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ESSENTIAL EMBEDDINGS OF ANNULI AND MÖBIUS BANDS IN 3-MANIFOLDS

BY

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ABSTRACT. In this paper we give conditions when the existence of an "essential" map of an annulus or Möbius band into a 3-manifold implies the existence of an "essential" embedding of an annulus or Möbius band into that 3-manifold.

Let λ_1 and λ_2 be disjoint simple "orientation reversing" loops in the boundary of a 3-manifold M and A an annulus. Let $f: (A, \partial A) \longrightarrow (M, \partial M)$ be a map such that $f_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic and $f(\partial A) = \lambda_1 \cup \lambda_2$. Then we show that there is an embedding $g: (A, \partial A) \longrightarrow (M, \partial M)$ such that $g(\partial A) = \lambda_1 \cup \lambda_2$.

Introduction. In 1968 F. Waldhausen reported in [8] that the existence of an "essential map" of an annulus into an orientable 3-manifold guarantees the existence of an essential embedding of an annulus in that 3-manifold. To our knowledge, a proof of this result has not yet appeared. As the result seems to be of considerable interest, we give here a proof of it and a number of related embedding theorems. We also prove that disjoint, simple, freely homotopic, "orientation reversing" loops embedded in the boundary of a 3-manifold must bound an annulus embedded in that 3-manifold *if either loop represents a fundamental group element of infinite order.*

Notation and conventions. Throughout this paper all spaces are *simplicial complexes* and all maps are *piecewise linear.* We also follow the "general practice" of Waldhausen on page 57 in [9]: in order to obtain a *regular neighborhood*, choose a triangulation in which all subspaces, previously mentioned in the argument, are subcomplexes; construct its second derived and take the closed star of the object in question. We denote the *boundary* of a manifold by ∂M and $M - \partial M$ by $\operatorname{int}(M)$. A manifold N is *properly embedded* in a manifold M if $N \cap \partial M = \partial N$. A simple loop embedded in a 3-manifold M is *orientation-preserving* if it has a regular neighborhood in M which is a solid torus; otherwise it is *orientation-reversing.* A two-sided surface F properly embedded in M is *incompressible* (in M) if for each disk D embedded in M such that $D \cap F = \partial D$, ∂D is homotopic to a point in F.

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Throughout the remainder of this paper M denotes a 3-manifold and A is an annulus. The components of ∂A are denoted by c_1 and c_2.

DEFINITION OF ESSENTIALITY. Let F be either an annulus or Möbius band. The arc α is a *spanning arc* of F if it is properly embedded in F and $F - \alpha$ is simply connected. Let M be a compact 3-manifold and α a spanning arc of F. A map f: $(F, \partial F) \longrightarrow (M, \partial M)$ is essential if

(1) f_*: $\Pi_1(F) \longrightarrow \Pi_1(M)$ is monic; and

(2) $f|\alpha$ is not homotopic rel its boundary to an arc in ∂M.

The *singular set* S_f of a map f: $M \longrightarrow N$ is the closure of $\{x \in M: f^{-1}f(x) \neq \{x\}\}$. The *complexity* $C(f)$ of f is the smallest total number of simplexes necessary to triangulate S_f. We choose this notion of complexity to avoid the problems presented by branch points. It is well known that if M has dimension 2 and N has dimension 3, we may suppose that a triangulation of S_f contains no simplexes of dimension greater than 1.

RESULTS. We state and prove below Theorem 1 which we believe to be the most difficult result in this paper.

THEOREM 1. *Let M be a compact, orientable 3-manifold and f: $(A, \partial A) \longrightarrow (M, \partial M)$ an essential map such that $f|\partial A$ is a homeomorphism. Then there is an essential embedding g: $(A, \partial A) \longrightarrow (M, \partial M)$ such that $g(\partial A) = f(\partial A)$.*

REVISION OF THEOREM 1. The proof of Theorem 1 involves a rather complex mixture of the tower building technique of Papakyriakopoulos (à la Shapiro-Whitehead-Stallings [3], [4], [6]) with other covering space and cut-out-paste techniques. In order to avoid a morass of notation, it is important to fracture the proof into well-defined steps, each independent of the others. The principle difficulty in so fracturing the proof is the difficulty in measuring the essentiality of a map, for essentiality can be destroyed in tower building processes. This difficulty is overcome by a refinement of the notion of essentiality.

REFINED DEFINITION OF ESSENTIALITY FOR THEOREM 1. The notions of groupoid and fundamental subgroupoid are defined in the appendix. If (X, Y) is a topological pair, then $P(X, Y)$ denotes the *fundamental subgroupoid* of X with *base points* Y, so that $P(X) = P(X, X)$ is the *fundamental groupoid* of X and $\Pi_1(X, x) = P(X, \{x\})$ is the *fundamental group* of X with *base point* x.

DEFINITION. A *triple* (M, f, G) consists of a 3-manifold M, a map f: $(A, \partial A) \longrightarrow (M, \partial M)$ and a subgroupoid G of $P(M, f(\partial A))$ containing $f_\#(P(\partial A))$. A triple is

(1) *essential* if $f_\# | \Pi_1(A)$ is monic and if, for each spanning arc α of A, $f_\#([\alpha]) \in P(M, f(\partial A)) - G$ (the requirement that G contain $f_\#(P(\partial A))$ is included to make this notion of essentiality independent of which spanning arc α of A one uses in checking the condition);

(2) *Dehn* if $S_f \cap \partial A = \emptyset$;

(3) *nonsingular* if f is an embedding.

If, in (1), $G = i_{\#}(\partial M, f(\partial A)) \subset P(M, f(\partial A))$, where i: $(\partial M, f(\partial A)) \longrightarrow (M, f(\partial A))$, then this definition of essentiality reduces to the original one.

THEOREM 1′. *Suppose that M is an orientable 3-manifold. If there is an essential Dehn triple (M, f, G) then there is an essential nonsingular triple (M, g, G) such that $g(\partial A) = f(\partial A)$.*

PROOF. The proof of Theorem 1′ is rather long. Although it could be arranged so as to be essentially constructive, an indirect proof seems to save most on notation. Hence we make the following (false) assumption:

BASIC ASSUMPTION. There is a counterexample (M, f, G) to Theorem 1′; i.e., M is orientable, (M, f, G) is an essential Dehn triple, and there is no essential nonsingular triple (M, g, G) such that $g(\partial A) = f(\partial A)$.

We show by means of two claims to be established that the basic assumption leads to a contradiction. The first of these claims (Claim 1) consists in showing, under the basic assumption, that there is a very nice counterexample (M, f, G) to Theorem 1′. Claim 2 states that M, whose existence was established by Claim 1, admits a very nice collection of properly embedded annuli. The existence of these annuli demonstrates very simply that the triple (M, f, G) could not have been essential and Theorem 1′ follows.

Claim 1. Under the basic assumption, there exist a counterexample (M, f, G) to Theorem 1′ and a properly embedded, two-sided surface F in M satisfying the following three conditions:

(1) for both $i = 1$ and $i = 2$, the intersection number $|sc(F, f(c_i))|$ of F with $f(c_i)$ is equal to the cardinality of $F \cap f(c_i)$.

(2) $f^{-1}(F)$ is the disjoint union of finitely many spanning arcs of A, each embedded by f, and $f^{-1}(F)$ divides A into complementary domains $D_1, \ldots, D_k$, each embedded by f; and for each pair i and j of distinct indices, $(f|D_i)^{-1}f(D_j)$ is the disjoint union of finitely many simple closed curves and of finitely many open arcs having disjoint arc closures, each such arc separating ∂D_i in D_i;

(3) F is connected; ∂M is incompressible in M and has total genus 1; there exist finitely many disjoint disks $E_1, \ldots, E_m$ in $(\operatorname{int} F) - f(A)$ such that $F - \bigcup \operatorname{int}(E_i)$ deformation retracts onto $F \cap f(A)$; and F is incompressible.

SATISFYING CONDITION (1) OF CLAIM 1. Let (M', f', G') be a counterexample having the smallest possible complexity. Let M be a regular neighborhood of $f'(A)$ in M'. Let $f = f'$: $A \longrightarrow M$. Let i: $(M, f(\partial A)) \longrightarrow (M', f'(\partial A))$ be the inclusion map. Let $G = i_{\#}^{-1}(G') \subset P(M, f(\partial A))$. Then (M, f, G) is a counterexample such that M deformation retracts to $f(A)$. The following three lemmas imply that there is a properly embedded, two-sided surface f in M satisfying condition (1) of Claim 1.

LEMMA 1. *Suppose that* (M, f, G) *is a counterexample such that* M *deformation retracts onto* $f(A)$, *and suppose that* M *has a two-sheeted connected covering* $(\widetilde{M}, p)$ *to which* f *lifts. Let* $\widetilde{f}\colon (A, \partial A) \longrightarrow (M, \partial M)$ *denote a lift of* f, *and let* $p_\#\colon P(\widetilde{M}, \widetilde{f}(\partial A)) \longrightarrow P(M, f(\partial A))$ *denote the groupoid morphism induced by* p. *Then the triple* $(\widetilde{M}, \widetilde{f}, p_\#^{-1}(G))$ *is a counterexample with smaller complexity than* (M, f, G).

