PIECEWISE LINEAR BUNDLES IN
THE METASTABLE RANGE

BY

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ABSTRACT. For numerable vector bundles a nonzero section determines
a unique trivial line subbundle containing the section and this subbundle is a di-
rect summand of the bundle. The main result, a consequence of concordance-
isotopy theory, states that in the metastable range a nonzero section to a piece-
wise linear R^n bundle determines a unique trivial line subbundle and that this is
the best possible result. This fact is then compared with the known failure of
the summand property below the stable range.

1. **Introduction.** One knows that the theories of linear and piecewise linear
bundles are quite similar in the stable range [10] and that below this range various
types of anomalous behaviour occur [7], [23]. Morlet [21] and others have
thought that these pathologies might not occur in the metastable range. Recent
results of Hatcher [11] and Volodin [26], however, have shown that this need not
be the case, cf. [25]. Several consequences of this observation are discussed in
light of the main purpose of this paper, a study of the consequences of nonzero
sections in the metastable range. The main theorem states that, in this range, a
nonzero section determines a unique trivial line bundle containing the section. We
shall also show that this is the best possible result.

The main ingredients in the analysis of these questions are the results con-
tained in my paper on piecewise linear concordances and isotopies [18]. Indeed
the genesis of that research was the desire to study the unstable geometry of
piecewise linear R^n bundles. The work of Kuiper and Lashof [17] provided the
impetus for this study and serves as a basic reference.

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the course of this research.

2. **Definitions and basic results.** All of the definitions and results concern
either the simplicial, block, or piecewise linear categories. For the definitions and

properties of the simplicial category see Curtis [9], for the block category see Rourke and Sanderson [22], and for the piecewise linear category see either Zeeman [27] or Hudson [14].

Objects in the simplicial and block categories are distinguished by a tilde ($\sim$) appearing over the objects in the block category. Thus $\Pi_i(X)$ and $\Pi_i(\widetilde{X})$ denote the ith homotopy groups of an object in the simplicial category and the ith block-homotopy groups of a related object in the block category. It will be important to compare these groups by including them in an exact sequence, cf. Morlet [21],

$$\rightarrow \Pi_i(X) \rightarrow \Pi_i(\widetilde{X}) \rightarrow \Pi_i^{\mathrm{rel}}(X) \rightarrow .$$

This is accomplished by defining $\Pi_i^{\mathrm{rel}}(X)$ to be the group of (simplicial) arc components of $\Omega^i\widetilde{X}$ which do not contain elements of $\Omega^i X$, where the "loops" are taken in the appropriate category.

Let M and N denote compact piecewise linear manifolds of dimensions m and n and boundaries ∂M and ∂N, respectively. Let Δ^s denote the standard s dimensional simplex, $I = [0, 1]$, $rJ = [-r, r]$, $(rJ)^n = rD^n$, and $\Sigma^{n-1} = \partial D^n$. A subspace (K, K_0) of a space (L, L_0) is said to be proper if $K \backslash K_0 \subset L \backslash L_0$. A piecewise linear map $f: (K, K_0) \rightarrow (L, L_0)$ is proper if $(f(K), f(K_0))$ is a proper subcomplex of (L, L_0).

DEFINITION (2.1). Given a proper piecewise linear embedding of a proper subcomplex of $(M, \partial M)$ in $(N, \partial N)$, $f: (K, K_0) \rightarrow (N, \partial N)$, let $E(M, N; f)$ denote the simplicial complex of locally unknotted proper piecewise linear embeddings of $(M, \partial M)$ into $(N, \partial N)$ extending f. That is the simplicial complex whose s simplices are proper piecewise linear embeddings

$$F: (M, \partial M) \times \Delta^s \longrightarrow (N, \partial N) \times \Delta^s$$
$$p\downarrow \qquad\qquad p\downarrow$$
$$1: \Delta^s \longrightarrow \Delta^s$$

such that

(i) the diagram is commutative;

(ii) $F^{-1}(\partial N \times \Delta^s) = \partial M \times \Delta^s$;

(iii) $F|(K, K_0) \times \Delta^s = f \times 1$;

(iv) for any simplex Δ linearly embedded in Δ^s, $(N \times \Delta, F(M \times \Delta))$ is a locally unknotted manifold pair.

If f is not specified this complex is denoted by $E(M, N)$.

The block complexes, $\widetilde{E}(M, N; f)$ and $\widetilde{E}(M, N)$, are defined as above except that condition (i) is replaced by

$(\widetilde{\mathrm{i}})$ for all faces Δ' of Δ^s, $F|(M, \partial M) \times \Delta'$ is an embedding of $(M, \partial M) \times \Delta'$ into $(N, \partial N) \times \Delta'$.

DEFINITION (2.2). Let (K, K_0) be a piecewise linear space. The complexes of proper piecewise linear homeomorphisms of (K, K_0) onto itself which are fixed on K_0 are denoted $H(K, K_0)$ and $\widetilde{H}(K, K_0)$. If K is a manifold the subcomplex of $H(K, K_0)$ of homeomorphisms which are fixed on $\partial K \cup K_0$ is denoted by $H_\partial(K, K_0)$. If K_0 is empty these spaces are denoted by $H(K)$ and $H_\partial(K)$, respectively.

