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COHOMOLOGY OF FINITE COVERS

BY

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ABSTRACT. For a finite dimensional CW-complex, X, and $q > 1$, it is shown that the qth Čech cohomology group based on finite open covers of X, $H_f^q(X)$, is naturally isomorphic to $H^q(X)$, the qth Čech cohomology of X (i.e. based on locally finite covers), and for reasonable X, $H^1(X)$ can be obtained algebraically from $H_f^1(X)$.

The Čech cohomology functor H_f^* based on finite open covers [2], [5] is not a cohomology theory. Dowker showed that H_f^1 of the real line is nontrivial and so H_f^1 does not satisfy the homotopy axiom [3].

In this paper I show that for finite dimensional CW-complexes, H_f^q is naturally isomorphic to the usual Čech cohomology functor H^q, i.e. based on locally finite covers, when $q > 1$ and for a nice space X, $H^1(X)$ can be obtained algebraically from $H_f^1(X)$. This greatly extends previous results [1].

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Let β be the Stone-Čech functor from the category of normal spaces and maps (continuous functions) to the category of compact ($\Rightarrow$ Hausdorff) spaces and maps. In other words, βX is the Stone-Čech compactification of X and $\beta(f\colon X \longrightarrow Y)\colon \beta X \longrightarrow \beta Y$ is the unique "extension" of f. We shall also use β to denote the embedding of X into βX.

For a space X and a topological group Y, $[X, Y]$ denotes the set of homotopy classes of maps from X to Y together with the group structure induced by Y.

A precise definition of Čech cohomology based on finite open covers can be found in [5, Chapter 9]. It is sufficient here to know that for a normal space X, the embedding $\beta\colon X \longrightarrow \beta X$ induces a natural isomorphism from $H_f^*(X)$ to $H^*(\beta X)$, [5, p. 282].

For paracompact spaces, $H^q(-; \pi)$ is naturally isomorphic to $[-, K(\pi, q)]$, where π is the coefficient group and $K(\pi, q)$ is an Eilenberg-Mac Lane space of type (π, q), [7, p. 423]. So for normal spaces, $H_f^q(X; \pi)$ is naturally isomorphic to $[\beta X, K(\pi, q)]$.

LEMMA 1. *For X a finite dimensional CW-complex, π a finitely generated*

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abelian group and $q > 0$, *the embedding* β: $X \longrightarrow \beta X$ *induces an epimorphism* β^*: $[\beta X, K(\pi, q)] \longrightarrow [X, K(\pi, q)]$.

PROOF. Let $\langle\cdot\rangle$ denote "equivalence class of", then $\beta^*\langle f\rangle = \langle f\beta\rangle$.

Since π is finitely generated and $q > 0$, we can choose $K(\pi, q)$ to be a CW-complex of finite type, i.e. with compact skeleta. Let $\langle f\rangle \in [X, K(\pi, q)]$. By the cellular approximation theorem we can assume that f is cellular. Since X is finite dimensional $f(X)$ is contained in a compact subspace of $K(\pi, q)$ and so f can be extended to a map F: $\beta X \longrightarrow K(\pi, q)$. Now $\beta^*\langle F\rangle = \langle f\rangle$.

To prove the main theorem we must show that for finite dimensional CW-complexes and $q > 1$, the kernel of β^* is trivial. The key to establishing that is the following

LEMMA 2. *Let* p: $E \longrightarrow B$ *be a principal G-bundle with G a CW-complex of finite type and B compact. If X is a finite dimensional CW-complex and* f: $X \longrightarrow E$ *is a map then there is a map* g: $X \longrightarrow E$ *such that* $pf = pg$ *and* $\overline{g(X)}$ *is compact.*

PROOF. Let $\{V_i\}$ be a finite atlas for the bundle, with coordinate functions φ_i: $G \times V_i \longrightarrow p^{-1}V_i$. Let $\{U_i\}$ be a shrinking of $\{V_i\}$, i.e. $\{U_i\}$ covers B and $\bar{U}_i \subset V_i$.

Let t_{ij}: $V_i \cap V_j \longrightarrow G$ be the coordinate transformations and let G^k denote the k-skeleton of G. Since $t_{ij}(\bar{U}_i \cap \bar{U}_j)$ is compact it is contained in $G^{m(i,j)}$ for some $m(i, j)$. Put $m = \max_{i,j}\{m(i, j)\}$ and for each k, choose $m(k)$ such that

$$G^mG^k = \{gg'\colon g \in G^m \text{ and } g' \in G^k\} \subset G^{m(k)}$$

and $m(k) > k$.

Let $E^k = \bigcup_i \varphi_i(G^k \times \bar{U}_i)$. Then E^k is compact and

$$E^k \cap p^{-1}U_i \subset \varphi_i(G^{m(k)} \times U_i).$$

[For if $y \in E^k \cap p^{-1}U_i$ then $y \in \varphi_j(G^k \times \bar{U}_j)$ for some j and $py \in U_i \cap \bar{U}_j$. So if $\varphi_j^{-1}y = (g, py) \in G^k \times \bar{U}_j$ then $\varphi_i^{-1}y = (t_{ij}(py)g, py) \in G^mG^k \times U_i \subset G^{m(k)} \times U_i$.]

Subdivide X so that for each cell u in X there is an i such that $pf\bar{u} \subset U_i$. Let X^n denote the n-skeleton of X in this subdivision and let $M^n = (X \times \{0\}) \cup (X^n \times I)$, where $I = [0, 1]$. Let $q_i = \theta\varphi_i^{-1}$: $p^{-1}V_i \longrightarrow G$, where θ: $G \times V_i \longrightarrow G$ is the projection. For $v \in X^0$, let λ_v: $I \longrightarrow G$ be a path from q_ifv to a 0-cell of G, where $pfv \in U_i$.

