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TOTAL MEAN CURVATURE OF IMMERSED SURFACES IN E^m

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ABSTRACT. Total mean curvature and value-distribution of mean curvature for certain pseudo-umbilical surfaces are studied.

1. **Introduction.** In the classical theory of surfaces in a euclidean m-space E^m, the two most important curvatures are the so-called Gauss curvature G and the mean curvature α. It is well known that the Gauss curvature is intrinsic. The integral of Gauss curvature gives the beautiful Gauss-Bonnet formula, which holds for orientable compact surfaces as well as nonorientable ones,

$$\int_M G\,dV = 2\pi\chi(M), \tag{1.1}$$

where dV and $\chi(M)$ denote the volume element and Euler-Poincaré characteristic of M. For the mean curvature of a compact surface M in E^m we have [2, I] (see also [5]),

$$\int_M \alpha^2\,dV \geqslant 4\pi. \tag{1.2}$$

The equality holds when and only when M is an ordinary 2-sphere in an affine 3-space. It is an interesting problem to improve inequality (1.2) for some special surfaces in E^m. In [2, III] the author obtains some results of this problem for surfaces in E^4. In this paper we shall study this problem for pseudo-umbilical surfaces in E^m (for the definition of pseudo-umbilicity see §2). In particular, we shall prove the following:

THEOREM 1. *Let M be a compact pseudo-umbilical surface in E^m with nonnegative Gauss curvature. If we have*

$$\int_M \alpha^2\,dV \leqslant (2+\pi)\pi, \tag{1.3}$$

then M is homeomorphic to a 2-sphere.

THEOREM 2. *Let M be a compact flat pseudo-umbilical surface in E^m.*

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Then we have

$$\int_M \alpha^2 \, dV \geqslant 2\pi^2. \tag{1.4}$$

The equality sign holds if and only if M is a Clifford torus, i.e., M is the product surface of two plane circles with the same radius.

2. **Preliminaries.** Let $x: M \longrightarrow E^m$ be an isometrical immersion of a surface M in an m-dimensional euclidean space E^m and let ∇ and ∇' be the covariant differentiations of M and E^m respectively. Let X and Y be two tangent vector fields on M. Then the second fundamental form h is given by

$$\nabla'_X Y = \nabla_X Y + h(X, Y). \tag{2.1}$$

It is well known that $h(X, Y)$ is a normal vector field on M and is symmetric on X and Y. Let ξ be a normal vector field on M; we write

$$\nabla'_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where $-A_\xi(X)$ and $D_X\xi$ denote the tangential and normal components of $\nabla'_X\xi$. Then we have

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in E^m. A normal vector field ξ on M is said to be *parallel* (in the normal bundle) if $D\xi = 0$. The *mean curvature vector* H is defined by

$$H = \tfrac{1}{2} \operatorname{trace} h. \tag{2.4}$$

The length of H, denoted by α, is called the *mean curvature* of M. If the mean curvature vector H is nowhere zero and the second fundamental form h satisfies

$$\langle h(X, Y), H\rangle = \lambda\langle X, Y\rangle, \tag{2.5}$$

for all tangent vectors X, Y on M, then M is said to be *pseudo-umbilical.*

Let R and R^N be the curvature tensors associated with connections ∇ and D, i.e., R and R^N are given respectively by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

and

$$R^N(X, Y) = [D_X, D_Y] - D_{[X,Y]}.$$

For a surface M in R^m, if R vanishes identically, then M is said to be flat. If R^N vanishes identically, then M is said to have *flat normal connection.* Let e_1 and e_2 be orthonormal vector fields tangent to M. Then the Gauss curvature G of M is a well-defined intrinsic function on M given by

$$G = \langle R(e_1, e_2)e_2, e_1\rangle.$$

The Gauss and Ricci equations are given respectively by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{2.6}$$

$$\langle R^N(X, Y)\xi, \eta\rangle = \langle h(A_\xi(Y), X), \eta\rangle - \langle h(A_\xi(X), Y), \eta\rangle, \tag{2.7}$$

where X, Y, Z, W are vector fields tangent to M and ξ, η are vector fields normal to M. For the second fundamental form h, we define the covariant derivative, denoted by $\bar{\nabla}_X$, to be

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{2.9}$$

Let $e_1, e_2, \xi_3, \ldots, \xi_m$ be orthonormal vector fields, defined along M, such that e_1, e_2 are tangent to M and $\xi_3, \ldots, \xi_m$ are normal to M. For each $r = 3, \ldots, m$, we simply denote A_{ξ_r} by A_r. Let

$$A_r = (h^r_{ij})_{ij=1,2}.$$

With respect to the basis e_1, e_2, we have $h^r_{12} = h^r_{21}$. From (2.4) and (2.6) we find

$$H = \frac{1}{2}\sum(h^r_{11} + h^r_{22})\xi_r, \tag{2.10}$$

$$G = \sum(h^r_{11}h^r_{22} - h^r_{12}h^r_{12}). \tag{2.11}$$

3. Mean curvature of pseudo-umbilical surfaces with $R = R^N = 0$. In this section we shall prove the following results for later use.

