L-FUNCTIONS OF A QUADRATIC FORM

BY

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ABSTRACT. Let Q be a positive definite integral quadratic form in n variables, with the additional property that the adjoint form $Q^\dagger$ is also integral. Using the functional equation of the Epstein zeta function, we obtain a symmetric functional equation of the L -function of Q with a primitive character $\omega \bmod q$ (additive or multiplicative) defined by $\sum \omega(Q(x))Q(x)^{-s}$, $\operatorname{Re}(s) > n/2$, where the summation extends over all $x \in \mathbb{Z}^n$, $x \neq 0$; our result does not depend upon the usual restriction that q be relatively prime to the discriminant of Q , but rather on a much milder restriction.

1. Introduction. Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ matrices in $\mathbb{Z}$ and let $SL(n, \mathbb{Z})$ denote those matrices in $M_n(\mathbb{Z})$ with determinant 1. For each $A \in M_n(\mathbb{Z})$, denote by $A^\dagger$ the classical adjoint of A , and by M' the transpose of M , where M is any matrix. Let $X_1, \dots, X_n$ be n independent indeterminants and set $X = (X_1, \dots, X_n)'$. For each $B \in M_n(\mathbb{Z})$, define

$$B[X] = X'BX.$$

To each integral quadratic form $Q(X)$ we can associate a unique symmetric matrix $A \in M_n(\mathbb{Z})$ such that

$$(1) \quad Q(X) = \frac{1}{2}A[X] = \frac{1}{2}X'AX,$$

where the diagonal elements of A are each divisible by 2. Define the discriminant of Q to be

$$D = D(Q) = \det A.$$

Associated with each quadratic form Q is its adjoint form $Q^\dagger$ defined by

$$(2) \quad Q^\dagger(X) = \frac{1}{2}A^\dagger[X].$$

Note that the discriminant of $Q^\dagger$ is $\det A^\dagger = D^{n-1}$, and that for each $n > 2$, $Q^\dagger$ may or may not be integral.

For each integral positive definite quadratic form Q in n variables and for each character $\omega \bmod q$ (additive or multiplicative, q a positive integer), the

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n -fold sum $\sum_{\mathbf{x} \in \mathbb{Z}^n - 0} \omega(Q(\mathbf{x}))Q(\mathbf{x})^{-s}$ ($s = \sigma + it$) converges absolutely for $\sigma > n/2$ and uniformly in every compact subset of the half-plane $\sigma > n/2$, and so represents an analytic function of s for $\sigma > n/2$. We therefore define for $\sigma > n/2$

$$(3) \quad L(s, \omega, Q) = \sum_{\mathbf{x} \in \mathbb{Z}^n - 0} \omega(Q(\mathbf{x}))Q(\mathbf{x})^{-s},$$

which we call the L -function of Q with the character ω defined mod q . We remark here that by an additive character ω defined mod q , we mean $\omega = e_q(p)$ for some integer p , where $\omega(n) = e_q(pn)$ for all $n \in \mathbb{Z}$; we say that ω is primitive mod q if $(p, q) = 1$. A multiplicative character defined mod q will have its usual meaning, i.e., a Dirichlet character mod q ; we shall insist that a primitive multiplicative character is nonprincipal.

It is easy to show that $L(s, \omega, Q)$ always has an analytic continuation into the entire s -plane and satisfies a functional equation (see the remark at the end of §5). In this paper, we shall prove that under suitable restrictions on q , ω and Q , then $L(s, \omega, Q)$ in fact satisfies a symmetric functional equation. Such results are found in the literature if D and q are relatively prime and ω^* is also primitive mod q , where ω^* is defined in (9) below when it exists. For example, Stark [8] has proved that $L(s, \omega, Q)$ satisfies a symmetric functional equation if ω is multiplicative and $(D, q) = 1$ when ω^* is primitive (cf. [1]); implicit in his paper is a similar result for ω additive. In general, however, the restriction $(D, q) = 1$ is too strong, as is indicated in [7], where a symmetric functional equation for $L(s, \omega, Q)$ with ω additive is found for $Q(X, Y) = X^2 + Y^2$ which holds for all q using a method of T. Estermann [3]. In the present investigation, we shall relax these restrictions. The idea of the proof is to represent $L(s, \omega, Q)$, with ω additive, as a linear combination of Epstein zeta functions for which the analytic continuation into the entire s -plane and the functional equation are well known, and then to evaluate the resulting Gaussian sums $G_Q(q, p, \mathbf{x})$ which appear (cf. (6) below). These Gaussian sums have been studied by Stark in [8] and [9] subject to the restrictions $(D, q) = 1$. In this paper, we shall study both $G_Q(q, p, \mathbf{x})$ and $L(s, \omega, Q)$ subject to the weaker restrictions

