)–4(j) in [10], then the reader may complete the proof by making the rather obvious changes in Weil's proof in [10].

THEOREM 6.2. *If f has a k th approximate Peano derivative $f_{(k)}$ everywhere on $[a, b]$ then $f_{(k)}$ has Property Z on $[a, b]$.*

PROOF. Let x be contained in $[a, b]$ and $\epsilon > 0$. It suffices to show that if given an $\eta > 0$ there exists a $\delta > 0$ such that if the closed interval $[\alpha, \beta]$ is contained in $(x - \delta, x + \delta) \cap [a, b]$, $x \notin [\alpha, \beta]$ and $f_{(k)}(y) \geq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$ or $f_{(k)}(y) \leq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$ then

$$(6.3) \quad \frac{m\{y \in [\alpha, \beta] : |f_{(k)}(y) - f_{(k)}(x)| \geq \epsilon\}}{(\beta - \alpha) + \text{dist}(x, [\alpha, \beta])} < \eta.$$

Let $\eta > 0$ be given and set

$$g(y) = f(y) - \sum_{n=0}^k \frac{(y-x)^n}{n!} f_{(n)}(x).$$

Then $g_{(k)}(y)$ exists for each $y \in [a, b]$ and furthermore

$$g_{(k)}(y) = f_{(k)}(y) - f_{(k)}(x).$$

From the existence of $f_{(k)}$, there exists a $\delta > 0$ and a measurable set $E \subset [a, b]$ such that x is a point of density of E , and so that

$$(6.4) \quad |g(y)| \leq \frac{\epsilon(\eta/2)^k}{k! \cdot 2^{k(k+1)}} \cdot |y - x|^k$$

for $|y - x| < \delta$ and $y \in E$,

$$(6.5) \quad m(J \cap E^c) \leq m(J) \cdot \eta/2$$

for J an interval contained in $(x - \delta, x + \delta) \cap [a, b]$ and $x \in J$, where $E^c = [a, b] - E$.

Let $[\alpha, \beta]$ be a closed interval contained in $(x - \delta, x + \delta) \cap [a, b]$ such that $x \notin [\alpha, \beta]$. First assume that $f_{(k)}(y) \geq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$. By Theorem 4.1, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Applying Lemma 6.1 to the function g , which satisfies $g^{(k)}(y) = f_{(k)}(y) - f_{(k)}(x)$ for each $y \in [\alpha, \beta]$, there exists a partition of $[\alpha, \beta]$, $\{\alpha = t_0 < t_1 < \dots < t_l = \beta\}$, with $l \leq 2^k$ such that for each $i = 1, 2, \dots, l$ and each $s, w \in [t_{i-1}, t_i]$ with $s \leq w$,

$$(6.6) \quad |g(w) - g(s)| \geq (\epsilon/k!)(m(A \cap [s, w]))^k$$

where $A = \{y \in [\alpha, \beta] : |g^{(k)}(y)| = |f_{(k)}(y) - f_{(k)}(x)| \geq \epsilon\}$. If $f_{(k)}(y) \leq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$, then consider $-g$ and apply Lemma 6.1 to obtain precisely the same inequality (6.6).

We first obtain an estimate for $m(A \cap E)$. For this purpose assume $[t_{i-1}, t_i] \cap E \neq \emptyset$. Let $t_{i-1} \leq t'_i \leq t''_i \leq t_i$ with $t'_i, t''_i \in E$. Then by (6.6) and (6.4)

$$\begin{aligned} m(A \cap [t'_i, t''_i]) &\leq (k!/\epsilon)^{1/k} |g(t''_i) - g(t'_i)|^{1/k} \\ &\leq (k!/\epsilon)^{1/k} (|g(t''_i)|^{1/k} + |g(t'_i)|^{1/k}) \\ &\leq \left(\frac{k!}{\epsilon}\right)^{1/k} \left(\frac{\epsilon(\eta/2)^k}{k! \cdot 2^{k(k+1)}}\right)^{1/k} (|t''_i - x| + |t'_i - x|) \\ &\leq (\eta/[2 \cdot 2^k]) [\text{dist}(x, [\alpha, \beta]) + (\alpha - \beta)] \\ &\leq (\eta/2l) [\text{dist}(x, [\alpha, \beta]) + (\alpha - \beta)]. \end{aligned}$$

If

$$s'_i = \inf\{t'_i : t'_i \in [t_{i-1}, t_i] \cap E\}$$

and

$$s''_i = \sup\{t''_i : t''_i \in [t_{i-1}, t_i] \cap E\}$$

then it follows from the above inequality that

$$\begin{aligned} m(A \cap E \cap [t_{i-1}, t_i]) &= m(A \cap E \cap [s'_i, s''_i]) \\ &\leq m(A \cap [s'_i, s''_i]) \leq (\eta/2l) [\text{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

Clearly the same estimate holds if $[t_{i-1}, t_i] \cap E = \emptyset$. Hence

$$\begin{aligned} m(A \cap E) &= m\left(A \cap E \cap \left(\bigcup_{i=1}^l [t_{i-1}, t_i]\right)\right) \\ &= \sum_{i=1}^l m(A \cap E \cap [t_{i-1}, t_i]) \\ (6.7) \quad &\leq \sum_{i=1}^l \frac{\eta}{2l} [\text{dist}(x, [\alpha, \beta]) + (\beta - \alpha)] \\ &\leq \frac{\eta}{2} [\text{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

Secondly, we obtain an estimate of $m(A \cap E^c)$. Let J be the smallest

closed interval in $[a, b]$ containing both x and $[\alpha, \beta]$. Using (6.5) we have the following estimate

$$(6.8) \quad \begin{aligned} m(A \cap E^c) &\leq m(J \cap E^c) \leq (\eta/2) \cdot m(J) \\ &\leq (\eta/2)[\text{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

Therefore by (6.7) and (6.8)

$$m(A) = m(A \cap E) + m(A \cap E^c) \leq [\text{dist}(x, [\alpha, \beta]) + (\beta - \alpha)] \cdot \eta$$

and (6.3) holds. Thus, $f_{(k)}$ has Property Z on $[a, b]$ and the proof is complete.

Property Z was first introduced by Weil [10]. He further showed in the same paper that if a function g has the Darboux property and Property Z then g also has the Zahorski property. (An example of a function having the Darboux property and the Zahorski property but not Property Z can be found in [10].) Hence, in the class of functions having the Darboux property, Property Z is strictly stronger than the Zahorski property.

Thus by Corollary 5.1 we have the following corollary to the last theorem.

COROLLARY 6.9. *If $f_{(k)}$ is defined on $[a, b]$, then $f_{(k)}$ has the Zahorski property on $[a, b]$.*

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