REMARK (2.3). Inasmuch as all the definitions have analogues in the block category which are apparent, given those in the simplicial category, only the simplicial definitions will be given.

DEFINITION (2.4). The complexes of germs of proper embeddings of $(M, \partial M)$ into $(N, \partial N)$ extending f and of germs of homeomorphisms of K which extend f are denoted by $GE(M, N; f)$ and $GH(K; K_0)$, respectively. The first, for example, consists of the simplices of the complexes of proper embeddings of neighborhoods (U, U_0) of (K, K_0) into $(N, \partial N)$ which extend $f|K_0$.

DEFINITION (2.5). The complex of proper piecewise linear maps of (L, L_0) into (L', L_0') extending $f: (K, K_0) \rightarrow (L', L_0')$ is denoted by $M((L, L_0), (L', L_0'); f)$. If $(L, L_0) = (L', L_0')$ this complex is denoted by $M(L, L_0; f)$.

DEFINITION (2.6). Let $i: (L_0, L_0) \rightarrow (L, L_0)$ denote the inclusion. The subcomplex of $M(L, L_0; i)$ consisting of homotopy equivalences is denoted by $N(L, L_0)$, that is to say if F is an s-simplex of $N(L, L_0)$ there is another s-simplex G of $N(L, L_0)$ such that FG and GF are homotopic to the identity through homotopies which are fixed on L_0, leave $L \backslash L_0$ setwise invariant, and respect the projection to $\Delta^s \times I$. If L is a manifold and $L_0 = \partial L$ then $N(L, L_0)$ is denoted by $N_\partial(L)$. The subcomplex of those equivalences keeping a subcomplex K fixed is denoted by $N_\partial(L)_K$.

REMARK (2.7). It is easy to see that $\Pi_j(N(L, L_0))$ is isomorphic to $\Pi_j(M(L, L_0; i))$, via the inclusion, for $j \geqslant 1$.

DEFINITION (2.8). The complex of proper concordances of (germs of) proper embeddings extending a proper embedding $f: (K, K_0) \rightarrow (N, \partial N)$ of a proper subcomplex of $(M, \partial M)$ is denoted by $C(M, N; f)$ $(GC(M, N; f))$. It is the subcomplex of $E(I \times M, I \times N; 1 \times f)$ $(GE(I \times M, I \times N; 1 \times f))$ whose s-simplices F satisfy

(i) $F^{-1}(\{0\} \times N) = \{0\} \times M$, and

(ii) $F^{-1}(\{1\} \times N) = \{1\} \times M$.

DEFINITION (2.9). The complex of (germs of) proper isotopies of proper embeddings extending a proper embedding $f: (K, K_0) \rightarrow (N, \partial N)$ of a proper subcomplex of $(M, \partial M)$ is denoted by $I(M, N; f)$ $(GI(M, N; f))$. It is the complex of paths in $E(M, N; f)$ $(GE(M, N; f))$ or, equivalently, the subcomplex of $E(I \times M, I \times N; 1 \times f)$ $(GE(I \times M, I \times N; 1 \times f))$ such that, for any s-simplex

F, the diagram

$$F: I \times M \times \Delta^s \longrightarrow I \times N \times \Delta^s$$

$$p \Big\downarrow \qquad\qquad p \Big\downarrow$$

$$1: I \times \Delta^s \quad \longrightarrow \quad I \times \Delta^s$$

is commutative.

DEFINITION (2.10). Let $f: (M, \partial M) \longrightarrow (N, \partial N)$ be a proper piecewise linear embedding. $C_\partial(M, N; f|K)(C_{\partial_0}(M, N; f|K))$ and $I_\partial(M, N; f|K)(I_{\partial_0}(M, N; f|K))$ are the subcomplexes of $C(M, N; f | \partial M \cup K)$ and $I(M, N; f | \partial M \cup K)$, respectively, whose simplices F satisfy the property that $F | (\{0, 1\} \times M \times \Delta^s) = 1 \times f \times 1$ $(F|(\{0\} \times M \times \Delta^s) = 1 \times f \times 1)$. The analogous definitions for concordances and isotopies of germs of embeddings are also assumed.

Several simplicial fibrations and quasi-fibrations will be required in the next section. These are recognized via standard methods, cf. [17], [21], by employing Hudson's s covering isotopy theorem.

DEFINITION (2.11). Let $p: E \longrightarrow B$ be a simplicial map and G a simplicial group acting on E. Then $p: E \longrightarrow B$ is a (weak) fiber bundle, with group G, if

(i) p is onto (the arc components of B which contain the image),

(ii) G acts freely and transitively on $p^{-1}(b)$ for all $b \in B$ (b in the arc components of B which contain the image).

DEFINITION (2.12). Let $p: E \longrightarrow B$ be a simplicial map and $F = p^{-1}(b_0)$. Then $p: E \longrightarrow B$ is a (weak) quasi-fibration if $p_*: \Pi_s(E, F) \longrightarrow \Pi_s(B, b_0)$ is an isomorphism for all s (for all $s > 0$ and 1-1 for $s = 0$).

(2.13) Note that Kuiper and Lashof define $PL(n, k) = GH(R^n, R^k)$ while here $PL(n, k) = H(R^n, R^k)$. As noted below these two groups have the same homotopy type. Let $\overline{PL}(n, k) = H_\partial(D^n, D^k)$.