$$(4) \quad (D_q, q) = 1 \quad \text{where} \quad D = \delta D_q \quad \text{and} \quad \delta = (D, q).$$

An immediate consequence of our symmetric functional equation for $L(s, \omega, Q)$ with ω additive and (4) holding is that we can write down a symmetric functional equation for $L(s, \omega, Q)$ with ω multiplicative, subject to (4) and ω^* being primitive when it exists. These results are the content of Theorem 3.

It is instructive to see the meaning of the restriction (4) in terms of valuations. Let ord_p denote the order valuation at the prime p . Then (4) is equivalent to

$$(5) \quad \text{ord}_p q \geq \text{ord}_p D \quad \text{or} \quad \text{ord}_p q = 0, \quad \text{for all primes } p.$$

Roughly speaking, these conditions mean that the form Q should not “degenerate” too much p -adically. Whether or not $L(s, \omega, Q)$ possesses a “nice” functional equation if Q is allowed to degenerate more than permitted by (5) is an open question, though the results in [7] must be kept in mind in looking into this matter.

2. Notation. Before we state the main results, we shall require some additional definitions to those given above. For any $t \in \mathbb{R}$ and $q \in \mathbb{Z}$, $q \geq 1$, write $e(t) = e^{2\pi i t}$ and $e_q(t) = e(t/q)$. If $p \in \mathbb{Z}$ is relatively prime to q , define $\bar{p} \in \mathbb{Z}$ by

$$p\bar{p} \equiv 1 \pmod{q}.$$

For any integral quadratic form Q in n variables, $p \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Z}^n$, define the Gaussian sum

$$(6) \quad G_Q(q, p, \mathbf{a}) = \sum_{\mathbf{x} \pmod{q}} e_q(pQ(\mathbf{x}) + \mathbf{a} \cdot \mathbf{x}),$$

where “ $\mathbf{x} \pmod{q}$ ” means that each component of $\mathbf{x} \in \mathbb{Z}^n$ runs through a complete set of residues mod q and $\mathbf{a} \cdot \mathbf{x}$ represents the ordinary inner product of $\mathbf{a}$ and $\mathbf{x}$. Also, we define

$$(7) \quad G_Q(q, p) = G_Q(q, p, \mathbf{0}).$$

Suppose there exists a multiplicative character χ_1 defined by

$$(8) \quad G_Q(q, p) = \chi_1(p)G_Q(q, 1),$$

where the defining modulus of χ_1 may depend on both q and Q . (Actually, if q is odd, the existence of χ_1 is guaranteed by Theorem 1, in which case it is described explicitly.) To each character ω , we now associate a new character ω^* defined by

$$(9) \quad \omega^* = \begin{cases} e_q(\overline{pD}_q) & \text{if } \omega = e_q(p), \\ \chi\bar{\chi}_1 & \text{if } \omega = \chi, \end{cases}$$

provided, in the second case, that χ_1 exists (cf. (8)). Note that ω^* depends on both ω and Q when it is defined.

We now introduce certain roots of unity which play a vital role in our functional equations. Since $G_Q(q, p) \neq 0$ when q, Q satisfy (4) (cf. (35), for example), we define

$$(10) \quad \epsilon_Q(q, p) = G_Q(q, p)/|G_Q(q, p)|.$$

For any multiplicative character $\chi \bmod q$, define

$$\tau(\chi) = \sum_{x \bmod q} \chi(x) e_q(x),$$

where $|\tau(\chi)| = \sqrt{q}$ if χ is primitive mod q . Therefore, if ω is a primitive character mod q , define

$$(11) \quad \epsilon(\omega, Q) = \begin{cases} \epsilon_Q(q, p) & \text{if } \omega = e_q(p), \\ \frac{\tau(\omega^*)}{\tau(\omega)} \epsilon_Q(q, 1) \chi^*(-D_q) & \text{if } \omega = \chi, \end{cases}$$

provided, in the second case, that χ_1 exists (cf. (8)). Note that $|\epsilon(\omega, Q)| = 1$ whenever ω^* is defined and is a primitive character mod q , which is no restriction for additive ω in view of (4) and (9).