PROOF. We observe that $C(f) \geqslant C(\widetilde{f})$ since S_f and $S_{\widetilde{f}}$ are both 1-complexes, $S_f \supseteq S_{\widetilde{f}}$ and ∂S_f and $\partial S_{\widetilde{f}}$ are empty since S_f and $S_{\widetilde{f}}$ are unions of collections of closed loops. Lemma 1 is now a consequence of Lemma 4.3 in [2].

LEMMA 2. *Suppose that* (M, f, G) *is a counterexample such that* $f(A)$ *is a deformation retract of* M *and* M *has no connected, two-sheeted cover to which* f *lifts. Then* ∂M *is incompressible in* M *and has total genus* 1, f *maps* ∂A *into the single torus boundary component of* M, *and* $f_*(H_1(A))$ *is an infinite cyclic subgroup of* $H_1(M)$.

PROOF. Except for the fact that ∂M is incompressible, this is a simple modification of an argument in the proof of Lemma 4.1 in [4]. Suppose $\mathcal{D}$ is a disk properly embedded in M and $\partial \mathcal{D}$ is essential in ∂M. It can be seen that $f(\partial A) \cap \partial \mathcal{D}$ is not empty and that $f^{-1}(\mathcal{D})$ contains a spanning arc α of A. Let T be the component of ∂M on which $\partial \mathcal{D}$ lies. Let A_1 and A_2 be the closures of the components of $T - f(\partial A)$. If we suppose that $\partial \mathcal{D}$ meets $f(\partial A)$ in a minimal number of points, then $f(\alpha)$ is homotopic in $\mathcal{D}$ to a sequence of simple arcs in $\partial \mathcal{D}$ and each of these arcs is a spanning arc of either A_1 or A_2. It follows that either A_1 or A_2 contradicts the assumption that the triple (M, f, G) is a counterexample of Theorem 1$'$ since (M, f, G) is essential. This completes the proof of Lemma 2.

LEMMA 3. *Suppose that* (M, f, G) *is a triple such that* M *is compact and* $f_*(H_1(A))$ *is an infinite cyclic subgroup of* $H_1(M)$. *Then there is a properly embedded, two-sided surface* F *in* M *such that condition* (1) *of Claim* 1 *is satisfied.*

PROOF. The proof of this is a modification of the third paragraph of the proof of Theorem 3.1 in [2].

SATISFYING CONDITION (2) OF CLAIM 1. Let (M, f', G) be a counterexample and F' a properly embedded, two-sided surface in M satisfying condition (1) of Claim 1. We shall show that, by changing $\operatorname{int}(F')$ and $f'(\operatorname{int} A)$ so as to obtain a new surface F and new map f, we can obtain a counterexample (M, f, G) and surface F satisfying both conditions (1) and (2). In particular, we note for future use in satisfying condition (3) that since we do not change f' near ∂A and do not change F' near $\partial F'$ the new intersection number $k = |\mathrm{sc}(F, f(c_1))|$ agrees with the old one $k' = |\mathrm{sc}(F', f'(c_1))|$.

By standard techniques, we may change F' to an incompressible surface F with compatible orientation and the same boundary. Let $(\widetilde{M}, p)$ be the k-sheeted cyclic covering of M associated with F (i.e., cut M along F, take k copies of the resulting space, and sew them together cyclically along the various copies of F; note that the construction uses only the two-sidedness of F but depends, for non-connected F, on the choice of those two sides; in particular the construction makes sense even in nonorientable manifolds; note, further, that $p^{-1}(F)$ is the disjoint union $F_1 \cup \ldots \cup F_k$ of k homeomorphic copies of F, determined by the construction and carried by p homeomorphically onto F; there is also a natural covering translation, cyclic of order k, which cyclically permutes the k copies of M; this covering translation need not generate all covering translations for general M and F but will do so if $\widetilde{M}$ is connected; a loop in M lifts to $\widetilde{M}$ if and only if its intersection number with F is some multiple of k). Since $|sc(F, f'(c_1))| = k$, there is a lift g': $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ of f'. Let $p_{\#}$: $P(\widetilde{M}, g'(\partial A)) \longrightarrow P(M, f'(\partial A))$ be the groupoid morphism induced by p. Note that the triple $(\widetilde{M}, g', p_{\#}^{-1}(G))$ is an essential Dehn triple. Note also that $g'(c_1)$ [and $g'(c_2)$] intersects each copy F_i of F tranversely in a single point. It follows in particular that $g'_*(H_1(A))$ is an infinite cyclic direct summand of $H_1(\widetilde{M})$. Hence we may apply the following lemma from [2] to conclude that there is an essential, nonsingular triple $(\widetilde{M}, g, p_{\#}^{-1}(G))$ such that $g(\partial A) = g'(\partial A)$.

LEMMA 4 [2, Theorem 3.1] (see the appendix for further discussion). *Let (M, f, G) be an essential Dehn triple such that M is orientable and $f_*(H_1(A))$ is an infinite cyclic direct summand of $H_1(M)$. Then there is an essential nonsingular triple (M, g, G) such that* $g(\partial A) = f(\partial A)$.

Since $p^{-1}(F)$ is incompressible in $\widetilde{M}$, we may assume that $g^{-1}p^{-1}(F)$ is the disjoint union of finitely many spanning arcs of A which divide A into complementary domains $D_1, \ldots, D_k$. After slight adjustment for general position, we may assume that $g(A)$, all of its images under the cyclic covering translations of $(\widetilde{M}, p)$ mentioned above, and all of the surfaces $F_1, \ldots, F_k$ are in mutual general position. If we let $f = pg$, then the only part of condition (2) possibly not satisfied by f is the requirement that each of the open arc components of $(f|D_i)^{-1}f(D_j)$ separate ∂D_i in D_i. We now show how to modify f so as to satisfy this final condition.

It can be seen that $g^{-1}p^{-1}(F)$ is the union of exactly k spanning arcs which we denote by $\alpha_1, \ldots, \alpha_k$. It follows from general position that

(1) $f(\alpha_i) \cap f(\alpha_j)$ is finite for $1 \leqslant i < j \leqslant k$;

(2) $f(\alpha_i)$ crosses $f(\alpha_j)$ at each point in $f(\alpha_i) \cap f(\alpha_j)$ for $1 \leqslant i < j \leqslant k$;

(3) $f(\alpha_i) \cap f(\alpha_j) \cap f(\alpha_s)$ is empty for $1 \leqslant i < j < s \leqslant k$.

Let $X_0 = \bigcup_{i \neq j}^{k} (f(\alpha_i) \cap f(\alpha_j))$.

We assume that f has been chosen to minimize the cardinality σ_0 of X_0. Suppose $\beta_1 \subset f(\alpha_r)$ and $\beta_2 \subset f(\alpha_s)$ are subarcs such that $\beta_1 \cup \beta_2$ bounds a disk $\mathcal{D}$ embedded in F and $\partial\beta_1 = \partial\beta_2 = \beta_1 \cap \beta_2$ where $1 \leqslant r < s \leqslant k$. Suppose further that for each arc β contained in $f(\alpha_i)$ and properly embedded in $\mathcal{D}$, $\beta \cap \beta_j$ is a single point for $j = 1, 2$ and $1 \leqslant i \leqslant k$. Then σ_0 could be reduced by pushing β_1 to β_2 across $\mathcal{D}$ and then pushing $f(\alpha_r)$ off of β_2. Since the reduction above could be accomplished with a homotopy of f in M covered by an isotopy of g in $\widetilde{M}$, the existence of β_1, β_2, and $\mathcal{D}$ contradicts the minimality of σ_0.

We claim that if one of the open arcs in $f|\mathcal{D}_j^{-1}(f(\mathcal{D}_i))$ fails to separate $\partial\mathcal{D}_j$, arcs β_1 and β_2 and a disk $\mathcal{D}$ as in the preceding paragraph exist. Suppose that δ is the closure of such an open arc. Then we may suppose that $\partial\delta$ lies on the spanning arc α_i where $1 \leqslant i \leqslant k$. Let δ_1 be the subarc of α_i bounded by $\partial\delta$. Now $\delta_1 \cup \delta$ bounds a disk E_1 on $\mathcal{D}_j$ and we may suppose that the closure of $E_1 \cap f^{-1}f(\mathcal{D}_i)$ contains no properly embedded arcs that have their boundary on δ_1. Let δ^* be an arc in D_i such that $f(\delta^*) = f(\delta)$. We may suppose that $\partial\delta^*$ lies on the spanning arc α_j ($\partial\delta^*$ must lie on some spanning arc since $f(\delta^*)$ is homotopic to an arc in F) where $1 \leqslant j \leqslant k$ and $j \neq i$, $\partial\delta^*$ cuts off an arc δ_2 on α_j, and $\delta_2 \cup \delta^*$ bounds a disk E_2 on D_i. Now $f(\delta_1 \cup \delta_2)$ is a simple loop on F that bounds the singular disk $f(E_1 \cup E_2)$. Since F is incompressible, $f(\delta_1 \cup \delta_2)$ bounds a disk E embedded in F. If β is an arc properly embedded in E such that $\partial\beta$ lies on $f(\delta_1)$ then β together with an arc on $f(\delta_1)$ bounds a subdisk of E. It follows from a standard argument that there exist arcs β_1 and β_2 in E and a subdisk $\mathcal{D}$ of E satisfying the claim above.

This establishes condition (2) of Claim 1.