To study these, and related spaces, it is necessary to consider several fibrations whose definitions are related. To begin, let $\gamma_5: H_\partial(D^m \times R^n, D^m \times R^k) \longrightarrow GH_\partial(D^m \times R^n, D^m \times R^k)$ be the projection of a homeomorphism to its germ. Next define

$$\gamma_1: H_\partial(D^m \times D^n, D^m \times D^k) \longrightarrow H_\partial(D^m \times R^n, D^m \times R^k),$$

following Kuiper and Lashof [17, II] by taking a piecewise linear homeomorphism, $1 \times \rho_1$, of $D^m \times R^n$ to $D^m \times \text{int}(\tfrac{1}{2}D^n)$, followed by the element of $H_\partial(D^m \times D^n, D^m \times D^k)$, followed by inclusion into $D^m \times R^n$, and followed by an extension of the piecewise linear version of Kister's theorem to push the image onto $D^m \times R^n$. Let $\gamma_3 = \gamma_5 \circ \gamma_1$.

Similarly one can define

$$\gamma_2: H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k) \longrightarrow H_\partial(D^m \times R^n, D^{m'} \times R^k)$$

by taking a piecewise linear homeomorphism, $1 \times \rho_2$, of $D^m \times R^n$ to $D^m \times \text{int}(\Sigma_+^n)$, followed by, etc. Let $\gamma_4 = \gamma_5 \circ \gamma_2$.

If ev denotes the appropriate evaluation and $N(K)$ denotes a variable neighborhood of K one has the following useful proposition.

PROPOSITION (2.14).

$$H_\partial(D^m \times D^n, N(D^m \times \{0\}) \cup (D^m \times D^k))$$

(2.15)
$$\longrightarrow H_\partial(D^m \times D^n, D^m \times D^k)$$

$$\xrightarrow{\gamma_3} GH_\partial(D^m \times R^n, D^m \times R^k)$$

is a (weak) fiber bundle if $m > 0$ $(m \geqslant 0)$;

$$H_\partial(D^m \times \Sigma^n, N(D^m \times (1, 0, \ldots, 0)) \cup (D^m \times \Sigma^k))$$

(2.16)
$$\longrightarrow H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k)$$

$$\xrightarrow{\gamma_4} GH_\partial(D^m \times R^n, D^m \times R^k)$$

is a (weak) fiber bundle if $k \geqslant 0$ and $m > 0$ $(m \geqslant 0)$;

$$H_\partial(D^m \times R^n, N(D^m \times \{0\}) \cup (D^m \times R^k))$$

(2.17)
$$\longrightarrow H_\partial(D^m \times R^n, D^m \times R^k)$$

$$\xrightarrow{\gamma_5} GH_\partial(D^m \times R^n, D^m \times R^k)$$

is a fiber bundle;

(2.18) $H_\partial(D^m \times D^n, D^m \times \{0\}) \rightarrow H_\partial(D^m \times D^n) \xrightarrow{\text{ev}} E(D^m, D^m \times D^n; j)$

is a (weak) fiber bundle if $n \geqslant 3$ $(n \geqslant 0)$.

$$H_\partial(D^m \times \Sigma^n, D^m \times (1, 0, \ldots, 0))$$
(2.19)
$$\rightarrow H_\partial(D^m \times \Sigma^n) \xrightarrow{\text{ev}} E(D^m, D^m \times \Sigma^n; j).$$

(2.20) PROOF OF PROPOSITION (2.14). The proofs of each of the cases follows the same format. First in each case it is clear that the indicated fiber is a group which acts transitively and freely on each fiber. Thus it need only be shown that the projection is onto (or onto the arc components of the base which meet the image in the case of a weak fibration). The first three cases follow from the techniques of *Microbundles and bundles*. I [17] and Hudson's s-isotopy extension theorem [13] as indicated in the proof of Lemma 1.1 in Kuiper and Lashof. The final two cases are handled in much the same fashion in that Zeeman's un-

knotting theorem [27] is employed in (2.18) with the s-isotopy extension theorem, while in (2.19) no statement concerning the image is given so that the s-isotopy extension theorem is sufficient.

PROPOSITION (2.21). *The following are contractible*:

$$H_\partial(D^m \times D^n, D^m \times D^k), \quad H_\partial(D^m \times \Sigma^n, D^m \times (\Sigma^n_+ \cup \Sigma^k)),$$

$$H_\partial(D^m \times \Sigma^n, N(D^m \times (1, 0, \ldots, 0)) \cup (D^m \times \Sigma^k)),$$

and $H_\partial(D^m \times R^n, N(D^m \times \{0\}) \cup (D^m \times R^k))$.

(2.22) The proof of Proposition (2.21) is a straightforward generalization of various forms of the Alexander isotopy [3] adapted to the piecewise linear category. For proofs, in a slightly less general case, the reader is referred again to Kuiper and Lashof. Their techniques are easily extended to prove (2.21).