Finally, for $x \in Z^n$, define

$$(12) \quad E_m(x) = \begin{cases} 1 & \text{if } x \equiv 0 \bmod m, \\ 0 & \text{otherwise,} \end{cases}$$

where m is any positive integer.

3. Statement of results. For convenience in stating our results, we introduce the following

DEFINITIONS. For each $n \geq 2$, the set H_n denotes the collection of all pairs (q, Q) , where q is a positive integer and Q is an integral quadratic form in n variables, satisfying $(D(Q)_q, q) = 1$; also, H_n^+ denotes the subset of pairs $(q, Q) \in H_n$ for which Q is positive definite. Further, the set H_n denotes the set of pairs $(q, Q) \in H_n$ for which the adjoint form $Q^\dagger$ is integral [cf. (2)], and finally let $H_n^+ = H_n \cap H_n^+$. We shall write $Q \in H_n$ instead of $(1, Q) \in H_n$.

For each $Q \in H_n$, let A denote the unique defining matrix of Q given in (1), and set

$$(13) \quad N(q, Q) = \text{card}\{x \bmod q: Ax \equiv 0 \bmod q\},$$

for any positive integer q .

If $[s_i \delta_{ij}]$, $s_i \in Z$, is the Smith Normal Form of A (cf. [5, p. 26]), it follows that $N(q, Q) = (s_1, q) \cdots (s_n, q)$ and $D(Q) = s_1 \cdots s_n$, whence $N(q, Q)$ divides $D(Q)$. Therefore, if $Q \in H_n^+$, there exists a unique positive integer $R(q, Q)$ such that

$$(14) \quad D(Q) = N(q, Q)R(q, Q).$$

THEOREM 1. Let $(q, Q) \in H_n$ and suppose $p \in Z$ is relatively prime to q .

(i) For any $a \in Z^n$, then

$$G_Q(q, p, \mathbf{a}) = E_\delta(A^\dagger \mathbf{a}) e_q(-\overline{pD}_q \delta^{-1} Q^\dagger(\mathbf{a})) G_Q(q, p),$$

where $A^\dagger$ is the adjoint of the defining matrix A of Q and $\delta = (D(Q), q)$, (cf. (2), (7) and (12)).

(ii) For q odd, then

$$(15) \quad G_Q(q, p) = \left(\frac{p}{q}\right)^n \left(\frac{p}{N(q, Q)}\right) G_Q(q, 1)$$

where $(p, N(q, Q)) = 1$ and

$$\left(\frac{p}{q}\right)$$

denotes the Jacobi symbol.

THEOREM 2. Let $Q \in H_n^+$ and assume that ω is any primitive character mod q , additive or multiplicative.

(i) If $\omega = e_q(p)$, then $L(s, e_q(p), Q)$ is a meromorphic function of s having a unique singularity at $s = n/2$ corresponding to a simple pole with residue

$$\operatorname{res}_{s=n/2} L(s, e_q(p), Q) = \frac{(2\pi)^{n/2} q^{-n}}{D(Q)^{1/2} \Gamma(n/2)} G_Q(q, p).$$

(ii) If $\omega = \chi$, then $L(s, \chi, Q)$ is an entire function of s , provided χ^* exists and is nonprincipal.

THEOREM 3. Let $(q, Q) \in H_n^+$ and assume that ω is a primitive character mod q such that ω^* is defined and primitive mod q (cf. (9) and Theorem 1(ii)). Then there exists an integral positive definite quadratic form Q^* , depending on q and Q , such that

$$(16) \quad Z(s, \omega, Q) = \epsilon(\omega, Q) Z(n/2 - s, \bar{\omega}^*, Q^*),$$

where

$$Z(s, \omega, Q) = (qR(q, Q))^{1/n} / (2\pi)^s \Gamma(s) L(s, \omega, Q)$$

and $\epsilon(\omega, Q)$ is defined by (11). Also, $R(q, Q)$ is defined in (14).