THEOREM 3. *Let M be a flat pseudo-umbilical surface in E^m with flat normal connection. Then the mean curvature α satisfies the following Laplace's equation,*

$$\Delta \ln \alpha = 0,$$

where Δ denotes the Laplacian on M.

PROOF. Since the normal connection of M is flat, the equation of Ricci implies that

$$[A_r, A_s] = 0, \qquad r, s = 3, \ldots, m. \tag{3.1}$$

Now, let $\xi_3, \ldots, \xi_m$ be chosen in the way that $H = \alpha\xi_3$. Then by the pseudo-umbilicity of M in E^m, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h^r_{11} & h^r_{12} \\ h^r_{12} & -h^r_{11} \end{pmatrix}, \qquad r = 4, \ldots, m. \tag{3.2}$$

By (3.1), we may choose orthonormal tangent vectors e_1, e_2 which diagonalize A_r simultaneously. With respect to such frame $e_1, e_2, \xi_3, \ldots, \xi_m$, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h^r_{11} & 0 \\ 0 & -h^r_{11} \end{pmatrix}, \qquad r = 4, \ldots, m. \tag{3.3}$$

Now, by the flatness of M and (2.11), we have

$$\alpha^2 = \sum_{r=4}^{m} (h^r_{11})^2. \tag{3.4}$$

For each $p \in M$, let N_p be the vector space consisting of all normal vectors of M in E^m at p which are perpendicular to the mean curvature vector H. On N_p we define a linear mapping into the set of all symmetric matrices of order 2 by $\rho(\xi) = A_\xi$. Let O_p denote the kernel of ρ. Then by (3.3) and (3.4) we see that $\dim O_p = m - 4$. Hence, we may choose a frame field $e_1, e_2, \tilde{\xi}_3, \tilde{\xi}_4, \ldots, \tilde{\xi}_m$ such that, with respect to this frame, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_4 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \qquad A_s = 0, s = 5, \ldots, m. \tag{3.5}$$

Since the normal connection is flat, there exist, at least locally, orthonormal normal vector fields $\bar{\xi}_3, \ldots, \bar{\xi}_m$ such that $\bar{\xi}_3, \ldots, \bar{\xi}_m$ are parallel (see [1, p. 99]), i.e.,

$$D\bar{\xi}_3 = \cdots = D\bar{\xi}_m = 0. \tag{3.6}$$

We put

$$\bar{\xi}_r = \sum_{s=3}^{m} a_{rs}\tilde{\xi}_s, \qquad r = 3, \ldots, m.$$

Then (a_{rs}) is an orthogonal matrix of order $m - 2$.

Since M is two dimensional and local study of M is sufficient, we may assume that M is covered by an isothermal coordinate (x, y) such that the metric on M has the form $ds^2 = E(dx^2 + dy^2)$. In the following, we shall denote the coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$ by X_1 and X_2 respectively. We put

$$L = h(X_1, X_1), \qquad M = h(X_1, X_2), \qquad N = h(X_2, X_2)$$

and

$$\nabla_{X_j} X_i = \sum_{k=1}^{2} \Gamma^k_{ji} X_k, \qquad i, j = 1, 2.$$

Then we have

$$\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = X_1E/2E, \qquad \Gamma_{22}^2 = \Gamma_{12}^1 = -\Gamma_{11}^2 = X_2E/2E.$$

Therefore the Codazzi equation reduces to

$$D_{X_2}L - D_{X_1}M = (X_2E)H,$$

$$D_{X_2}M - D_{X_1}N = -(X_1E)H.$$

Since X_1 and X_2 are orthonormal, we may define a function $\theta = \theta(x, y)$ by

$$\begin{aligned} X_1 &\equiv \partial/\partial x = \cos\theta e_1 + \sin\theta e_2, \\ X_2 &\equiv \partial/\partial y = -\sin\theta e_1 + \cos\theta e_2. \end{aligned} \tag{3.7}$$