COROLLARY (2.23). (i) $\Omega G H_\partial(D^m \times R^n, D^m \times R^k)$ *is homotopy equivalent to* $H_\partial(D^m \times D^n, N(D^m \times \{0\}) \cup (D^m \times D^k))$, *via* λ,

(ii) $H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k)$ *is homotopy equivalent to* $G H_\partial(D^m \times R^k, D^n \times R^k)$, *via* γ_4 *modulo arc components*,

(iii) $H_\partial(D^m \times R^n, D^m \times R^k)$ *is homotopy equivalent to* $G H_\partial(D^m \times R^n, D^m \times R^k)$, *via* γ_5,

(iv) $E(D^m, D^m \times D^n; j)$ *is contractible if* $n \geqslant 3$.

This corollary follows easily from Propositions (2.14) and (2.21), and standard facts for simplicial fibrations.

COROLLARY (2.24). *If* $f: (D^m, \Sigma^{m-1}) \longrightarrow (D^{m+n}, \Sigma^{m+n-1})$ *is a proper piecewise linear embedding and* $n \geqslant 3$, *then* $E(D^m, D^{m+n}; f)$ *is contractible*.

(2.25) PROOF OF (2.24). Zeeman's unknotting theorem [27] implies that there is a proper piecewise linear homeomorphism $F: D^{m+n} \longrightarrow D^{m+n}$ extending f, i.e., $F | D^m \times \{0\} = f$. F induces a homeomorphism $F': E(D^m, D^m \times D^n; j) \longrightarrow E(D^m, D^{m+n}; f | \Sigma^{m-1})$ and thereby proves the corollary.

An extension of the proof of Lemma (1.7) of Kuiper and Lashof [17] gives the following proposition which is required for the description of several useful quasi-fibrations.

PROPOSITION (2.26). *The following inclusions are homotopy equivalences*:

(i)

$$H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1})$$

$$\overset{i_1}{\hookrightarrow} H_\partial(D^m \times D^n, N(D^m \times \{0\}) \cup (D^m \times D^k)),$$

(ii)
$$H_\partial(D^m \times \Sigma^n, D^m \times (\Sigma^n_+ \cup \Sigma^k))$$

$$\xrightarrow{\ i_2\ } H_\partial(D^m \times \Sigma^n, N(D^m \times (1, 0, \ldots, 0)) \cup (D^m \times \Sigma^k)),$$

where i_1 takes $D^{m+1} \times \Sigma^{n-1}$ to $D^m \times (D^n \backslash \text{int } \tfrac{1}{2}D^n)$ and extends by the identity and i_2 is the usual inclusion.

PROPOSITION (2.27). *The following are quasi-fibrations*:

(2.28)
$$H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1}) \longrightarrow H_\partial(D^m \times D^n, D^m \times D^k)$$

$$\xrightarrow{\ \gamma_1\ } H_\partial(D^m \times R^n, D^m \times R^k),$$

(2.29)
$$H_\partial(D^m \times \Sigma^n, D^m \times (\Sigma^n_+ \cup \Sigma^k)) \longrightarrow H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k)$$

$$\xrightarrow{\ \gamma_2\ } H_\partial(D^m \times R^n, D^m \times R^k),$$

where $j: D^m \longrightarrow D^m \times \Sigma^n$ is defined by $j(x) = (x, (1, 0, \ldots, 0))$ and γ_1 and γ_2 are defined below.

(2.30) *Quasi-fibrations* (2.28) *and* (2.29). Let 1 denote the germ of the identity homeomorphism as well as the identity homeomorphism. Since the fiber γ_5 is contractible

$$\gamma_{5*}: \Pi_s(H_\partial(D^m \times R^n, D^m \times R^k), \{1\}) \longrightarrow \Pi_s(GH_\partial(D^m \times R^n, D^m \times R^k), \{1\})$$

is an isomorphism. Consider the diagrams:

(2.31)

$$H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1}) \overset{i_1}{\hookrightarrow} H_\partial(D^m \times D^n, N(D^m \times \{0\}) \cup (D^m \times D^k))$$

$$\downarrow$$

$$H_\partial(D^m \times D^n, D^m \times D^k)$$

$$\gamma_1 \swarrow \qquad \qquad \downarrow \gamma_3$$

$$H_\partial(D^m \times R^n, D^m \times R^k) \xrightarrow{\ \gamma_5\ } GH_\partial(D^m \times R^n, D^m \times R^k)$$

(2.32)

$$H_\partial(D^m \times \Sigma^n, D^m \times (\Sigma^n_+ \cup \Sigma^k)) \overset{i_2}{\hookrightarrow} H_\partial(D^m \times \Sigma^n, N(D^m \times (1, 0, \ldots, 0)) \cup (D^m \times \Sigma^k))$$

$$\downarrow$$

$$H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k)$$

$$\gamma_2 \swarrow \qquad \qquad \downarrow \gamma_4$$

$$H_\partial(D^m \times R^n, D^m \times R^k) \xrightarrow{\ \gamma_5\ } GH_\partial(D^m \times R^n, D^m \times R^k)$$

Clearly,

$$\gamma_1^{-1}(1) = i_1(H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1})) \quad \text{and}$$

$$\gamma_2^{-1}(1) = i_2(H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1})).$$