Note that the right-hand side of (16) is defined even though (q, Q^*) may not belong to H_n^+ , in spite of Q^* being integral (cf. Remark 5 below).

REMARKS. 1. From the obvious identity

$$(17) \quad L(s, \chi, Q) = \frac{1}{\tau(\chi)} \sum_{p \bmod q} \bar{\chi}(p) L(s, e_q(p), Q)$$

for primitive χ , we obtain

$$Z(s, \chi, Q) = \frac{1}{\tau(\chi)} \sum_{p \bmod q} \bar{\chi}(p) Z(s, e_q(p), Q).$$

Therefore, the functional equation for $Z(s, \chi, Q)$ follows immediately from the functional equation for $Z(s, e_q(p), Q)$, which we establish in §6.

2. The decisive step in establishing (16) for $(q, Q) \in H_n^+$ and ω additive is the explicit evaluation of $G_Q(q, p, a)$ given in Theorem 1(i).

3. In his paper, Stark does not assume that $Q^\dagger$ is integral. In the present paper, however, not only does this assumption simplify to some extent the proof of Theorem 1, it is essential. Indeed, the form Q defined by $X_1^2 + X_2^2 + X_3^2 + X_2X_3$, whose discriminant is 6, does not belong to H_3 since $Q^\dagger$ is not integral, and furthermore Theorem 1(i) is easily seen to be false with $q = 2$ and $a = (1, 0, 0)'$.

4. Q^* is defined explicitly in (31) and (32) in terms of the Smith Normal Form of $A^\dagger$. If $\delta = 1$, then $Q^* = Q^\dagger$, so that (16) is essentially Stark's result in [8] for ω multiplicative. For $n = 2$ and $Q(X, Y) = aX^2 + 2bXY + cY^2$ with $(a, b) = 1$, say, then

$$Q^*(X, Y) = \delta((1 + Ds^2)/a)X^2 + 2sDXY + aD_qY^2,$$

where $s \in Z$ is defined by $bs \equiv 1 \pmod{a}$. This is a fairly direct consequence of (29), since the congruence conditions on the sum are easily seen to be equivalent to the single congruence condition $ax + by \equiv 0 \pmod{\delta}$. Observe that Q^* is an integral form with $D(Q^*) = D(Q)$.

5. The statement of Theorem 3 contains a defect, namely, our functional equation cannot be iterated except in special instances, since $(q, Q) \in H_n^+$ does not necessarily imply that $(q, Q^*) \in H_n^+$. However, if $\delta = 1$ or $n = 2$, this defect does not arise in view of earlier remarks.

4. **Proof of Theorem 1.** For each $\gamma \in Z^n$, the automorphism of the group of residue classes mod q defined by $x \rightarrow x + \bar{p}\gamma$ transforms $G_Q(q, p, a)$ defined by (6) into

$$(18) \quad G_Q(q, p, a) = e_q(\bar{p}Q(\gamma) + \bar{p}a \cdot \gamma)G_Q(q, p, A\gamma + a)$$

where A is the defining matrix of Q . In order to evaluate $G_Q(q, p, a)$, the following result is found to be useful. The proof is an easy consequence of the well-known matrix identity $AA^\dagger = DI_n$, $D = \det A$.

LEMMA 1. $AX + a \equiv 0 \pmod{q}$ is solvable in Z^n iff $A^\dagger a \equiv 0 \pmod{\delta}$, $\delta = (D, q)$. If the congruence is solvable, then $-\bar{D}_q \delta^{-1} A^\dagger a$ is a solution.

To evaluate $G_Q(q, p, a)$, we distinguish two cases.