Then with respect to the frame field $X_1, X_2, \bar{\xi}_3, \ldots, \bar{\xi}_m$, the second fundamental tensors are given by

$$A_r = \begin{pmatrix} \alpha(a_{r1} + a_{r2}\cos 2\theta) & -\alpha a_{r2}\sin 2\theta \\ -\alpha a_{r2}\sin 2\theta & \alpha(a_{r1} - a_{r2}\cos 2\theta) \end{pmatrix}.$$

Since M is flat, we may assume that $E = 1$. Hence by (3.6) equations of Codazzi reduce to

$$\frac{\partial}{\partial y}[\alpha(a_{r1} + a_{r2}\cos 2\theta)] = -\frac{\partial}{\partial x}[\alpha a_{r2}\sin 2\theta], \tag{3.8}$$

$$-\frac{\partial}{\partial y}[\alpha a_{r2}\sin 2\theta] = \frac{\partial}{\partial x}[\alpha(a_{r1} - a_{r2}\cos 2\theta)]. \tag{3.9}$$

Multiplying a_{r1} to (3.8) and summing over r, then, by the fact that $(a_{rs}) \in O(m-2)$, we find

$$\frac{\partial \ln \alpha}{\partial y} = \sum_{r=3}^{m}\left\{\left(\frac{\partial a_{r1}}{\partial y}\right)a_{r2}\cos 2\theta + \left(\frac{\partial a_{r1}}{\partial x}\right)a_{r2}\sin 2\theta\right\}. \tag{3.10}$$

Similarly, multiplying a_{r1} to (3.9) and summing over r, we have

$$\frac{\partial \ln \alpha}{\partial x} = \sum_{r=3}^{m}\left\{\left(\frac{\partial a_{r1}}{\partial y}\right)a_{r2}\sin 2\theta - \left(\frac{\partial a_{r1}}{\partial x}\right)a_{r2}\cos 2\theta\right\}. \tag{3.11}$$

Multiplying a_{r2} to (3.8) and summing over r, we find

$$\sum a_{r2}\left(\frac{\partial a_{r1}}{\partial y}\right) + \left(\frac{\partial \ln \alpha}{\partial x}\right)\sin 2\theta + \left(\frac{\partial \ln \alpha}{\partial y}\right)\cos 2\theta$$

$$= 2\frac{\partial \theta}{\partial y}\sin 2\theta - 2\frac{\partial \theta}{\partial x}\cos 2\theta.$$

Hence, by substituting (3.10) and (3.11) into this equation, we find

$$\sum a_{r2} \frac{\partial a_{r1}}{\partial y} = \sin 2\theta \frac{\partial \theta}{\partial y} - \cos 2\theta \frac{\partial \theta}{\partial x}. \tag{3.12}$$

Similarly, by multiplying a_{r2} to (3.9), summing over r, and by using (3.10) and (3.11), we get

$$\sum a_{r2} \frac{\partial a_{r1}}{\partial x} = -\cos 2\theta \frac{\partial \theta}{\partial y} - \sin 2\theta \frac{\partial \theta}{\partial x}. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.10) and (3.11), we may find

$$\frac{\partial \ln \alpha}{\partial x} = \frac{\partial \theta}{\partial y}, \quad \frac{\partial \ln \alpha}{\partial y} = -\frac{\partial \theta}{\partial x}. \tag{3.14}$$

From this we get $(\partial^2/\partial x^2)\ln \alpha + (\partial^2/\partial y^2)\ln \alpha = 0$. Since $E = 1$, this implies that $\Delta \ln \alpha = 0$. Q.E.D.

It shall be remarked that for a pseudo-umbilical surface in E^4, the normal connection is always flat.

As an application of Theorem 3, we have the following result concerning about the *value-distribution* of mean curvature α.

THEOREM 4. *Let M be a complete flat pseudo-umbilical surface in E^m with flat normal connection. Then we have either*

(1) *the mean curvature α of M takes every value in* $(0, \infty)$, *or*

(2) *the mean curvature α of M takes only one value in* $(0, \infty)$.

If case (2) *holds, M is the product of two curves C_1 and C_2 where C_1 is a curve in E^n for some n; $1 < n < m$; and C_2 is a curve in E^{m-n} so that the first curvatures of C_1 and C_2 are equal.*

PROOF. Since M is flat and complete, M is parabolic in the sense that there exists no nonconstant negative subharmonic function on M. Thus every subharmonic function on M which is bounded from above on M must be a constant function. By Theorem 3, $\ln \alpha$ is a continuous harmonic function and so a subharmonic and also a superharmonic function on M. Hence, if α does not take every value in $(0, \infty)$, then α must be constant. This proves the first part of the theorem.