Therefore

$$\Pi_s(H_\partial(D^m \times D^n, D^m \times D^k), i_1(H_\partial(D^{m+1} \times \Sigma^{n-1}, D^{m+1} \times \Sigma^{k-1})))$$

$$\gamma_{1*} \Big\downarrow$$

$$\Pi_s(H_\partial(D^m \times R^n, D^m \times R^k), \{1\})$$

and

$$\Pi_s(H_\partial(D^m \times \Sigma^n, D^m \times \Sigma^k), i_2(H_\partial(D^m \times \Sigma^n, D^m \times (\Sigma_+^n \cup \Sigma^k))))$$

$$\gamma_{2*} \Big\downarrow$$

$$\Pi_s(H_\partial(D^m \times R^n, D^m \times R^k), \{1\})$$

are isomorphisms since $\gamma_3 = \gamma_5 \circ \gamma_1$ and $\gamma_4 = \gamma_5 \circ \gamma_2$. Thus (2.28) and (2.29) are quasi-fibrations if $m > 0$. If $m = 0$ then γ_3 and γ_4 fail to be onto, they miss the orientation reversing component, and therefore give only a weak quasi-fibration.

PROPOSITION (2.33). *As far as homotopy groups are concerned, there is an exact sequence for a "fibration"*

$$H_\partial(D^m \times \Sigma^n, D^m \times (1, 0, \ldots, 0)) \longrightarrow \Omega H_\partial(D^{m-1} \times R^{n+1}, D^{m-1} \times \{0\})$$
(2.34)

$$\longrightarrow E(D^m, D^m \times \Sigma^n; j).$$

(2.35) PROOF. Let $k = 0$. By Corollary (2.20) and Proposition (2.26) there is a homotopy equivalence "(i_1^{-1})" $\circ \lambda \circ (\Omega\gamma_5) = \alpha$ from $\Omega H_\partial(D^{m-1} \times R^{n+1}, D^{m-1} \times \{0\})$ to $H_\partial(D^m \times \Sigma^n)$. Also there is a germ map, analogous to γ_4, giving the fibration

$$H_\partial(D^m \times \Sigma^n, N(D^m \times (1, 0, \ldots, 0)))$$

$$\Big\downarrow$$

(2.36) $$H_\partial(D^m \times \Sigma^n, D^m \times (1, 0, \ldots, 0))$$

$$\Big\downarrow \gamma_6$$

$$GH_\partial(D^m \times R^n, D^m \times \{0\})$$

which is a homotopy equivalence since its fiber is contractible, by (2.21). Thus "(γ_6^{-1})" $\circ \gamma_5 = \beta$ is a homotopy equivalence from $H_\partial(D^m \times R^n, D^m \times \{0\})$ to

$H_\partial(D^m \times \Sigma^n, D^m \times (1, 0, \ldots, 0))$. Since (2.19) is a fiber bundle, $m > 0$, and α and β are homotopy equivalences there is an exact sequence for

$$H_\partial(D^m \times R^n, D^m \times \{0\}) \xrightarrow{\text{``}(\alpha^{-1})\text{''} \circ \beta} \Omega H_\partial(D^{m-1} \times R^{n+1}, D^{m-1} \times \{0\})$$

$$\xrightarrow{(\text{ev}) \circ \alpha} E(D^m, D^m \times \Sigma^n; j).$$

A fundamental result of the study of concordances and isotopies [18] is the following theorem which provides the key to the metastable theory of nonzero sections.

THEOREM (2.37). *Let f: $(D^m, \Sigma^{m-1}) \to (N, \partial N)$ be a proper piecewise linear embedding. If $\Pi_j(N) = 0$, $j \leqslant k$ and $n - m \geqslant 3$, then $\Pi_s(C_\partial(D^m, N), I_\partial(D^m, N)) = 0$ for $s \leqslant n + k - m - 3$.*

3. The metastable theory of nonzero sections. In employing the methods developed in the previous section to the study of nonzero sections to piecewise linear bundles a key piece of information is supplied by the following theorem. Let G_{n+1} denote $h(\Sigma^n)$ and F_n denote $h(\Sigma^n, (1, 0, \ldots, 0))$. Note that suspension gives an inclusion of G_n as a subcomplex of F_n.

PROPOSITION (3.1). *If $n \geqslant 3$ there is a homomorphism*

$$\Pi_s(E(D^m, D^m \times \Sigma^n; j)) \to \Pi_{s+m}(\Sigma^n) \oplus \Pi_{s+m}(F_n, G_n),$$

where j: $D^m \to D^m \times \Sigma^n$ is given by $j(x) = (x, (1, 0, \ldots, 0))$, which is an isomorphism if $s < 2n - 3$ and an epimorphism if $s = 2n - 3$.

(3.2) PROOF OF PROPOSITION (3.1). The proof is by induction on s. The case $s = 0$ is precisely Corollary 2.18, page 454, of Rourke and Sanderson [22] which states, in effect, that $\Pi_0(E(D^m, D^m \times \Sigma^n; j))$ is isomorphic to $\Pi_m(G_{n+1}, G_n)$ for $n > 2$. It is easily noted that $\Pi_m(G_{n+1}, G_n) \cong \Pi_m(\Sigma^n) \oplus \Pi_m(F_n, G_n)$ and that the homomorphism from $\Pi_0(E(D^m, D^m; j))$ onto the $\Pi_m(\Sigma^n)$ summand is defined by taking the homotopy class of the projection of the embedding, relative to its boundary, in Σ^n.