SATISFYING CONDITION (3) OF CLAIM 1. Among all counterexamples to Theorem 1′ and surfaces satisfying condition (1), choose a counterexample (M_0, f_0, G_0) and surface F_0 such that F_0 is incompressible and $k = |sc(F_0, f_0(c_1))|$ is minimal. As noted in the discussion of condition (2), we may assume that (M_0, f_0, G_0) and F_0 also satisfy condition (2). Among all such counterexamples, we may also assume that $f_0(S_{f_0}) \cap F_0$ has the smallest cardinality, and, among such, that f_0 has the smallest complexity. After slight modification, as we shall see, (M_0, f_0, G_0) and F_0 will also satisfy condition (3).

Let M be a regular neighborhood of $f_0(A)$ in M_0. Let $f = f_0\colon (A, \partial A) \longrightarrow (M, \partial M)$. Let $i\colon (M, f(\partial A)) \longrightarrow (M_0, f_0(\partial A))$ be the inclusion map. Let $G = i_\#^{-1}(G_0)$. Let $F_1 = F_0 \cap M$. Since k was minimal, at most one component of F_1 intersects $f(\partial A)$, and that component contains $f_0 f_0^{-1}(F_0)$. Hence we may assume F_1 is connected and that (M, f, G) and F_1 still satisfy conditions (1) and (2).

By Lemma 1, M has no connected two-sheeted cover to which f lifts. By Lemma 2, ∂M is incompressible in M and has total genus 1, and f maps ∂A into the single torus boundary component T of M. Hence only the last two parts of condition (3) remain to be satisfied. Note that it is a consequence of Lemma 4 that $k \neq 1$ and of Lemma 1 that $k \neq 2$. Thus $k \geqslant 3$. Note that if $d_1, \ldots, d_k$ denote the components of ∂F_1 intersecting $f(c_1) \cup f(c_2)$, then $Y = \partial M - (f(c_1) \cup f(c_2) \cup d_1 \cup \ldots \cup d_k)$ is simply connected. Let $e_1, \ldots, e_m$ denote the boundary components of F_1 in Y. It is possible to fill these in with disks $E_1, \ldots, E_m$ in Y and to push these slightly into $\operatorname{int} M$ so as to form a new properly embedded two-sided surface F with boundary $d_1 \cup \ldots \cup d_k$. We claim that (M, f, G) and F satisfy all of conditions (1), (2) and (3). It obviously remains only to check the fact that F is incompressible in M.

Let $S \subset F_0$ be a regular neighborhood in F of $f(A) \cap F$. Let l be a simple loop properly embedded in S that bounds a disk $\mathcal{D}$ in M such that $\mathcal{D} \cap F = \partial \mathcal{D} = l$. It is sufficient to show that l bounds a disk in F or that l is nullhomotopic in F.

Let $(\hat{M}, p)$ be the k-sheeted cover of M associated with F_1 and g an embedding such that $pg = f$. Let $\hat{\mathcal{D}}$ be a disk embedded in $\hat{M}$ so that $p\hat{\mathcal{D}} = \mathcal{D}$. Let $\hat{l} = \partial\hat{\mathcal{D}}$. Clearly F lifts to $\hat{M}$. Let $\hat{F}$ be the component of $p^{-1}(F)$ on which $\hat{l}$ lies, and $\hat{S} = p^{-1}(S) \cap \hat{F}$.

It can be seen that $\hat{\mathcal{D}}$ meets exactly one component of $\hat{M} - p^{-1}(F)$. We denote the closure of this component by $\hat{B}$ and observe that $\hat{B} \neq \hat{M}$ since $k \geqslant 3$. For $i = 1, \ldots, k$, we observe that the closure of $(p^{-1}(\mathcal{D}_i) \cap \hat{B})$ is a disk properly embedded in $\hat{B}$ and denote this disk by $\hat{\mathcal{D}}_i$. We suppose that l and $\mathcal{D}$ have been so chosen that the number of points in $\hat{l} \cap \hat{\mathcal{D}}_1$ is minimal (finite), that $\hat{l}$ is not nullhomotopic in $\hat{F}$, and that $\hat{\mathcal{D}}$ is in general position with respect to $\bigcup_{i=1}^{k} \hat{\mathcal{D}}_i$. Suppose that $\hat{l} \cap \hat{\mathcal{D}}_1$ is not empty. After the usual argument, we may suppose that $\hat{\mathcal{D}} \cap \hat{\mathcal{D}}_1$ contains no simple loops. Note that this can be done by adding disks that lie in a regular neighborhood of $\hat{\mathcal{D}}_1$ to a punctured subdisk of $\hat{\mathcal{D}}$. Thus there is an arc $\beta \subset \hat{\mathcal{D}} \cap \hat{\mathcal{D}}_1$ properly embedded in $\hat{\mathcal{D}}_1$ that cuts off a disk $\hat{E}$ on $\hat{\mathcal{D}}_1$ such that int $(\hat{E})$ does not meet $\hat{\mathcal{D}}$ and $\partial\hat{E} - \beta$ lies on $\hat{F}$. Now β separates $\hat{\mathcal{D}}$ into two disks $\hat{E}'$ and $\hat{E}''$. Let $\hat{E}_1 = \hat{E} \cup \hat{E}'$ and $\hat{E}_2 = \hat{E} \cup \hat{E}''$. Both $\hat{E}_1$ and $\hat{E}_2$ are disks embedded in $\hat{B}$ and either $\partial\hat{E}_1$ or $\partial\hat{E}_2$ is essential in $\hat{F}$. Suppose $\partial\hat{E}_1$ is essential in $\hat{F}$. Deform $\hat{E}_1$ slightly pushing $\partial\hat{E} \cap \partial\hat{E}_1$ off of $\partial\hat{\mathcal{D}}_1$ to obtain a disk $\hat{\mathcal{D}}^*$. Now $\partial\hat{\mathcal{D}}^* \cap \hat{\mathcal{D}}_1$ contains fewer points than $\partial\hat{\mathcal{D}} \cap \hat{\mathcal{D}}_1$ and this contradicts our choice of $\mathcal{D}$ and l.

We proceed under the added assumption that $\hat{l} \cap \hat{\mathcal{D}}_1$ is empty and assume that l and $\mathcal{D}$ have been chosen so that

(1) $\hat{l} \cap \hat{\mathcal{D}}_2$ contains a minimal number of points;

(2) $\hat{\mathcal{D}}$ is in general position with respect to $\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2, \ldots, \hat{\mathcal{D}}_k$;

(3) $\hat{\mathcal{D}} \cap \hat{\mathcal{D}}_2$ contains no simple loops (note that $\hat{\mathcal{D}} \cap \hat{\mathcal{D}}_1$ may not be empty).

Suppose that $\hat{l} \cap \hat{\mathcal{D}}_2$ is not empty. As above we can find an arc $\beta \subset \hat{\mathcal{D}}_2$ and disk $\hat{E} \subset \hat{\mathcal{D}}_2$ such that $\hat{E} \cap \hat{\mathcal{D}} = \beta$ and $\partial \hat{E} - \beta \subset \hat{F}$. Let $\hat{E}'$ and $\hat{E}''$ be the closures of the components of $\hat{\mathcal{D}} - \beta$. Let $\hat{E}_1 = \hat{E}' \cup \hat{E}$ and $\hat{E}_2 = \hat{E}'' \cup \hat{E}$ and observe that $\hat{E}_1$ and $\hat{E}_2$ are disks embedded in $\hat{M}$. We may suppose that $\partial \hat{E}_1$ is essential in F. We deform $\hat{E}_1$ slightly to obtain a disk $\hat{\mathcal{D}}^*$ such that $\partial \hat{\mathcal{D}}^* \cap \hat{\mathcal{D}}_2$ contains fewer points than $\partial \hat{\mathcal{D}} \cap \hat{\mathcal{D}}_2$. If $\partial \hat{\mathcal{D}}^*$ does not meet $\hat{\mathcal{D}}_1$, the existence of $\hat{\mathcal{D}}^*$ contradicts our choice of $\mathcal{D}$ and l. Thus if $\beta_1 = \partial \hat{E} - \beta$ cannot meet $\hat{\mathcal{D}}_1$, we will conclude that $\mathcal{D}$ and l have been chosen so that $\hat{l}$ misses not only $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ but also $\hat{\mathcal{D}}_3, \ldots, \hat{\mathcal{D}}_k$.

Consider the loop $p\partial \hat{\mathcal{D}}^*$ as a subset of F_0. This loop is nullhomotopic in M_0 and thus also in F_0. Thus $p\partial\hat{\mathcal{D}}$ bounds a disk E' embedded in F_0. If $\beta_1 \cap \partial \hat{\mathcal{D}}_1$ is not empty, there are arcs δ_1 and δ_2 embedded in E' such that

(1) $\delta_1 \cap \delta_2 = \partial \delta_1 = \partial \delta_2$;

(2) $\delta_1 \subset f(\alpha_j)$ and $\delta_2 \subset f(\alpha_i)$ where $1 \leqslant i \leqslant j \leqslant k$;

(3) $\delta_1 \cup \delta_2$ is the boundary of a disk E embedded in F_0

since $p\beta_1 \subset ff^{-1}(F)$ and $\partial \hat{\mathcal{D}}_1$ does not meet $\partial \hat{\mathcal{D}}^* - \beta_1 \subset \hat{l}$. Note that the existence of δ_1, δ_2, and E has been shown to contradict the minimality of the cardinality of $f_0(S_{f_0}) \cap F_0$. It follows that β_1 could not meet $\hat{\mathcal{D}}_2$ and by induction that $\hat{l}$ does not meet $\bigcup_{i=1}^k (\hat{\mathcal{D}}_i \cap \hat{F})$ or that $l \cap f(A)$ is empty. But then l lies either in an annular neighborhood of a component of ∂S or in a disk on S. In either case l is nullhomotopic on F which contradicts our hypothesis that l was essential on F. Thus F is incompressible and condition (3) in Claim 1 is established.