Case 1. $AX + a \equiv 0 \pmod{q}$ is solvable. By Lemma 1, $A^\dagger a \equiv 0 \pmod{\delta}$. Therefore, $Da'A^\dagger a = 2Q(A^\dagger a) \equiv 0 \pmod{2\delta^2}$, which may be rewritten as $a'A^\dagger a \equiv 0 \pmod{2\delta}$ when δ is even, since $(D_q, q) = 1$ implies $(D_q, 2\delta) = 1$; for δ odd, this is trivial. Hence, $A^\dagger a \equiv 0 \pmod{\delta}$ implies $Q^\dagger(a) \equiv 0 \pmod{\delta}$, noting that $Q^\dagger$

is an integral form since $Q \in H_n$. Using the solution of $AX + a \equiv 0 \pmod q$ given in Lemma 1, then (18) becomes

$$G_Q(q, p, a) = e_q(-\overline{pD_q} \delta^{-1} Q^\dagger(a)) G_Q(q, p, 0),$$

where $\delta^{-1} Q^\dagger(a) \in Z$. Furthermore, it is easy to verify that this result is independent of which solution of $AX + a \equiv 0 \pmod q$ is chosen.

Case 2. $AX + a \equiv 0 \pmod q$ is not solvable. We shall prove that $G_Q(q, p, a) = 0$. To verify this, replace γ in (18) by $q_1 A^\dagger \gamma$, where q_1 is defined by $q = \delta q_1$, so that

$$(19) \quad G_Q(q, p, a) = e_q(\overline{p} q_1 A^\dagger a \cdot \gamma) G_Q(q, p, a).$$

If $A^\dagger a \cdot \gamma \equiv 0 \pmod \delta$ for all $\gamma \in Z^n$, then we would have $A^\dagger a \equiv 0 \pmod \delta$, contrary to Lemma 1. Therefore, there exists $\gamma \in Z^n$ so that $\overline{p} q_1 A^\dagger a \cdot \gamma \not\equiv 0 \pmod q$, from which the result follows by (19). This completes the proof of (i).

To prove (15), we employ a variation of a technique used by Stark [8]. By Jones [4, p. 65, Theorem 25], we know that for each odd prime l , and for each positive integer e , there exists $E \in SL(n, Z)$ such that $EAE' \equiv 2K \pmod{l^e}$, where $K \in M_n(Z)$ is a diagonal matrix. By the Chinese Remainder Theorem and the fact that q is odd, there exists $E \in M_n(Z)$ with $\det E \equiv 1 \pmod q$ such that $EAE' \equiv 2C \pmod q$, where $C = [c_i \delta_{ij}] \in M_n(Z)$ is a diagonal matrix. Therefore, the automorphism of the group of residue classes mod q defined by $x \rightarrow Ex$ transforms $G_Q(q, p)$ into

$$(20) \quad G_Q(q, p) = \sum_{x \pmod q} e_q(p(c_1 x_1^2 + \cdots + c_n x_n^2)) = \prod_{j=1}^n G(q, pc_j),$$

where $G(q, h)$ is the ordinary Gaussian sum defined by

$$G(q, h) = \sum_{x \pmod q} e_q(hx^2).$$

It is well known that

$$G(q, ph) = \left(\frac{p}{r}\right) G(q, h) \quad \text{and} \quad |(G(q, h))| = \sqrt{dq},$$

where $(p, q) = 1$, $(q, h) = d$, $q = dr$ and

$$\left(\frac{p}{r}\right)$$

is the Jacobi symbol. Therefore, (20) implies

$$(21) \quad G_Q(q, p) = \prod_{j=1}^n \left(\frac{p}{q_j}\right) \cdot G_Q(q, 1)$$

and

$$(22) \quad |G_Q(q, p)|^2 = t_1 \cdots t_n q^n,$$

where $t_j = (c_j, q)$ and $q = t_j q_j$, $j = 1, \dots, n$. In order to obtain an invariant interpretation of the product $t_1 \cdots t_n$, we evaluate $|G_Q(q, p)|$ in another way, which is given by

LEMMA 2. *Let $Q \in H_n$ and $p, q \in Z$ be relatively prime, $q \geq 1$. If $G_Q(q, p) \neq 0$, then*

$$(23) \quad |G_Q(q, p)|^2 = q^n N(q, Q),$$

where $N(q, Q)$ is defined by (13).