Assume, inductively, that the proposition has been verified for $s = 0$, $1, \ldots, k - 1$. First $\Pi_k(E(D^m, D^m \times \Sigma^n; j)) \cong \Pi_{k-1}(\Omega E(D^m, D^m \times \Sigma^n; j))$. Since $\Omega E(D^m, D^m \times \Sigma^n; j) = I_\partial(D^m, D^m \times \Sigma^n)$, $\Pi_{k-1}(\Omega E(D^m, D^m \times \Sigma^n; j)) \cong \Pi_{k-1}(I_\partial(D^m, D^m \times \Sigma^n))$. By Theorem (2.37), if $n \geqslant 3$ and $k < 2n - 3$ ($k = 2n - 3$), the inclusion of $I_\partial(D^m, D^m \times \Sigma^n)$ in $C_\partial(D^m, D^m \times \Sigma^n)$ induces an isomorphism (epimorphism) on Π_{k-1}. Furthermore $C_\partial(D^m, D^m \times \Sigma^n)$ is $E(D^{m+1}, D^{m+1} \times \Sigma^n; j)$ so that by induction $\Pi_{k-1}(C_\partial(D^m, D^m \times \Sigma^n))$ is isomorphic to $\Pi_{k+m}(\Sigma^n) \oplus \Pi_{k+m}(F_n, G_n)$. The composition of these isomorphisms gives the isomorphism

$$\Pi_k(E(D^m, D^m \times \Sigma^n; j)) \cong \Pi_{k+m}(\Sigma^n) \oplus \Pi_{k+m}(F_n, G_n).$$

One also notes that the projection of $\Pi_k(E(D^m, D^m \times \Sigma^n; j))$ onto the $\Pi_{k+m}(\Sigma^n)$ summand is defined by taking the homotopy class of the projection of the k simplex F of embeddings, relative to the total boundary, $p(2) \circ F: D^m \times \Delta^k \longrightarrow \Sigma^n$.

Let $1PL(n) = H(R^n, \{0, (1, 0, \ldots, 0)\})$ and $PL(n, k) = H(R^n, R^k)$ denote the simplicial groups of piecewise linear R^n bundles with nowhere zero sections and trivial sub-R^k-bundles, respectively.

PROPOSITION (3.3). *If $n \geqslant 4$ there is a homomorphism*

$$\Pi_s(1PL(n), PL(n, 1)) \longrightarrow \Pi_s(F_{n-1}, G_{n-1}),$$

which is an isomorphism if $s < 2n - 4$ and an epimorphism if $s = 2n - 4$.

COROLLARY (3.4). *A nowhere zero section to a $PL(n)$ bundle, $n \geqslant 4$, over a complex K lies in a (unique) subline bundle if $\dim(K) < 2n - 3$ ($\dim(K) < 2n - 4$).*

Above this range this need not be the case as $\Pi_s(F_{n-1}, G_{n-1})$ may be non-zero. The following corollary of (3.3) contradicts a result of T. Akiba [1], [2].

COROLLARY (3.5). *There is a $PL(n)$ bundle, $n \geqslant 4$, over a sphere having a nowhere zero section which does not lie in a unique subline bundle.*

Another result of these fibrations and Proposition (3.2) takes a somewhat different, but equivalent, form. Let the quotient space $PL(n + k)/PL(n + k, k)$ denote the piecewise linear Stiefel manifold $V_{n+k,k}^{PL}$ of [32].

COROLLARY (3.6). *For $n \geqslant 4$ there is a homomorphism*

$$\Pi_s(V_{n,1}^{PL}) \longrightarrow \Pi_s(G_n, G_{n-1}),$$

which is an isomorphism if $s < 2n \leqslant 4$ and an epimorphism if $s = 2n - 4$.

(3.7) PROOF OF PROPOSITION (3.3) AND COROLLARIES (3.4), (3.5) AND (3.6). Taking $m = 0$ and $k = 0, 1$ in (2.28) there are homotopy equivalences of $\Omega PL(n) = \Omega H(R^n, \{0\})$ and $\Omega PL(n, 1) = \Omega H(R^n, R^1)$ with $H_\partial(D^1 \times \Sigma^{n-1})$ and $H_\partial(D^1 \times \Sigma^{n-1}, D^1 \times \Sigma^0)$, respectively. By (2.36) "(γ_6^{-1})" $\cdot \gamma_4$ gives a homotopy equivalence of $H_\partial(D^1 \times \Sigma^{n-1}, D^1 \times \Sigma^0)$ to

$$H_\partial(D^1 \times \overset{n-1}{\Sigma}, D^1 \times \{(1, 0, \ldots, 0)\},$$

thereby giving the sequence

(3.8) $\Omega PL(n, 1) \longrightarrow \Omega PL(n) \longrightarrow C(D^1, D^1 \times \Sigma^{n-1}; j).$