Claim 2. Suppose (M, f, G) and F satisfy Claim 1 and the basic assumption. Then there is a collection $A_1, \ldots, A_n$ of annuli properly embedded in M such that

(1) $\partial A_i = f(\partial A)$ for $i = 1, \ldots, n$;

(2) $F \cap A_i$ is a collection of arcs properly embedded in F for $i = 1, \ldots, n$;

(3) $A_i \cap A_j = \partial A_i$ for $1 \leqslant i < j \leqslant n$;

(4) $F - \bigcup_{i=1}^n A_i$ is simply connected.

Step 1. Let $(\hat{M}, p)$ be the k-sheeted cover of M associated with F. Let $\hat{F}$ be a component of $p^{-1}(F)$ and β an arc properly embedded in $\hat{F}$ such that $\partial\beta$ meets both $p^{-1}f(c_1)$ and $p^{-1}f(c_2)$. Then we claim there is an annulus $\hat{A}$ properly embedded in $\hat{M}$ such that $\hat{A} \cap \hat{F} = \beta$ and $\hat{A} \cap \partial \hat{M} \subset p^{-1}f(\partial A)$.

Let g be an embedding such that $pg = f$ and ρ a generator of the group of covering translations of $(\hat{M}, p)$. It is a consequence of condition (3) of Claim 1 that we may suppose that β lies on $p^{-1}f(A) \cap \hat{F}$ and thus that β is homotopic rel its boundary to the product of a sequence of simple arcs $\beta_1 * \beta_2 * \ldots * \beta_w$ such that $\beta_i \subset \rho^{j(i)}g(A) \cap \hat{F}$ where $1 \leqslant i \leqslant w$ and $0 \leqslant j(i) < k$. We may assume that $\beta_i \cap \beta_{i+1}$ is a single point in $\partial\beta_i \cap \partial\beta_{i+1}$ and that point determines a loop λ_i in $\rho^{j(i)}g(A) \cap \rho^{j(i+1)}g(A)$ for $1 \leqslant i \leqslant w-1$ since $\rho^{j(i)}g(A) \cap \rho^{j(i+1)}g(A)$ is the union of a collection of simple loops such that each component of the intersection that meets F meets F in a single point (by condition (2) of Claim 1). We may assume that β_1 has one endpoint on a loop $\lambda_0 \subset p^{-1}f(c_1)$ and β_w has one endpoint on $\lambda_w \subset p^{-1}f(c_2)$. Let $I = [0, 1]$. We define a map ϕ on the boundary of $I \times I$ by requiring that

(1) ϕ carry the interior of $I \times \{0\}$ homeomorphically into λ_0;

(2) ϕ carry the interior of $I \times \{1\}$ homeomorphically into λ_w;

(3) ϕ carry each component of $\{0, 1\} \times [i/w, (i+1)/w]$ homeomorphically onto β_{i+1} for $i = 0, \ldots, w-1$;

(4) the intersection number of the loop $\phi(I \times \{j\})$ and F is one for $j = 0, 1$.

The reader should observe that if we can show that the loop $\phi(\partial(I \times I))$ is nullhomotopic, the existence of the desired annulus follows from Dehn's lemma [3]. The proof of the above statement is especially easy, since one will be able to see that ϕ could be constructed so that $\phi^{-1}(F) = \{0, 1\} \times I$.

We extend ϕ to carry the interior of $I \times \{i/w\}$ homeomorphically into λ_i for $i = 1, \ldots, w-1$ so that the loop $\phi(I \times i/w)$ has intersection number one with $\hat{F}$ for $i = 1, \ldots, w-1$. We observe that ϕ carries the loop $I \times \{i/w, (i+1)/w\} \cup \{0, 1\} \times [i/w, (i+1)/w]$ into the annulus determined by β_{i+1} and that the image loop is nullhomotopic, by construction, on that annulus for $i = 0, \ldots, w-1$. Thus ϕ can be extended to $I \times I$. This shows the existence of the annulus required in Step 1 of Claim 2.

Step 2. We show the existence of the annuli required in Claim 2.

First we prove a lemma useful in establishing the existence of desired collection of annuli.

LEMMA 5. *Let M be a compact, orientable 3-manifold such that ∂M is incompressible in M and has total genus* 1, *and let F be an incompressible, two-sided, nonseparating surface properly embedded in M. Let $(\widetilde{M}, p)$ be a k-sheeted cyclic covering space of M associated with F. Let $g: (A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ be an embedding such that*

(1) $g_{\#}: \pi_1(A) \longrightarrow \pi_1(\widetilde{M})$ *is monic*;

(2) $g(A)$ *meets each component of $p^{-1}(F)$ in a simple spanning arc of* $g(A)$;

(3) *there is a component F_1 of $p^{-1}(F)$ which $g(A)$ does not separate*;

(4) $g(c_1)$ *crosses $p^{-1}(F)$ at each point in* $g(c_1) \cap p^{-1}(F)$.

Then if $pg|\partial A$ is embedding, there is an embedding $h: (A, \partial A) \longrightarrow (M, \partial M)$ such that

(5) $h(\partial A) = pg(\partial A)$;

(6) $h^{-1}(F)$ *is a collection of spanning arcs* $\alpha_1, \ldots, \alpha_k$;

(7) *$h(\alpha_i)$ is not homotopic rel its boundary to an arc in ∂M for $i = 1, \ldots, k$.*

PROOF. Let $N(F)$ be a collar neighborhood on F in M. Since F is incompressible in M, it is possible to kill $\pi_2(M)$ by attaching 3-cells to $M - N(F)$. Let M^+ be the resulting space (which is locally a 3-manifold near F). Let $(\widetilde{M}^+, p^+)$ be the k-sheeted cyclic covering of M^+ associated with F, and note that we may think of $\widetilde{M}$ as $p^{+-1}(M)$ and of p as $p^+ | \widetilde{M}$.

Let T be the boundary component of M having genus 1. Let T^+ be a solid torus with boundary T such that the curves $f(c_i)$ ($i = 1$ and 2) are nullhomotopic in T^+. Let $M^* = M^+ \cup_T T^+$. Since the curves $f(c_i)$ lift homeomorphically to the regular covering space $\widetilde{M}^+$, it is an easy matter to extend the covering $(\widetilde{M}^+, p^+)$ to a covering $(\widetilde{M}^*, p^*)$ of M^*.

Form a 2-sphere S^2 from A by attaching disjoint 2-cells D_1 and D_2 to ∂A along c_1 and c_2. Extend $f = pg: A \longrightarrow M$ to a map $f^*: S^2 \longrightarrow M^*$ which takes D_1 and D_2 homeomorphically onto disjoint meridional disks E_1 and E_2 in T^+. Let $g^*: S^2 \longrightarrow \widetilde{M}^*$ be the lift of f^* to $\widetilde{M}^*$ which extends the embedding g. We see that g^* (hence also f^*) is essential as follows: Recall that F_1 is a component of $p^{-1}F$ which $g(A)$ does not separate. There is therefore a simple closed curve J in int F_1 which crosses $g^*(S^2)$ transversely at a single point. Since $\widetilde{M}^*$ is locally a 3-manifold near F_1, the existence of J implies that g^* is essential in $\widetilde{M}^*$. Hence f^* is essential in M^*.

By the proof of the sphere theorem in [3, 9], there is an embedding h^*: $S^2 \longrightarrow M \cup T^+$, essential in M^*, which meets the solid torus T^+ only in some subset of the two meridional disks E_1 and E_2. Since $\pi_2(M^+) = 0$, $h^*(S^2)$ contains at least one of the two disks. If $h^*(S^2)$ contained only one of the two disks, T would be compressible in M. Thus $h^*(S^2) \cap T^+ = E_1 \cup E_2$, and $h^*(S^2) \cap M$ is a properly embedded annulus in M with the same boundary as $pg(A)$.

Let $h: A \longrightarrow h^*(S^2) \cap M$ be a homeomorphism. It is clear that $h^{-1}(F)$ is the union of a collection of spanning arcs $\alpha_1, \ldots, \alpha_k$ and simple closed curves $J_1, \ldots, J_m$. If, for any i, $h(\alpha_1)$ were homotopic rel its boundary to an arc in M, then, since T is incompressible in M and both $\pi_2(M^+)$ and $\pi_2(T^+)$ are 0, it would follow that $h^*(S^2)$ is inessential in M^*, a contradiction. The simple closed curves $J_1, \ldots, J_m$ can be removed since F is incompressible. Hence the proof of the lemma is complete.

We continue with the proof of Step 2 after making the following fundamental observation:

OBSERVATION 1. Suppose (M, f, G) is an essential Dehn triple and for $i = 1, \ldots, n$, $g_i: A \longrightarrow M$ is a collection of proper embeddings such that $f(\partial A) = g_i(\partial A)$ for $i = 1, \ldots, n$. Then if there is a spanning arc α of A such that $f(\alpha)$ is homotopic rel its boundary to a sequence of spanning arcs on the $g_i(A)$ at least one of the triples (M, g_i, G) is essential and thus (M, f, G) does not satisfy the basic assumption.