We shall complete the proof of Theorem 1(ii) before proving Lemma 2. By (20) and the results immediately following it, it is clear that $G_Q(q, p) \neq 0$, so that on comparing (22) and (23), we have $t_1 \cdots t_n = N(q, Q)$. Since $(p, q) = 1$ and t_j divides q for each $j = 1, \dots, n$, then $(p, N(q, Q)) = 1$. Therefore, (15) follows immediately from (21), as required.

We now prove Lemma 2. By the definition of $G_Q(q, p)$, we clearly have

$$|G_Q(q, p)|^2 = \sum_{\mathbf{x} \bmod q} \sum_{\mathbf{y} \bmod q} e_q(p[Q(\mathbf{y}) - Q(\mathbf{x})]);$$

applying the automorphism $\mathbf{y} \rightarrow \mathbf{y} + \mathbf{x}$ to the inner sum transforms the sum into

$$\sum_{\mathbf{x}, \mathbf{y} \bmod q} e_q(p[Q(\mathbf{y}) + \mathbf{x} \cdot A\mathbf{y}]).$$

By appealing to the obvious identity

$$\sum_{\mathbf{x} \bmod q} e_q(\mathbf{a} \cdot \mathbf{x}) = q^n E_q(\mathbf{a}),$$

$\mathbf{a} \in Z^n$ (cf. (12)), it follows that

$$(24) \quad |G_Q(q, p)|^2 = q^n \psi,$$

where

$$\psi = \sum_{\mathbf{x} \bmod q; A\mathbf{x} \equiv 0 \bmod q} e_q(pQ(\mathbf{x})).$$

Now consider

$$\psi^2 = |\psi|^2 = \sum_{\mathbf{x}, \mathbf{y} \bmod q; A\mathbf{x} \equiv A\mathbf{y} \equiv 0 \bmod q} e_q(p[Q(\mathbf{y}) - Q(\mathbf{x})]).$$

The same argument used above (i.e., $\mathbf{y} \rightarrow \mathbf{y} + \mathbf{x}$) transforms this sum into $\psi^2 = N(q, Q)\psi$. Since $G_Q(q, Q) \neq 0$ by hypothesis, $\psi \neq 0$, whence $\psi = N(q, Q)$, as required.

REMARK. For $(q, Q) \in H_n$, it is easy to see that $Ax \equiv 0 \pmod q$ implies $Dx'Ax = 2Q^+(Ax) \equiv 0 \pmod{q^2}$ from which $x'Ax \equiv 0 \pmod q$. Therefore, for q odd, (23) follows from (24) directly. However, for q even, this argument does not work. The proof of (23) given above has been designed to avoid these difficulties.

5. **Proof of Theorem 2.** Before beginning the proof of this theorem, we shall require the following information regarding the Epstein zeta function.

For each $Q \in H_n^+$ and $u, v \in \mathbb{R}^n$, the associated Epstein zeta function is defined as

$$(25) \quad \zeta(s, u, v, Q) = \sum_{x \in Z^n, x+v \neq 0} e(x \cdot u) Q(x+v)^{-s},$$

which is an analytic function of s for $\sigma > n/2$ and has the following additional properties due to Epstein [2] (for a convenient version, see Siegel [6, p. 69]).

LEMMA 3. $\zeta(s, u, v, Q)$ has an analytic continuation into the entire s -plane, which is an entire function of s if $u \notin Z^n$. If $u \in Z^n$, then $\zeta(s, u, v, Q)$ is meromorphic in the entire s -plane possessing a unique singularity at $s = n/2$ corresponding to a simple pole with residue $(2\pi)^{n/2} |D(Q)|^{1/2} \Gamma(n/2)$. In either case, $\zeta(s, u, v, Q)$ satisfies the following symmetric functional equation:

$$\left(\frac{D(Q)^{1/n}}{2\pi} \right)^s \Gamma(s) \zeta(s, u, v, Q) = e(-u \cdot v) \left(\frac{D(Q^+)^{1/n}}{2\pi} \right)^{s'} \Gamma(s') \zeta(s', v, -u, Q^+)$$

where $s' = n/2 - s$ for fixed n .

REMARK. Lemma 3 is true in a much wider context, though the above is sufficient for the present needs (cf. [6]).