On the other hand one has the standard fibration

(3.9) $1PL(n) \longrightarrow PL(n) \longrightarrow E((1, 0, \ldots, 0), R^n \backslash \{0\})$

where the base space is homotopically Σ^{n-1}, cf. Kuiper and Lashof. The loop of this fibration can be then compared with (3.8) by

$$\Omega 1PL(n) \longrightarrow \Omega PL(n) \longrightarrow \Omega \Sigma^{n-1}$$

$$\Omega PL(n, 1) \longrightarrow \Omega PL(n) \longrightarrow E(D^1, D^1 \times \Sigma^{n-1}; j)$$

where $\widetilde{p}$ is the projection given by $\widetilde{p}(f) = p(2) \circ f$ and i is the inclusion. The homotopy lattice of this diagram [19] gives an isomorphism

$$\Pi_{s-1}(E(D^1, D^1 \times \Sigma^{n-1}; j), i(\Omega \Sigma^{n-1})) \cong \Pi_{s-1}(\Omega 1PL(n), \Omega PL(n, 1)).$$

The exact sequence of this pair of base complexes and Proposition (3.1) implies that there is a homomorphism

$$\Pi_{s-1}(E(D^1, D^1 \times \Sigma^{n-1}; j), i(\Omega \Sigma^{n-1})) \longrightarrow \Pi_s(F_{n-1}, G_{n-1})$$

which is an isomorphism if $s < 2n - 4$ and an epimorphism if $s = 2n - 4$ proving the proposition.

A result of James [15] implies that $\Pi_s(F_{n-1}, G_{n-1}) = 0$ if $s < 2n - 4$ so that Corollary (3.4) follows from Proposition (3.3) by the usual obstruction theory for fiber bundles. Since $\Pi_{2n-4}(F_{n-1}, G_{n-1})$ can be nonzero, Corollary (3.5) is proved in the same manner.

Corollary (3.6) follows directly from the homotopy sequence of the fibration (3.8) recalling that

$$\Pi_s(V_{n,1}^{PL}) = \Pi_s(PL(n)/PL(n, 1)) \cong \Pi_s(PL(n), PL(n, 1))$$

$$\cong \Pi_{s-1}(\Omega PL(n), \Omega PL(n, 1)) \cong \Pi_{s-1}(E(D^1, D^1 \times \Sigma^{n-1}; j))$$

and that $\Pi_s(\Sigma^{n-1}) \oplus \Pi_s(F_{n-1}, G_{n-1}) \cong \Pi_s(G_n, G_{n-1})$.

4. Anomalous behavior near the stable range. The study of piecewise linear bundles is facilitated by a modification of an exact sequence of Kuiper and Lashof [17].

PROPOSITION (4.1). *There is an exact homotopy sequence*

(4.2) $\cdots \longrightarrow \Pi_i(C_{n-1}) \longrightarrow \Pi_i(PL(n - 1)) \longrightarrow \Pi_i(1PL(n)) \longrightarrow \cdots .$

(4.3) *Construction of the exact sequence.* In the same manner that the quasi-fibration (2.28) was constructed via Proposition (2.14) in the proof of (2.27) one may construct a quasi-fibration

(4.4) $H_{\partial_0}(D^1 \times \Sigma^{n-1}, D^1 \times \Sigma^{k-1}) \longrightarrow H(D^n, D^k) \longrightarrow GH(R^n, R^k)$

which, by taking $k = 0$, becomes

(4.5) $H_{\partial_0}(D^1 \times \Sigma^{n-1}) \longrightarrow H(D^n, \{0\}) \overset{\gamma}{\longrightarrow} GH(R^n, \{0\}).$

A consequence of the Alexander Proposition (2.21) and the fibration

$$H_\partial(D^n, \{0\}) \longrightarrow H(D^n, \{0\}) \overset{\partial}{\longrightarrow} H(\Sigma^{n-1})$$

is that ∂ is a homotopy equivalence while Corollary (2.23)(iii) shows that $GH(R^n, \{0\})$ is homotopy equivalent to $H(R^n, \{0\})$. Up to homotopy γ in (4.5) is equivalent to the inclusion $H(\Sigma^{n-1}) \hookrightarrow H(R^n, \{0\})$ given by taking the open cone. Evaluation then gives a commutative diagram of fibrations:

$$(4.6) \quad \begin{array}{ccc}
H(\Sigma^{n-1}, \{(1, 0, \ldots, 0)\}) & \longrightarrow & H(R^n, \{0, (1, 0, \ldots, 0)\}) \\
\downarrow & & \downarrow \\
H(\Sigma^{n-1}) & \longrightarrow & H(R^n, \{0\}) \\
\downarrow & & \downarrow \\
E((1, 0, , \ldots, 0), \Sigma^{n-1}) & \longrightarrow & E((1, 0, \ldots, 0), R^n \backslash \{0\})
\end{array}$$

where the base spaces are homotopically equivalent (to Σ^{n-1}). One can identify $H(\Sigma^{n-1}, \{(1, 0, \ldots, 0)\})$ as $PL(n-1)$ by noticing that the base space of the fibration

$$H(\Sigma^{n-1}, \Sigma^0) \longrightarrow H(\Sigma^{n-1}, \{(1, 0, \ldots, 0)\})$$

$$\longrightarrow E((-1, 0, \ldots, 0), \Sigma^{n-1} \backslash \{(1, 0, \ldots, 0)\})$$

is contractible and applying (2.23), parts (ii) and (iii) with $m = 0$. The inclusion of the fibers in (4.6) is just a disguised form of the inclusion of $PL(n-1)$ in $1PL(n)$. The homotopy lattice of (4.6) then shows that $\Pi_i(1PL(n), PL(n-1))$ is isomorphic to $\Pi_i(H(R^n, \{0\}), H(\Sigma^{n-1}))$. This is isomorphic to Π_i (mapping cylinder γ, $H(D^n, \{0\})$) which, from (4.5), is isomorphic to $\Pi_{i-1}(H_{\partial_0}(D^1 \times \Sigma^{n-1}))$. If we let $C_{n-1} = C_{\partial_0}(\Sigma^{n-1}, \Sigma^{n-1}; 1) = H_{\partial_0}(D^1 \times \Sigma^{n-1})$ and insert the isomorphism between $\Pi_i(1PL(n), PL(n-1))$ and $\Pi_{i-1}(C_{n-1})$ into the exact sequence of the pair $(1PL(n), PL(n-1))$ we have the sequence (4.2).