We suppose again that (M, f, G) and F are an essential Dehn triple and surface that satisfy the fundamental assumption and the conditions of Claim 1 and that $(\hat{M}, p)$ is a k-sheeted covering space of M associated with F. Let $\hat{F}$ be a component of $p^{-1}(F)$. Since F is not a disk there is an arc β_1 properly embedded in $\hat{F}$ that does not separate $\hat{F}$ such that $\partial\beta_1$ meets both $p^{-1}f(c_1)$ and $p^{-1}f(c_2)$. It is a consequence of Step 1 that there is an annulus A_1 properly embedded in $\hat{M}$ such that $A_1 \cap \hat{F} = \beta_1$ and $\partial A_1 \subset p^{-1}f(\partial A)$. After the usual argument we suppose A_1 meets each component of $p^{-1}(F)$ in a spanning arc. Thus by Lemma 5 there is an essential embedding $g_1: (A, \partial A) \longrightarrow (M, f(\partial A))$ such that $g_1(A) \cap F$ is a collection of disjoint simple arcs and none of these arcs is homotopic rel its boundary to an arc in ∂F. We observe that f is homotopic to a map $\bar{f}$ such that

(1) $\bar{f}^{-1}g_1(A)$ is the union of a collection of disjoint essential simple loops $\lambda_1, \ldots, \lambda_m$;

(2) $\bar{f}(\bigcup_{i=1}^m \lambda_i) \subset f(\partial A) = g_1(\partial A)$;

(3) $\bar{f}^{-1}(F)$ is a collection of k spanning arcs;

(4) $\bar{f}|\lambda_i$: $\lambda_i \longrightarrow f(\partial A)$ is a homeomorphism.

Note that the restriction of $\bar{f}$ to the closure of each component of $A - \bigcup_{i=1}^m \lambda_i$ determines a map of an annulus A into M. Denote these maps by f_i where $1 \leqslant i \leqslant m - 1$ and observe that since (M, f, G) is an essential triple, (M, f_j, G) must be an essential triple for some j where $1 \leqslant j \leqslant m - 1$. Thus we may suppose that (M, f_1, G) is an essential triple and $f_1(A) \cap g_1(A) = \partial g_1(A)$.

Suppose $F - g_1(A)$ is not simply connected. Then there is an arc β_2 properly embedded in $\hat{F}$ such that $\beta_2 \cap (\hat{F} \cap p^{-1}g_1(A)) = \partial\beta_2$ and $\beta_2 \cup (p^{-1}g_1(A) \cap \hat{F})$ does not separate $\hat{F}$. As above there is an annulus A_2 properly embedded in $\hat{M}$ such that $A_2 \cap \hat{F} = \beta_2$ and $\partial A_2 \subset p^{-1}f(\partial A)$. Since $A_2 \cap (p^{-1}g_1(A)) \cap \hat{F} = \partial\beta_2$, we may suppose after the usual argument that $A_2 \cap p^{-1}g_1(A) = \partial A_2$ and that A_2 meets each component of $p^{-1}(F)$ in a single spanning arc of A_2.

We would like to apply Lemma 5 to the manifolds M_1 and $\hat{M}_1$ obtained by splitting M along $g_1(A)$ and $\hat{M}$ along $p^{-1}g_1(A)$. This can be done if ∂M_1 is incompressible and of total genus 1. Let P: $M_1 \longrightarrow M$ be the natural identification map and $\bar{f}_1: A \longrightarrow M_1$ the map induced by f_1.

Suppose there is a disk $\mathcal{D}$ properly embedded in M_1 such that $\partial\mathcal{D}$ is essential in ∂M_1. We may assume that $\partial\mathcal{D}$ is in general position with respect to $P^{-1}f(\partial A)$ and that the cardinality of the set $\partial\mathcal{D} \cap P^{-1}f(\partial A)$ is minimal. Since $\bar{f}_{1*}: \pi_1(A) \longrightarrow \pi_1(M_1)$ is monic, $f_1(\partial A)$ must meet $\partial\mathcal{D}$. Thus there is a spanning arc α in $\bar{f}_1^{-1}(\mathcal{D})$. But $\bar{f}_1(\alpha)$ is homotopic rel its boundary to a sequence of spanning arcs of the annuli which are the closures of the components of $\partial M_1 - P^{-1}f(\partial A)$. Thus it can be seen that $f_1(\alpha)$ is homotopic rel its boundary to a product of spanning arcs in the annuli which are the closures of components of $\partial M - f(\partial A)$ and the annulus $g_1(A)$. Thus (M, f_1, G) and (M, f, G) do not satisfy the basic assumption because of Observation 1. It follows that ∂M_1 is incompressible.

If ∂M_1 has two components T_1 and T_2 of genus 1 and M_1 is connected, we can assume that β_2 above runs from $p_1^{-1}(T_1)$ to $p_1^{-1}(T_2)$ where p_1 is the covering map associated with $\hat{M}_1$. It follows that $p_1 A_2$ is a singular annulus and by the theorem in [4] there is a proper embedding $g_2'\colon A \longrightarrow M_1$ such that $g_2'(\partial A) = p_1(\partial A_2)$. Let $g_2 = Pg_2'\colon A \longrightarrow M$. It is easily seen that $g_2^{-1}(F)$ may be taken to be the union of a nonempty collection of spanning arcs and that at least one of these, say α, has the property that $g_2'(\alpha)$ is not homotopic in $P^{-1}(F)$ to an arc in $\partial P^{-1}(F)$.

It should now be clear that we can use an inductive procedure to construct the annuli required in Step 2 and Claim 1 since F (or any compact 2-manifold) becomes simply connected after a finite number of nontrivial cuts.

Thus Theorem 1′ follows from Observation 1 and Theorem 1 follows from Theorem 1′.

THEOREM 2. *Let M be a compact, orientable 3-manifold. Let $f\colon (A, \partial A) \longrightarrow (M, \partial M)$ be an essential map such that $f(c_1)$ does not meet $f(c_2)$. Then there is an essential embedding $g\colon (A, \partial A) \longrightarrow (M, \partial M)$ such that $g(c_j)$ lies in any prespecified neighborhood of $f(c_j)$ for $j = 1, 2$.*

PROOF. The proof of this theorem is a simplification of the proof of Theorem 3 and we omit it. We state and prove below Waldhausen's "annulus theorem".

THEOREM 3. *Let M be a compact orientable 3-manifold. Let $f\colon (A, \partial A) \longrightarrow (M, \partial M)$ be an essential map. Then there is an essential embedding $g\colon (A, \partial A) \longrightarrow (M, \partial M)$.*

PROOF. We will show that there is an essential map $f_1\colon (A, \partial A) \longrightarrow (M, \partial M)$ such that $f_1 | \partial A$ is an embedding. Theorem 3 will then follow immediately from Theorem 1.

Since the inverse image under f of any disk properly embedded in M cannot contain a spanning arc of A, we may assume that ∂M is incompressible. We may assume that f carries ∂A to a single component F of ∂M since otherwise Theorem 3 follows from the Satz in [7] Let $(\widetilde{M}, p)$ be the covering space of M associated with $\pi_1(F, f(x)) \subset \pi_1(M, f(x))$ where x is some point in ∂A. Since $\pi_1(F, f(x)) \subset \pi_1(F, f(x))$, there is an embedding $\widetilde{F}$ of F in $\widetilde{M}$ such that $p | \widetilde{F}$ is a homeomorphism onto F. It can be seen that there is a map $\widetilde{f}\colon (A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ such that $p\widetilde{f} = f$ and $\widetilde{f}(c_1) \subset \widetilde{F}$. Since $\widetilde{f}$ is essential, $\widetilde{f}(c_2)$ does not meet $\widetilde{F}$. Thus $\widetilde{f}(c_1) \cap \widetilde{f}(c_2)$ is empty and it follows from the Satz in [7] that we may suppose that $\widetilde{f}$ is an embedding. We may now suppose that $f|c_1$ is an embedding since $p|\widetilde{F}$ is a homeomorphism.

It can be seen that there is a map f_1: $(A, \partial A) \longrightarrow (\widetilde{M}. \partial\widetilde{M})$ such that $pf_1 = f$ and $f_1(c_2)$ lies on $\widetilde{F}$. Since f is essential, $f_1(c_1)$ does not lie on $\widetilde{F}$. It follows from the Satz in [7] that there is an embedding h_1: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ such that $h_1(c_1)$ lies in an annular neighborhood of $f_1(c_1)$ in $\partial\widetilde{M}$ and $h_1(c_2)$ lies on $\widetilde{F}$. It can be seen that we may assume $f_1(c_1) = h_1(c_1)$. Since $h_1(c_1)$ and $h_1(c_2)$ lie on distinct components of $\partial\widetilde{M}$, ph_1 is an essential map. Since $p|\widetilde{F}$ is an embedding, we may suppose that $f|c_j$ is a homeomorphism for $j = 1, 2$. After a general position argument, we may suppose that $f(c_1) \cap f(c_2)$ is a finite set. We also assume that f has been chosen so that $f(c_1) \cap f(c_2)$ has minimal cardinality.