We now prove Theorem 2. By definition of $L(s, e_q(p), Q)$, we may partition Z^n into residue classes mod q so that

$$L(s, e_q(p), Q) = \sum_{d \bmod q} e_q(pQ(d)) \sum_{x \in Z^n - 0; x \equiv d \bmod q} Q(x)^{-s},$$

which by (25) can be rewritten as

$$(26) \quad L(s, e_q(p), Q) = q^{-2s} \sum_{d \bmod q} e_q(pQ(d)) \zeta(s, 0, q^{-1}d, Q).$$

By Lemma 3, it is clear that $L(s, e_q(p), Q)$ has an analytic continuation into the entire s -plane, and further that it is meromorphic in the entire s -plane with a unique singularity occurring at $s = n/2$ corresponding to a simple pole with residue as given in Theorem 2. To prove (ii), apply (i) to (17), which shows that $L(s, \chi, Q)$ has in fact a removable singularity at $s = n/2$ when χ^* is defined and nonprincipal mod q .

REMARK. In view of the functional equation for the Epstein zeta func-

tion, it is clear from (26) that a functional equation for $L(s, e_q(p), Q)$ can be written down. It turns out that the functional equation for $\zeta(s, u, v, Q)$, together with the condition $(q, Q) \in H_n^+$, forces the functional equation of $L(s, e_q(p), Q)$ to be symmetric. This is the content of the next section.

6. **Proof of Theorem 3 for $\omega = e_q(p)$, $(p, q) = 1$.** By multiplying both sides of (26) by $(D(Q)^{1/n}/2\pi)^s \Gamma(s)$ and applying the functional equation of $\zeta(s, 0, q^{-1}d, Q)$ given in Lemma 3 we obtain

$$(27) \quad \left(\frac{D(Q)^{1/n}}{2\pi} \right)^s \Gamma(s) L(s, e_q(p), Q) = q^{-2s} \left(\frac{D(Q^\dagger)^{1/n}}{2\pi} \right)^{s'} \Gamma(s') W(s', e_q(p), Q)$$

where $s' = n/2 - s$ and

$$(28) \quad W(s', e_q(p), Q) = \sum_{d \bmod q} e_q(pQ(d)) \zeta(s', q^{-1}d, 0, Q^\dagger).$$

Inserting the definition of $\zeta(s', q^{-1}d, 0, Q^\dagger)$ into (28) for $\sigma < 0$ and changing the order of summation, we find that

$$(29) \quad \begin{aligned} W(s', e_q(p), Q) &= \sum_{x \in \mathbb{Z}^n - 0} G_Q(q, p, x) Q^\dagger(x)^{-s'} \\ &= G_Q(q, p) \sum_{x \in \mathbb{Z}^n - 0; A^\dagger x \equiv 0 \bmod \delta} e_q(-\overline{pD}_q \delta^{-1} Q^\dagger(x)) Q^\dagger(x)^{-s'} \end{aligned}$$

by Theorem 1.

Let $S = [s_i \delta_{ij}]$ denote the Smith Normal Form of $A^\dagger$, $s_i \in \mathbb{Z}$ (cf. [5, p. 26]). Then there exists $U, V \in SL(n, \mathbb{Z})$ such that $A^\dagger = U^{-1} S V^{-1}$. Substituting this into (29) and applying the automorphism $x \rightarrow Vx$ of $\mathbb{Z}^n$, we obtain

$$(30) \quad \begin{aligned} W(s', e_q(p), Q) &= G_Q(q, p) \sum_{x \in \mathbb{Z}^n - 0; Sx \equiv 0 \bmod \delta} e_q(-\overline{pD}_q \delta^{-1} Q^\dagger(Vx)) Q^\dagger(Vx)^{-s'}. \end{aligned}$$

As S is diagonal, the restriction on the sum in (30) is easily dealt with. For each $i = 1, \dots, n$, define b_i by

$$\delta = b_i(\delta, s_i) \quad \text{and put} \quad B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}.$$