Now, suppose that case (2) holds. Then α is a nonzero constant. From (3.5) we see that with respect to the frame field $e_1, e_2, \xi'_3, \xi'_4, \xi'_5, \ldots, \xi'_m$, we have

$$A_3 = \begin{pmatrix} \sqrt{2}\alpha & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\alpha \end{pmatrix}, \quad A_5 = \cdots = A_m, \tag{3.15}$$

where $\xi_3' = \cos\theta\tilde{\xi}_3 + \sin\theta\tilde{\xi}_4$, $\xi_4' = \sin\theta\tilde{\xi}_3 - \cos\theta\tilde{\xi}_4$, $\xi_5' = \tilde{\xi}_5, \ldots, \xi_m' = \tilde{\xi}_m$. From (3.15) and the equations of structure we may easily find that both the distributions $T_i = \{ae_i: a \in R\}$, $i = 1, 2$ are parallel. By the de Rham decomposition theorem we see that $M = C_1 \times C_2$ where C_1 (respectively, C_2) is the maximal integral manifold of T_1 (resp. T_2). Moreover, (3.15) implies that $h(e_1, e_2) = 0$. Hence, M is the product of C_1 and C_2 such that C_1 is a curve in an affine n-space E^n and C_2 is a curve in an affine $(m-n)$-space E^{m-n} in E^m. Let h^i be the second fundamental form of C_i, $i = 1, 2$. Then, (3.15) implies $h^i(e_i, e_i) = \sqrt{2}\alpha\xi_{i+2}'$, $i = 1, 2$. Hence the first curvatures of C_1 and C_2 are equal. Q.E.D.

4. **Proofs of Theorems 1 and 2.** Let M be a compact pseudo-umbilical surface in E^m. Let $e_1, e_2, \xi_3, \ldots, \xi_m$ be a local field of orthonormal frame defined along M such that e_1, e_2 are tangent to M and $\xi_3, \ldots, \xi_m$ normal to M. For a unit normal vector ξ at $p \in M$, the Lipschitz-Killing curvature $K(p, \xi)$ is defined by $K(p, \xi) = \det A_\xi$. Let $\xi = \sum_{r=3}^m \cos\theta_r \xi_r$ and $A_r = (h_{ij}^r)$. Then we have

$$K(p, \xi) = \left(\sum \cos\theta_r h_{11}^r\right)\left(\sum \cos\theta_s h_{22}^s\right) - \left(\sum \cos\theta_t h_{12}^t\right)^2.$$

The right-hand side of this equation is a quadratic form of $\cos\theta_3, \ldots, \cos\theta_m$. Hence, by choosing a suitable local frame field $\bar{\xi}_3, \ldots, \bar{\xi}_m$, we may write

$$K(p, \xi) = \sum \lambda_{r-2}(p)\cos^2\theta_r, \tag{4.1}$$

where $\lambda_1, \ldots, \lambda_{m-2}$ are continuous functions defined on M and satisfy the following relations: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{m-2}$. From (2.11) we find

$$G = \lambda_1 + \cdots + \lambda_{m-2}. \tag{4.2}$$

Since M is pseudo-umbilical, if we choose ξ_3 in the direction of the mean curvature vector, then (3.2) holds. From this we may easily see that $K(p, \xi)$ takes its maximal value at ξ_3. Hence we have

$$\lambda_1 = \alpha^2, \qquad \lambda_2 \leqslant 0, \ldots, \lambda_{m-2} \leqslant 0. \tag{4.3}$$

Now, let S_p be the unit $(m-3)$-sphere of all unit normal vectors at $p \in M$ and $d\sigma$ the volume element of S_p. Then the total absolute curvature $K^*(p)$ at p is given by

$$K^*(p) = \int_{S_p} |K(p, \xi)|\, d\sigma$$

and the total absolute curvature $TA(M)$ of M in E^m (in the sense of Chern-Lashof [3]) is given by $TA(M) = \int_M K^*(p)\, dV$. Let $H_i(M; F)$ be the ith homology group of M over a field F and $\beta_i(M; F)$ the dimension of $H_i(M; F)$. Then we have [3]

$$TA(M) \geqslant c_{m-1}\beta(M), \tag{4.4}$$

where $\beta(M) = \max\{\Sigma_{i=0}^2 \beta_i(M; F)\colon F \text{ fields}\}$ and c_{m-1} the area of unit $(m-1)$-sphere.