Let $\hat{f}_j$: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ be maps such that $\hat{f}_j(c_j) \subset \widetilde{F}$ and $p\hat{f}_j = f$ for $j = 1, 2$. We may suppose that $\hat{f}_1$ is an embedding. We assume that $p\hat{f}_1 = p\hat{f}_2 = f$ is in general position with respect to itself. It can be seen that S_f contains double curves and triple points but no branch points since $\hat{f}_1$ is an embedding and p is a local homeomorphism. We claim that we may assume that both $\hat{f}_1$ and $\hat{f}_2$ are embeddings. Let h: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ be an embedding such that h carries exactly one component of ∂A to $\widetilde{F}$. Then ph is an essential map. If $\hat{f}_2$ is not an embedding, we can apply the argument of [4] to find an embedding h: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ such that $h(\partial A) = f_2(\partial A)$ and $S_{ph} \subsetneq S_f$. We observe that the complexity (see pp. 14–15 in [3]) of S_{ph} is less than that of S_f since $f = p\hat{f}_2$. Our claim follows since the complexity of S_f is finite.

We assume that $\hat{f}_1$ and $\hat{f}_2$ are embeddings and that $\hat{f}_1(A) \cap \hat{f}_2(A)$ has been minimized by a homotopy constant on $\partial\widetilde{M}$. Of course if one moves $\hat{f}_1$, one also moves $\hat{f}_2$; but general position involves only local movement. It follows that we may suppose that $\hat{f}_1(A) \cap \hat{f}_2(A)$ is a collection of disjoint simple arcs and loops properly embedded in $\hat{f}_1(A)$ and $\hat{f}_2(A)$. If $\hat{f}_1(c_1) \cap \hat{f}_2(c_2)$ is empty we are finished.

Claim 1. We may suppose that no arc in $\hat{f}_1(A) \cap \hat{f}_2(A)$ has both its endpoints on $\hat{F}$ and thus each arc in $\hat{f}_1(A) \cap \hat{f}_2(A)$ is a spanning arc of both $\hat{f}_1(A)$ and $\hat{f}_2(A)$.

If some arc β in $\hat{f}_1(A) \cap \hat{f}_2(A)$ has both its endpoints on $\widetilde{F}$, there is a disk $\mathcal{D}_j$ embedded in $\hat{f}_j(A)$ such that $\partial\mathcal{D}_j$ is the union of β and a simple arc β_j on $\widetilde{F}$ for $j = 1, 2$. It can be shown that we may suppose that $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2 = \beta$. Since $\widetilde{F}$ is incompressible and $\beta_1 \cup \beta_2$ bounds the singular disk $\mathcal{D}_1 \cup \mathcal{D}_2$, $\beta_1 \cup \beta_2$ bounds a disk embedded in $\widetilde{F}$. It can now be seen that this contradicts our choice of f and Claim 1 is established.

Claim 2. If $f(c_1) \cap f(c_2)$ is not empty, there are spanning arcs α_1 and α_2 of A such that $\hat{f}_1(\alpha_1) = \beta = \hat{f}_2(\alpha_2)$, $\hat{f}_1(\alpha_2) = \beta^* = \hat{f}_2(\alpha_1)$, and $\beta \cap \beta^*$ is empty.

Let $\beta \subset \hat{f}_1(A) \cap \hat{f}_2(A)$ be an arc properly embedded in $\hat{f}_1(A)$. (If such an arc does not exist, $f|\partial A$ is a homeomorphism and we are finished.) Let α_j be the arc on A such that $\hat{f}_j(\alpha_j) = \beta$ for $j = 1, 2$. Suppose $\alpha_1 = \alpha_2$. Then $\hat{f}_2^{-1}\hat{f}_1: \alpha_1 \longrightarrow \alpha_1$ is a map of α_1 onto itself. It follows that there is a point $x \in \alpha_1$ such that $\hat{f}_2^{-1}\hat{f}_1(x) = \{x\}$. Since $\hat{f}_1(x) = \hat{f}_2(x)$ and $p\hat{f}_1 = p\hat{f}_2$, $\hat{f}_1 = \hat{f}_2: A \longrightarrow \widetilde{M}$. Thus $\alpha_1 \neq \alpha_2$. Let x_j, y_j be the endpoints of α_j for $j = 1, 2$. We may suppose that $x_j \in c_1$ and $y_j \in c_2$ for $j = 1, 2$. Now $x_1 \neq x_2$ since a point in ∂A is an endpoint of at most one double arc in S_f. We observe that $\hat{f}_1(x_2) = \hat{f}_2(y_1)$ since $p|F$ is a homeomorphism. Thus there is an arc β^* embedded in $\widetilde{M}$ such that $p\beta^* = p\beta$ and $\hat{f}_1(x_2) = \hat{f}_2(y_1)$ is an endpoint of β^*. Note that $\hat{f}_1(\alpha_2) = \beta^*$ and $\hat{f}_2(\alpha_1) = \beta^*$. Since $\hat{f}_1(A) \cap \hat{f}_2(A)$ is composed of disjoint simple arcs and loops, $\beta \cap \beta^*$ is empty. Since $\hat{f}_1$ is an embedding and $\beta \cap \beta^*$ is empty, $\alpha_1 \cap \alpha_2$ is empty. This establishes Claim 2.

Suppose α_1 and α_2 are arcs properly embedded in A such that $\hat{f}_j(\alpha_1)$ and $\hat{f}_j(\alpha_2)$ are contained in $\hat{f}_1(A) \cap \hat{f}_2(A)$ but $f(\alpha_1) \neq f(\alpha_2)$. We claim $\alpha_1 \cap \alpha_2$ is empty. Let β_j and β_j^* be disjoint arcs in $\hat{f}_1(A) \cap \hat{f}_2(A)$ such that $f(\alpha_j) = p\beta_j = p\beta_j^*$ for $j = 1, 2$. If $\alpha_1 \cap \alpha_2$ is not empty, then $\hat{f}_1(\alpha_1) \cap \hat{f}_1(\alpha_2)$ is not empty. But this is impossible since $\beta_1, \beta_2, \beta_1^*$, and β_2^* are pairwise disjoint. Our claim follows.

In the following paragraph, we use the facts about $\hat{f}_1(A) \cap \hat{f}_2(A)$ established above to construct a simpler singular map h of an annulus. The arcs a_1 and a_2 defined below are used to insure that $h|c_1$ and $h|c_2$ are homeomorphisms.

Suppose $f(c_1) \cap f(c_2)$ is nonempty. Then we can find arcs a_1 and a_2 contained in c_1 and c_2 respectively such that $\hat{f}_1(a_1) \cap \hat{f}_2(c_2) = \partial\hat{f}_1(a_1)$ and $\hat{f}_1(a_1) \cap \hat{f}_2(a_2) = \partial\hat{f}_2(a_2)$. Let β_1 and β_2 be the arc components of $\hat{f}_1(A) \cap \hat{f}_2(A)$ which meet $\hat{f}_1(a_1)$. Let $\mathcal{D}_1$ be the closure of the component of $A - \hat{f}_1^{-1}(\beta_1 \cup \beta_2)$ which meets a_1. Let $\mathcal{D}_2$ be the closure of the component of $A - \hat{f}_2^{-1}(\beta_1 \cup \beta_2)$ which does not meet a_1. Now $A_1 = \hat{f}_1(\mathcal{D}_1) \cup \hat{f}_2(\mathcal{D}_2)$ is a singular annulus in $\widetilde{M}$. Let $\hat{h}: (A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ be an embedding such that $\hat{h}(\partial A) = \hat{f}_1(\mathcal{D}_1 \cap \partial A) \cup \hat{f}_2(\mathcal{D}_2 \cap \partial A)$ and $\hat{h}(A)$ lies in a regular neighborhood of A_1. Let $h = p\hat{h}$. If $\mathcal{D}_2$ is the closure of $A - \mathcal{D}_1$, it can be seen that $h(c_1) \cap h(c_2) = f(c_1) \cap f(c_2)$ but that one can reduce the cardinality of $h(c_1) \cap h(c_2)$ by pulling $h(c_1)$ and $h(c_2)$ apart at $f(\partial a_1)$. If $\mathcal{D}_2$ is properly contained in $A - \mathcal{D}_1$, $h(c_1) \cap h(c_2)$ is properly contained in $f(c_1) \cap f(c_2)$. Note that $\mathcal{D}_2$ does not meet the interior of $\mathcal{D}_1$ since $\hat{f}_2(A)$ does not meet the interior of $f_1(a_1)$.

It remains to be shown that h is essential. Clearly we need only show that $h_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic. Note that $f(a_1) \cup f(a_2)$ is a simple loop l. If h_*

is not monic, l is nullhomotopic on F. Since F is incompressible, l would bound a disk on F. But then it is easily seen that f was not chosen so that $f(c_1) \cap f(c_2)$ would have minimal cardinality.

This completes the proof of Theorem 3.

We remark that the example in [1] shows that the existence of an essential map of an annulus may not imply the existence of an essential embedding of an annulus so that the conclusion of Theorem 4 is the best that we can expect.