COROLLARY (4.7). *If $n \geqslant 4$, then $\iota_*: \Pi_i(PL(n-1)) \to \Pi_i(1PL(n))$ is an isomorphism if $i \leqslant n$ and an epimorphism for $i = n + 1$.*
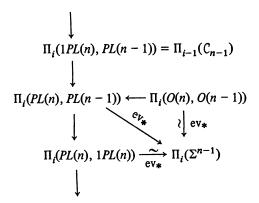
COROLLARY (4.8). *The evaluation homomorphism*

$$\text{ev}_*: \Pi_i(PL(n), PL(n-1)) \to \Pi_i(\Sigma^{n-1})$$

is an isomorphism for $i \leqslant n + 1$ and an epimorphism for $i = n + 2$.

(4.9) PROOF OF COROLLARIES (4.7) AND (4.8). In Theorem (5.3) of [20] it is shown that $\Pi_i(C_{\partial_0}(\Sigma^{n-1}, \Sigma^{n-1}; 1))$ is trivial for $i \leqslant n$. Thus Corollary (4.7) follows immediately from (4.2).

Recalling that in the proof of (4.1) it was shown that $\Pi_i(1PL_n, PL_{n-1}) \cong \Pi_{i-1}(C_{n-1})$ Corollary (4.8) follows from the exact sequence of the triple $(PL(n), 1PL(n), PL(n-1))$,

$$\downarrow$$

$$\Pi_i(1PL(n), PL(n-1)) = \Pi_{i-1}(C_{n-1})$$

$$\downarrow$$

$$\Pi_i(PL(n), PL(n-1)) \longleftarrow \Pi_i(O(n), O(n-1))$$

$$\downarrow \qquad\qquad ev_* \searrow \quad \wr \downarrow ev_*$$

$$\Pi_i(PL(n), 1PL(n)) \xrightarrow[ev_*]{\sim} \Pi_i(\Sigma^{n-1})$$

$$\downarrow$$

and Theorem (5.3) of [20].

(4.10) REMARK. Of course ev_* always gives a splitting of the exact sequence of the square $(PL(n); O(n), PL(n-1); O(n-1)) = \square$ so that $\Pi_i(PL(n)/PL(n-1)) \cong \Pi_i(O(n)/O(n-1)) \oplus \Pi_i(\square)$. In a personal communication R. Stern reminded me that Morlet [21] and Kirby and Siebenmann [16] have shown that for $n \geq 5$

$$\Pi_{i+n+1}(\square) \cong \Pi_i(C_{\partial_0}^{DIFF}(D^{n-1}, D^{n-1}; 1 \,|\, \Sigma^{n-2}))$$

where the superscript DIFF denotes the extension of the definitions given in §2 to the differentiable category with the usual modifications near boundaries, cf. [4], [21]. Cerf [8] has shown that $\Pi_0(C_{\partial_0}^{DIFF}(D^{n-1}, D^{n-1}; 1 \,|\, \Sigma^{n-2})) = 0$ if $n \geq 5$ while Hatcher [11] and Volodin [26] have shown that

$$\prod_1 (C_{\partial_0}^{DIFF}(D^{n-1}, D^{n-1}; 1 \,|\, \overset{n-2}{\Sigma} \,)) \cong Z_2 + wh_3(0)$$

if $n \geq 9$. As a consequence one notes that $\Pi_i(C_{n-1}) \cong \Pi_{i+1}(\square)$, which is nontrivial for $i = n+1$.

PROPOSITION (4.11). *For* $n \geq 9$, $\Pi_{n+2}(PL(n, 1), PL(n-1)) \neq 0$.

(4.12) PROOF OF (4.11). Proposition (3.3) states that $\Pi_s(1PL(n), PL(n, 1)) \cong \Pi_s(F_{n-1}, G_{n-1})$ if $s < 2n - 4$. As observed before, this latter group is trivial in this range so that (4.2) implies that $\Pi_{n+2}(PL(n, 1), PL(n-1)) \cong \Pi_{n+1}(C_{n-1})$. However, the previous remark shows that this group is nonzero.

(4.13) In summary we have shown in §3 that in the metastable range a nonzero section gives rise to a unique trivial sub-R^1-bundle, but that this fails just below the metastable range. The previous proposition shows that in the stable range a trivial sub-R^1-bundle splits the bundle but that this fails to hold just below the stable range. This failure of the splitting also shows that $\Pi_{n+2}(PL(n), \overline{PL(n)}) \neq 0$ if $n \geq 9$, cf. [25].

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