Define a quadratic form Q^* by

$$(31) \quad Q^*(X) = \frac{1}{2} A^* [X],$$

where A^* is the symmetric positive definite matrix defined by

$$(32) \quad A^* = \delta^{-1} B V' A^\dagger V B.$$

Since $SB \in \delta M_n(Z)$, $A^\dagger(VBx) = U^{-1}SBx \equiv 0 \pmod{\delta}$ for all $x \in Z^n$. Therefore, the argument at the beginning of the proof of Theorem 1(i), Case 1, shows that $\delta Q^*(x) = Q^\dagger(VBx) \equiv 0 \pmod{\delta}$, for all $x \in Z^n$. Therefore, Q^* is an integral quadratic form belonging to H_n^+ , so that (30) may be rewritten as

$$W(s', e_q(p), Q) = \delta^{-s'} G_Q(q, p) L(s', e_q(-pD_q), Q^*).$$

Consequently, (27) can now be written as

$$(33) \quad \left(\frac{qD(Q)^{1/n}}{2\pi} \right)^s \Gamma(s) L(s, e_q(p), Q) \\ = q^{-n/2} G_Q(q, p) \left(\frac{qD(Q^*)^{1/n}}{2\pi\delta} \right)^{s'} \Gamma(s') L(s', e_q(-\overline{pD_q}), Q^*).$$

The functional equation in (33) for our L -functions clearly lacks symmetry in its present form. However, in the special case in which $\delta = 1$, we can readily deduce (16) from (33) as follows. First we note that we may essentially take $Q^* = Q^\dagger$ in view of (29), in which case $N(q, Q) = 1$ since $\delta = 1$. Using (10) and (23), we obtain (16). The general case is more subtle and appears to essentially depend on our tentative functional equation (33) and the residue of our L -functions at $s = n/2$. First note that the functional equation of the Epstein zeta function of Lemma 3 immediately implies that

$$\zeta(0, u, v, Q) = \begin{cases} -e(-u \cdot v) & \text{if } v \in Z^n, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with (26), we find that $L(0, e_q(p), Q) = -1$. Multiplying both sides of (33) by s and letting $s \rightarrow 0$, we find that Theorem 2(i) implies

$$G_Q(q, p) G_{Q^*}(q, -\overline{pD_q}) = q^n (\delta^n D(Q^*)/D(Q^\dagger))^{1/2}.$$

By (32), we find

$$(34) \quad \delta^n D(Q^*) = D(Q^\dagger) \det(B)^2,$$

so that

$$(35) \quad G_Q(q, p) G_{Q^*}(q, -\overline{pD_q}) = q^n \det B.$$

Taking absolute values of (35), and applying Lemma 2, as we may since the right-hand side of (35) is nonzero, we obtain

$$N(q, Q) N(q, Q^*) = \det(B)^2,$$

whence (34) becomes

$$(36) \quad \delta^n D(Q^*) = D(Q^\dagger) N(q, Q) N(q, Q^*).$$

Combining this result with (14), our tentative functional equation (33) can now be rewritten as

$$Z(s, e_q(p), Q) = q^{-n/2} N(q, Q)^{-1/2} G_Q(q, p) Z(s', e_q(-\overline{pD_q}), Q^*),$$

as claimed, in view of (10) and (23).

7. **Some final observations.** (35) may be interpreted as a kind of reciprocity law for our Gaussian sums $G_Q(q, p)$, which may be described equivalently by $\epsilon_{Q^*}(q, \overline{pD_q}) = \epsilon_Q(q, p)$.

It is worth noting that Stark [8, p. 43, Lemma 10] has introduced another type of reciprocity law for these Gaussian sums, namely,

$$(37) \quad G_Q(q, 1) = q^{n/2} D(Q)^{1/2-n} e_8(n) G_{Q^\dagger}(D, -q)$$

which holds for all $Q \in H_n^+$; the proof of this is a consequence of the analytic properties of the theta function of Q . In fact Stark introduced (37) precisely so that he could obtain the results in Theorem 1(ii) for arbitrary q , subject to his restriction that $(q, D(Q)) = 1$. Furthermore (37) can be used to extend Theorem 1(ii) to include the case of even q , provided $(q, Q) \in H_n$, but at the expense of considerable complications (cf. [8]).

From (37), one can deduce

$$D(Q^\dagger) N(q, Q) = \delta^n N(D_q, Q^\dagger),$$

so that on combining this with (36), we obtain the following interpretation of $R(q, Q^*)$, which can also be verified directly:

$$R(q, Q^*) = N(D_q, Q^\dagger).$$

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