THEOREM 4. *Let M be a compact (possibly nonorientable) 3-manifold and f: $(A, \partial A) \longrightarrow (M, \partial M)$ an essential map. Then there is an essential embedding g: $(F, \partial F) \longrightarrow (M, \partial M)$ where F is either an annulus or a Möbius band. Furthermore, if $f|\partial A$ is an embedding and $f(c_1)$ is an orientable loop, we may assume that $g(\partial F) \subset f(\partial A)$.*

PROOF. As in the proof of Theorem 3 we may suppose that ∂M is incompressible. Let $(\widetilde{M}, p)$ be the orientable double cover of M. If $f_*\pi_1(A)$ is not contained in $p_*\pi_1(\widetilde{M})$, we can find a two-sheeted cover $(\overline{A}, p_1)$ of A and $(fp_1)_*\pi_1(\overline{A})$ is contained in $p_*\pi_1(\widetilde{M})$. Thus we may assume that there is a map $\hat{f}$: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ such that $p\hat{f} = f$ for if $f_*\pi_1(A)$ is not contained in $p_*\pi_1(\widetilde{M})$, we can let f be fp_1. Now $\hat{f}$ is an essential map so it follows either from Theorem 1 or Theorem 3 that we may suppose that $\hat{f}$ is an embedding. Let ρ be the nontrivial covering translation of $\widetilde{M}$. We may suppose that $\rho\hat{f}(A)$ and $\hat{f}(A)$ are in general position with respect to one another. Thus $J = \rho\hat{f}(A) \cap \hat{f}(A)$ may be taken to be a collection of disjoint simple arcs and loops. If some arc in J has both endpoints in a single component of $\hat{f}(\partial A)$, one can simplify J by a standard cutting argument. If $\hat{f}^{-1}(J)$ contains a spanning arc of A, we can follow the proof of Lemma 4.4 in [2] to produce an essential annulus or Möbius band embedded in M. Thus we may suppose that J contains no simple arcs properly embedded in $\widetilde{M}$. If J contains distinct loops λ_1 and λ_2 such that $\rho\lambda_1 = \lambda_2$ one can use a cutting argument as in the proof of Lemma 4.2 in [2] to simplify J.

Thus we may suppose that for each loop $\lambda \subset J$, $\rho\lambda = \lambda$. If $\lambda \subset J$ is a simple loop and λ is nullhomotopic in $\widetilde{M}$, λ bounds a disk $\mathcal{D}$ on $\hat{f}(A)$. We may suppose that $\mathcal{D} \cap J = \lambda$. Then $\mathcal{D} \cup \rho\mathcal{D}$ is a 2-sphere S^2 embedded in $\widetilde{M}$ and we can modify $\hat{f}(A)$ inside a regular neighborhood of S^2 so that J is simplified. Thus we may suppose that each simple loop λ in J is not nullhomotopic on $\hat{f}(A)$.

We suppose that f has been chosen so that J will contain a minimal number of simple loops.

Suppose there is a simple loop $\lambda \subset J$. Let A_1 be the subannulus of A bounded by $\hat{f}^{-1}(\lambda)$ and c_1. We may assume that $\hat{f}(A_1) \cap J = \lambda$. Let α be a spanning arc of A which meets $\hat{f}^{-1}(\lambda)$ in a single point x_1. Let $\alpha_1 = A_1 \cap \alpha$.

Let A_2 be the closure $A - A_1$ and $\alpha_2 = A_2 \cap \alpha$. Let $x_2 = (\rho\hat{f})^{-1}\hat{f}(x_1)$ and β_1 a spanning arc of A_1 with one end at x_2. Since $\hat{f}(\alpha)$ is not homotopic rel its boundary to an arc in $\partial\widetilde{M}$, either the arc $\hat{f}(\alpha_1) \cup \rho\hat{f}(\beta_1)$ or the arc $\rho\hat{f}(\beta_1) \cup \hat{f}(\alpha_2)$ has the same property since the arc $\hat{f}(\alpha_1) * \rho\hat{f}(\beta_1) * \rho\hat{f}(\beta_1)^{-1} * \hat{f}(\alpha_2)$ is homotopic rel its boundary to $\hat{f}(\alpha)$ where $*$ denotes the natural product of arcs and α_1, β_1 and α_2 are given the appropriate parametrizations. If $\hat{f}(\alpha_1) \cup \rho\hat{f}(\beta_1)$ is not homotopic rel its boundary to an arc in $\partial\widetilde{M}$, p projects the annulus $\hat{f}(A_1) \cup \rho\hat{f}(A_1)$ to an essential Möbius band in M. Otherwise one deforms the annulus $\hat{f}(A_1) \cup \rho\hat{f}(A_2)$ to obtain an annulus whose projection is an essential map of an annulus into M. Denote this essential map by h. Now S_h contains fewer double curves than S_f. This contradicts our hypothesis and the proof of Theorem 4 is complete.

We prove below two lemmas which are necessary in the proof of Theorem 5.

LEMMA 6. *Let K be a Klein bottle and λ_1 and λ_2 disjoint, orientation reversing loops embedded in K. Let μ_1 be a simple loop embedded in K. Then there is a simple loop μ_2 isotopic to μ_1 in K such that $\mu_2 \cap \lambda_j$ contains at most a single point for $j = 1, 2$.*

PROOF. Let F_1 and F_2 be regular neighborhoods of λ_1 and λ_2. Then F_1 and F_2 are Möbius bands and we may suppose that the closure of $K - (F_1 \cup F_2)$ is an annulus A. After an isotopy, we may assume that μ_1 meets $\partial F_1 \cup \partial F_2$ in a finite collection of points. We suppose that $\mu_1 \cap \partial F_j$ is isotopically minimal for $j = 1, 2$. Then each arc in $A \cap \mu_1$, $F_1 \cap \mu_1$, and $F_2 \cap \mu_1$ must be a spanning arc. If μ_1 fails to meet F_1 or F_2 the lemma follows immediately. It can be seen that the endpoints of each of the spanning arcs in $F_j \cap \mu_1$ may be taken to be antipodal points on the 1-sphere which is the boundary of F_j for $j = 1, 2$. We let x_i and x_i' be the endpoints of the ith spanning arc in $F_1 \cap \mu_1$ and y_i and y_i' be the endpoints of the ith spanning arc in $F_2 \cap \mu_1$. Clearly the number n of spanning arcs in $F_1 \cap \mu_1$ is the same as that in $F_2 \cap \mu_1$. Thus we may place these points on ∂A as is shown in Figure 1.

We may suppose that there is a spanning arc α_1 in A from x_1 to y_1. If there is a spanning arc α_2 in A from x_2 to y_i or y_j' where $i \neq 2$ or $1 \leqslant j \leqslant n$, $\alpha_1 \cup \alpha_2$ cuts off a disk in A containing y_z but none of the x_i or x_j' for $i = 3, \ldots, n$ and $j = 1, \ldots, n$. This is impossible since no arc in $\mu_1 \cap A$ could have both its endpoints on a single component of ∂A and μ_1 is simple. It follows that if α is a spanning arc with one endpoint x_j (x_j'), its other endpoint is y_j (y_j'). This completes the proof of Lemma 6.

LEMMA 7. *Let M be a compact nonorientable 3-manifold. Let f: $(A, \partial A) \longrightarrow (M, \partial M)$ be a map such that*

(1) *$f|\partial A$ is a homeomorphism*;

(2) *$f(c_1)$ is an orientation reversing loop.*

Let $(\widetilde{M}, p)$ be a 2-sheeted cover of M and $\hat{f}$: $(A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ an embedding such that $p\hat{f} = f$. Then there is an embedding g: $(A, \partial A) \longrightarrow (M, \partial M)$ such that $g(\partial A) = f(\partial A)$.

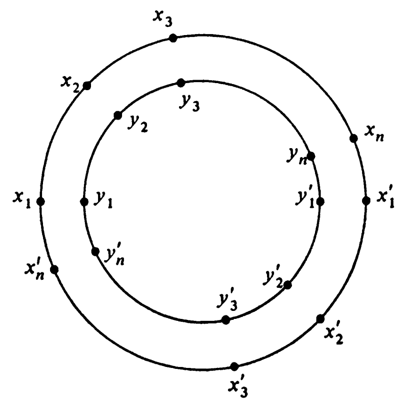

FIGURE 1

PROOF. Let ρ be the nontrivial covering translation of $(\widetilde{M}, p)$. Then we may insist that $\hat{f}(A) \cap \rho\hat{f}(A) = J$ is a collection of disjoint simple loops. We assume that f has been chosen so that the number of loops in J is minimal. Suppose λ_1 and λ_2 are disjoint loops in J such that $\rho\lambda_1 = \lambda_2$. Then the usual cutting argument shows that f was not chosen so that the number of loops in J is minimal. Let λ be a loop in J. Then $\rho\lambda = \lambda$. Suppose λ is nullhomotopic in $\widetilde{M}$. Then λ bounds a disk $\mathcal{D}$ on $\hat{f}(A)$. We may insist that $\mathcal{D} \cap J = \lambda$. Let $\mathcal{D}_1$ be a regular neighborhood of $f^{-1}(\mathcal{D})$ in A. Then we may suppose that $\hat{f}(\mathcal{D}_1) \cap J = \lambda$. Thus we can apply Dehn's lemma [3] so that $f|\mathcal{D}_1$ is an embedding and it can be seen that f was not chosen so that the number of loops in J would be minimized.

It follows that if λ is a loop in J, $\rho\lambda = \lambda$ and $\mu = f^{-1}(\lambda)$ is a generator of $\pi_1(A)$, and thus $(\mu, f|\mu)$ is a double cover of $f(\mu)$. We note that a loop representing the square of any element in $\pi_1(M)$ can be lifted to the orientable double cover of M. It would follow that $f(c_1)$ can be lifted to the orientable double cover of M, but this is impossible. This completes the proof of Lemma 7.