Define a map F^0: $M^0 \longrightarrow E$ by

$$F^0(x, 0) = fx, \qquad x \in X,$$

$$F^0(v, t) = \varphi_{i_v}(\lambda_v t, pfv), \qquad v \in X^0, \qquad t \in I \text{ and } pfv \subset U_{i_v}.$$

Then $pF^0(x, t) = pfx$ and $F^0(X^0 \times \{1\}) \subset E^0$.

Assume that there exists a map $F^{n-1}: M^{n-1} \longrightarrow E$ such that $pF^{n-1}(x, t) = pfx$ and $F^{n-1}(X^{n-1} \times \{1\}) \subset E^k$ for some $k \geqslant n$. For each n-cell u in X, choose i_u such that $pf\bar{u} \subset U_{i_u}$. Then

$$F^{n-1}(ux\{0\}) \subset p^{-1}U_{i_u}$$

and

$$F^{n-1}(\dot{u}x\{1\}) \subset E^k \cap p^{-1}U_{i_u} \subset \varphi_{i_u}(G^{m(k)} \times U_{i_u}).$$

[$\dot{u}$ denotes the boundary of u in X.] Let

$$h_u = q_{i_u}F^{n-1}|(ux\{0\}) \cup (\dot{u} \times I).$$

Then $h_u(\dot{u} \times \{1\}) \subset G^{m(k)}$. Since $m(k) \geqslant n$, $\pi_n(G, G^{m(k)}) = 0$ and therefore h_u can be extended to a map $H_u: u \times I \longrightarrow G$ such that $H_u(u \times \{1\}) \subset G^{m(k)}$. Define $F^n: M^n \longrightarrow E$ by $F^n|M^{n-1} = F^{n-1}$ and $F^n(x, t) = \varphi_{i_u}(H_u(x, t), pfx)$, for $x \in u$. Then F^n is a map such that $pF^n(x, t) = pfx$, $F^n|M^{n-1} = F^{n-1}$ and $F^n(X^n \times \{1\})$ is contained in $E^{m(k)}$. So by induction such a map exists for all n.

Define $g: X \longrightarrow E$ by $gx = F^n(x, 1)$ for all $x \in X^n$. Then $pf = pg$ and $\overline{g(X^n)}$ is compact for each n. Thus if X is finite dimensional $\overline{g(X)}$ is compact.

THEOREM 1. *For X a finite dimensional CW-complex, π a finitely generated abelian group and $q > 1$,*

$$\beta^*: H_f^q(X; \pi) \longrightarrow H^q(X; \pi)$$

is an isomorphism.

PROOF. Assume $K(\pi, q)$ is of finite type and let $p: E \longrightarrow K(\pi, q)$ be the universal bundle. Let $\langle f \rangle \in \ker \beta^*$. Then $f\beta$ can be lifted to a map $f': X \longrightarrow E$. Now $f\beta(X) \subset f(\beta X)$ which is compact and so contained in the k-skeleton K^k of $K(\pi, q)$, for some k. Let $E' \longrightarrow K^k$ be the $K(\pi, q-1)$-bundle over K^k obtained by restricting $p: E \longrightarrow K(\pi, q)$. We can now apply Lemma 2 to f' restricted to E' to obtain a map $g: X \longrightarrow E$ such that $\overline{g(X)}$ is compact and $pg = f\beta$. Now g can be extended to a map $g': \beta X \longrightarrow E$ and $pg' = f$. So f is null homotopic and hence $\ker \beta^* = 0$. The result now follows by Lemma 1.

We now consider the case $q = 1$, and X a normal space.

Let $j: X \longrightarrow CX$ be the embedding of X as the base of the cone CX on X. Then $\alpha = \beta j: \beta X \longrightarrow \beta CX$ is also an embedding. Consider the sequence

$$[\beta CX, S^1] \xrightarrow{\alpha^*} [\beta X, S^1] \xrightarrow{\beta^*} [X, S^1],$$

where α^* is induced by α and S^1 is the circle. If $\langle f \rangle \in \operatorname{im} \beta^*\alpha^*$, then f can be extended to βCX and thus to CX and so is null homotopic. So $\operatorname{im} \alpha^* \subset \ker \beta^*$.

If $\langle g\rangle \in \ker \beta^*$, then $g\beta$ is null homotopic, so can be extended to CX and hence to βCX. Thus $\ker \beta^* = \operatorname{im} \alpha^*$.

Because CX is contractible it follows from [4], or see [8, p. 225], that $[\beta CX, S^1]$ is isomorphic to the quotient of the additive group of real valued maps on CX by the subgroup of bounded maps. That is a divisible group [6, p. 163] and so $\operatorname{im} \alpha^*$ is a divisible group. Hence the short exact sequence

$$0 \longrightarrow \ker \beta^* \longrightarrow [\beta X, S^1] \xrightarrow{\beta^*} [X, S^1] \longrightarrow 0$$

splits. (β^* is onto since S^1 is compact.)

Thus if $H^1(X) \cong [X, S^1]$ is reduced, i.e. has no divisible subgroups, $\ker \beta^*$ is the unique maximal divisible subgroup, [6, p. 164], of $H^1_f(X) \cong [\beta X, S^1]$. In particular this will be the case if X has the homotopy type of a CW-complex with a finite 1-skeleton. Hence we have proved the following result.

THEOREM 2. *If X is a normal space and $H^1(X)$ is a reduced group then $H^1(X) \cong H^1_f(X)/G$, where G is the maximal divisible subgroup of $H^1_f(X)$.*

Theorems 1 and 2 can easily be extended to a relative CW-complex (X, A) since $\beta(X/A)$ is homeomorphic to $\beta X/\beta A$.

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