From (4.2) and (4.3) we find

$$\begin{aligned} |K(p, \xi)| &= \left| G\cos^2\theta_3 + \sum_{r=4}^m \lambda_{r-2}(\cos^2\theta_r - \cos^2\theta_3)\right| \\ &\leqslant |G|\cos^2\theta_3 - \sum_{r=4}^m \lambda_{r-2}|\cos^2\theta_r - \cos^2\theta_3|. \end{aligned} \tag{4.5}$$

This implies that

$$K^*(p) \leqslant \frac{c_{m-1}}{2\pi}|G| - \sum_{r=4}^m \lambda_{r-2}\int_{S_p}|\cos^2\theta_r - \cos^2\theta_3|\,d\sigma.$$

On the other hand, since

$$\int_{S_p}|\cos^2\theta_r - \cos^2\theta_3|\,d\sigma = 2c_{m-1}/\pi^2,$$

(4.2), (4.3) and (4.5) imply

$$\alpha^2 \geqslant (\pi^2/2c_{m-1})K^*(p) + G(p) - (\pi/4)|G(p)|. \tag{4.6}$$

Case (1). $G = 0$. In this case, (4.4) and (4.6) imply

$$\int \alpha^2\,dV \geqslant (\pi^2/2)\beta(M). \tag{4.7}$$

On the other hand, the flatness of M implies that M is either homeomorphic to a torus or a Klein bottle; in both cases, $\beta(M) = 4$. Hence (4.7) implies inequality (1.4). Now, if the equality of (1.4) holds, then we have

$$TA(M) = 4c_{m-1}. \tag{4.8}$$

Moreover, the inequality in (4.5) is actually an equality for all $(\theta_3, \ldots, \theta_m)$ satisfying $\cos^2\theta_3 + \cos^2\theta_4 + \cdots + \cos^2\theta_m = 1$. Hence we have

$$\lambda_{m-2} = -\alpha^2, \quad \lambda_2 = \cdots = \lambda_{m-3} = 0. \tag{4.9}$$

From this we see that $[A_r, A_s] = 0$ for all $r, s = 3, \ldots, m$. By using equation (2.7) of Ricci, we see that the normal connection of M in E^m is flat. Thus, Theorem 4 implies that $M = C_1 \times C_2$, where C_1 and C_2 are two closed curves in E^n and E^{m-n}, respectively, with the same first curvature for some n, $1 < n < m$. On the other hand by a result of Kuiper [4], we have

$$TA(M)/c_{m-1} = TA(C_1)TA(C_2)/c_{n-1}c_{m-n-1},$$

where $TA(C_i)$ is the total absolute curvature of C_i, $i = 1, 2$. Since $TA(C_1) \geqslant 2c_{n-1}$ and $TA(C_2) \geqslant 2c_{m-n-1}$, (4.8) implies $TA(C_1) = 2c_{n-1}$ and $TA(C_2) = 2c_{m-n-1}$. From these we know that both C_1 and C_2 are two plane circles with the same radius [3]. Thus M is a Clifford torus. This proves Theorem 2.

Case (2). $G \geqslant 0$ *and* $G \not\equiv 0$. In this case, M is either homeomorphic to a sphere or a real projective plane. Now, suppose that M is homeomorphic to a real projective plane. Then we have $\chi(M) = 1$ and $\beta(M) = 3$. Hence inequality (4.4) implies

$$\int_M \alpha^2 \, dV \geqslant \frac{3}{2}\pi^2 + \left(1 - \frac{\pi}{2}\right)\int_M G \, dV.$$

This, combining with Gauss-Bonnet's formula, gives

$$\int_M \alpha^2 \, dV \geqslant (2 + \pi)\pi. \tag{4.10}$$

In the equality of (4.10) holds, then the inequality in (4.5) is actually an equality for all $(\theta_3, \ldots, \theta_m)$ satisfying $\cos^2\theta_3 + \cdots + \cos^2\theta_m = 1$. Hence, we have either $\lambda_2 = \cdots = \lambda_{m-2} = 0$ or $G = \lambda_3 = \cdots = \lambda_{m-2} = 0$ pointwise. Now, let $U = \{p \in M: G(p) \neq 0\}$. Then U is a nonempty open subset of M. By the assumption of pseudo-umbilicity, U is totally umbilical in E^m. Hence the Gauss curvature G is positive constant on every component of U (see [1, p. 49]). From this we know that U is also a closed subset of M. Thus $U = M$ and M is an ordinary 2-sphere in E^m. This is a contradiction. This proves Theorem 1. Q.E.D.

REMARK 1. The real projective plane can be immersed in E^5 as a pseudo-umbilical surface with positive constant Gauss curvature and total mean curvature 6π.

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