THEOREM 5. *Let M be a compact nonorientable 3-manifold and f: $(A, \partial A) \longrightarrow (M, \partial M)$ a map such that*

(1) *$f|\partial A$ is a homeomorphism*;

(2) *f_*: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic*;

(3) *$f(c_1)$ is an orientation reversing loop.*

Then there is an embedding g: $(A, \partial A) \longrightarrow (M, \partial M)$ such that $g(\partial A) = f(\partial A)$.

PROOF. We suppose that S_f is the union of a collection of 1-simplexes, and that f has been chosen so that $C(f)$ is minimal. Let M_0 be a regular neighborhood of $f(A)$. It is a consequence of Lemma 7 that we may suppose that M_0 does not admit a two-sheeted cover to which f can be lifted. We assume also that $M_0 = M$. Since $f(c_1)$ is an orientation reversing loop in ∂M and $f_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic, there is at least one nonorientable component of ∂M. By Lemma 4.5 in [2], $f_*(H_1(A))$ is of finite index in $H_1(M)$. Since ∂M contains a component of Euler characteristic less than or equal to zero, it can be shown that the rank of $H_1(M)$ is at least one. If the Euler characteristic of any component of ∂M is negative, the rank of $H_1(M)$ is at least two. It follows that the Euler characteristic of the component K of ∂M on which $f(\partial A)$ lies is zero and K must be a Klein bottle. As in the proof of Theorem 1, we construct an incompressible, two-sided, connected surface F properly embedded in M such that the intersection number of $f(c_j)$ and F is positive and equal to the cardinality k of $f(c_j) \cap \partial F$ for $j = 1, 2$.

Suppose that F is simply connected. Then F is a disk and ∂F has exactly one component. It is a consequence of Lemma 6 that we may suppose $\partial F \cap f(c_j)$ is a single point for $j = 1, 2$. After a standard argument, we may assume that $f^{-1}(F)$ is a simple arc α and that $f|\alpha$ is a homeomorphism. One then applies Dehn's lemma to obtain an embedding $g: (A, \partial A) \longrightarrow (M, \partial M)$ such that $g|\partial A = f|\partial A$. This would complete the proof of Theorem 5.

Suppose K is compressible. Then there is a disk $\mathcal{D}$ properly embedded in M such that $\mathcal{D} \cap k = \partial\mathcal{D}$ and $\partial\mathcal{D}$ is not nullhomotopic on K. If there is an isotopy moving $\partial\mathcal{D}$ off $f(\partial A)$, $[f(c_1)]$ is of order two in $\pi_1(M)$. It follows that if K is compressible, F can be chosen to be a disk and Theorem 5 follows.

Suppose $f: (A, \partial A) \longrightarrow (M, \partial M)$ is not essential. Then after a homotopy, we may assume that f carries a spanning arc α to ∂M. Now f induces a map $\bar{f}$: $(\mathcal{D}, \partial\mathcal{D}) \longrightarrow (M, \partial M)$ where $\mathcal{D}$ is a disk and $\bar{f}(\partial\mathcal{D}) = f(\partial A \cup \alpha)$. If $\bar{f}(\partial\mathcal{D})$ is inessential in ∂M, $f(c_1)$ is homotopic to $f(c_2)$ in ∂M which is impossible since $f(c_1)$ is nonorientable and $f|\partial A$ is a homeomorphism. Thus by the loop theorem in [6] K would be compressible and Theorem 5 would follow.

Assume that $f: (A, \partial A) \longrightarrow (M, \partial M)$ is essential. Let $f': (A, \partial A) \longrightarrow (M, \partial M)$ be a map such that $f'(A) = f(A)$ and $f'_*\pi_1(A)$ is of index two in $f'_*(\pi_1(A)) \subset \pi_1(M)$. We deform f' slightly so that $f'|\partial A$ is a homeomorphism. Clearly f' is essential. By Theorem 4 there is an essential embedding $h_1: (F_1, \partial F_1) \longrightarrow (M, \partial M)$ where F_1 is either an annulus or a Möbius band and $h_1(\partial F_1) \subset f'(\partial A)$.

Suppose F_1 is a Möbius band. After a general position argument, we may take the intersection of $h_1(F_1)$ with the properly embedded incompressible surface F to be a collection of disjoint simple arcs. We put $f(A)$ in general position with respect to $h_1(F_1)$. Since $f(\partial A)$ does not meet $h_1(\partial F_1)$, we may assume that

$f^{-1}h_1(F_1)$ is a collection of disjoint simple loops. Since $h_1(F_1)$ is incompressible in M, we may insist that none of these loops are nullhomotopic on A. Let λ be a central circle on the Möbius band $h_1(F_1)$. We may require that $f(A) \cap h_1(F_1) \subset \lambda$. Now the intersection number k of λ and F can be seen to be the same as that of $f(c_1)$ and F. Thus if μ is any loop in $f^{-1}(h_1(F_1))$, the intersection number of $f(\mu)$ and F is k. Thus we can require that $f|\mu$ is a homeomorphism onto λ. After a cutting argument, we may suppose that $f^{-1}(h_1(F_1))$ is at most a single simple loop. It is not difficult to see that if the restriction of f to each component of the complement of $A - f^{-1}(h_1(F_1))$ is a homeomorphism, one can find an embedding g: $(A, \partial A) \longrightarrow (M, \partial M)$ such that $g(\partial A) = f(\partial A)$. This would establish Theorem 5. Thus we need only consider the restriction of f to each component of $A - f^{-1}(h_1(F_1))$ and the manifold obtained by splitting M across $f^{-1}h_1(F_1)$. Note that this manifold will have two boundary components homeomorphic to Klein bottles and that the arcs in $h_1(F_1) \cap F$ are not homotopic in F rel their boundaries to arcs in ∂F.

Suppose that F_1 is an annulus. After a general position argument, we may require that $f^{-1}(h_1(F_1))$ is a collection of disjoint simple loops. Since f_*: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic and $f(c_1)$ is not homotopic to any loop on $h_1(F_1)$, each loop in $f^{-1}(h_1(F_1))$ is inessential on A. It follows that we may suppose that $f^{-1}(h_1(F_1))$ is empty. We observe that we may assume that $h_1(F_1) \cap F$ is a nonempty collection of disjoint simple arcs and that these arcs are not homotopic in F rel their boundaries to arcs in ∂F.

It can now be seen that one can cut the manifold M along the surface $h_1(F_1)$ and that this cutting simplifies F if F_1 is either a Möbius band or an annulus. Since F is compact, it admits only finitely many simplifications. If the surface resulting from F is ever simply connected or the resulting map of an annulus is not essential, it can be seen from the argument above that Theorem 5 follows. Since one of the above eventually must happen, this completes the proof of Theorem 5.

THEOREM 6. *Let M be a compact 3-manifold, F a Möbius band, and f: $(F, \partial F) \longrightarrow (M, \partial M)$ an essential map. Then there exists an essential embedding g: $(F_1, \partial F_1) \longrightarrow (M, \partial M)$ where F_1 is either an annulus or a Möbius band.*

PROOF. Let (A, p) be a two-sheeted cover of F. Then fp: $(A, \partial A) \longrightarrow (M, \partial M)$ is an essential map. Theorem 6 can now be seen to be a consequence of Theorem 4.

Appendix.

GROUPOIDS. A *groupoid* is a small category in which each morphism is invertible. A *subgroupoid* is a subcategory which is also a groupoid. A *groupoid* *morphism* is a covariant functor between groupoids. (See for example [Spanier, Chapter 1].)

If X is a topological space and Y is a subspace, then there is a groupoid which has as its objects the points of Y and has as its morphisms the homotopy classes of paths in X with endpoints in Y. This groupoid, denoted $P(X, Y)$, is called the *fundamental subgroupoid* of X with *base points* Y.

If $g\colon (X, Y) \longrightarrow (X', Y')$ is a map of pairs, then g induces a groupoid morphism $g_\#\colon P(X, Y) \longrightarrow P(X', Y')$. If G is a subgroupoid of $P(X', Y')$, then $g_\#^{-1} G$ is a subgroupoid of $P(X, Y)$. If g is injective on base points and if H is a subgroupoid of $P(X, Y)$, then $g_\# H$ is a subgroupoid of $P(X', Y')$.

MODIFIED LOOP THEOREM. *Lemma* 4, *as stated in this paper, is not exactly* [2, *Theorem* 3.1]. *However, the proof as given in* [2] *applies without change. The only critical point to check, perhaps, is the point at which one applies the proof of Stallings' Loop Theorem. It is perhaps appropriate to give a precise statement of the requisite theorem.*

LOOP THEOREM. *Suppose*

(1) $f\colon B \longrightarrow M$ *is a PL, general position map from a disk B into a* 3-*manifold M such that* $f^{-1}(\partial M) = \partial B$;

(2) Y *is an open subinterval of* ∂B, $X = (\partial B) - Y$, $f(Y) \cap f(X) = \emptyset$, *and* $S_f \cap \partial Y = \emptyset$; *and*

(3) G *is a subgroupoid of* $P(M; f(\partial Y))$ *such that, if* α *is a path in* Cl Y *with interior equal to* Y, *then* $f_\#([\alpha]) \notin G$.

Then there is a PL embedding $g\colon B \longrightarrow M$, *containing* $f(\partial Y)$, $g(B)$ *obtained from* $f(B)$ *by cut-and-paste, such that, if* α' *is the subset of* $\partial(gB)$ *obtained from* $f(\alpha)$ *by cut-and-paste,* $\alpha' \neq \emptyset$ *and* α' *represents an element of* $P(M, f(\partial Y)) - G$.

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