ANALYTIC HYPOELLIPTICITY OF CERTAIN SECOND-ORDER EVOLUTION EQUATIONS WITH DOUBLE CHARACTERISTICS

BY

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ABSTRACT. The present article establishes the analytic hypoellipticity (Definition 1.2) of a class of abstract evolution equations of order two, with double characteristics, under the hypothesis that the coefficients are analytic (in a suitable sense; see §2). The noteworthy feature of the main result (Theorem 4.1) is that analytic hypoellipticity holds whenever hypoellipticity does, even when one of the asymptotic eigenvalues $c^j(A)$ fails to be elliptic of order one.

Introduction. The present paper establishes the analytic hypoellipticity (Definition 1.2) of a class of (abstract) evolution equations of order two, with double characteristics—precisely the same class as that studied in the work [1], but under the hypothesis that the coefficients are analytic (in a suitable sense; see §2). We adapt the concatenation method of [1] and use the method of Grushin [2] to derive analytic hypoellipticity from suitable a priori estimates. The noteworthy feature of the main result (Theorem 4.1) is that analytic hypoellipticity holds whenever hypoellipticity does (under the analyticity hypothesis concerning the coefficients)—*even when one of the asymptotic eigenvalues $c^j(A)$ fails to be elliptic of order one* (in the latter case our results are essentially particular cases of those of Grushin). Since the equations studied here are microlocal models for the pseudodifferential equations of the kind studied in [4], our main result strongly suggests that, when their total symbols are analytic, the operators Π studied in [4] are hypoelliptic analytic if and only if all their asymptotic eigenvalues are.

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1. Analytic hypoellipticity in the abstract set-up. Let A be a linear operator, densely defined in a Hilbert space H . We shall assume that A is *unbounded* but it is *selfadjoint*, *positive-definite* and that it has a *bounded* inverse A^{-1} .

Let J be a given open subset of the real line. We denote by $\mathcal{Q}_A(J)$ the ring of the series in the nonnegative powers of A^{-1} with coefficients in $C^\infty(J)$ which converge in $L(H, H)$ (the Banach space of bounded linear operators of H into H)

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as well as each one of their derivatives in t , uniformly with respect to t on compact subsets of J .

$\mathcal{Q}_A(J)$ will denote the formal power series in A^{-1} , with coefficients in $C^\infty(J)$.

Now we want to state precisely what we mean for analytic hypoellipticity of an operator of the kind

$$(1.1) \quad P = \sum_{J+k=m} a_{J,k}(t, A) A^J \partial_t^k$$

where $a_{J,k}(t, A) \in \mathcal{Q}_A(J)$. Then we have to introduce a scale of "Sobolev spaces" H^s , $s \in \mathbb{R}$, defined by A :

If $s \geq 0$, H^s is the space of elements $u \in H$ such that $A^s u \in H$, equipped with the norm $\|u\|_s = \|A^s u\|_0$, where $\|\cdot\|_0$ denotes the norm in H .

If $s < 0$, H^s is the completion for the norm $\|u\|_s = \|A^s u\|_0$.

The inner product in H^s will be denoted by $(\cdot, \cdot)_s$.

Whatever $s, m \in \mathbb{R}$, A^m is an isomorphism (for the Hilbert space structure) of H^s onto H^{s-m} .

We denote by H^∞ the intersection of the spaces H^s and by $H^{-\infty}$ their union, the former equipped with the projective limit topology and the latter with the inductive limit topology.

H^∞ and $H^{-\infty}$ with their topology are the strong dual of each other as H^s and H^{-s} are.

We denote by $C^\infty(J, H^\infty)$ the space of the C^∞ functions in J valued in H^∞ , equipped with its natural C^∞ topology. It is the intersection of the spaces $C^h(J, H^k)$ as h and k tend to ∞ , of the h -continuously differentiable functions defined in J and valued in the Hilbert space H^k .

If K is any compact subset of J , we denote by $C_c^\infty(K, H^\infty)$ the subspaces of $C^\infty(J, H^\infty)$ consisting of the functions which vanish identically outside of K . this is a closed linear subspace of $C^\infty(J, H^\infty)$, then a Fréchet space and we denote by $C_c^\infty(J, H^\infty)$ the inductive limit of the $C_c^\infty(K, H^\infty)$ as K ranges over the compact subsets of J .

Now we can define the space $\mathcal{D}'(J, H^{-\infty})$ of the distributions in J valued in $H^{-\infty}$ as the dual of $C_c^\infty(J, H^\infty)$.

Finally we denote by $A(J, H^\omega)$, $[C^\infty(J, H^\omega)]$, the subspace of $C^\infty(J, H^\infty)$ given by the set of the functions $u(t) \in C^\infty(J, H^\infty)$ such that for every point $t_0 \in J$, there is an open neighborhood J' of t_0 relatively compact contained in J and a constant $C > 0$ such that for every $\alpha, \beta \in \mathbb{N}$

$$(1.2) \quad \begin{aligned} \sup_{t \in J'} \|\partial_t^\alpha A^\beta u(t)\|_0 &\leq C^{\alpha+\beta+1} (\alpha + \beta)! \\ \left[\sup_{t \in J'} \|A^\beta u(t)\|_0 &\leq C^{\beta+1} \beta \right]! \end{aligned}$$

DEFINITION 1.1. Let P be as in (1.1), P is said to be hypoelliptic in J if given any open subset J' of J and any distribution $u(t) \in \mathcal{D}'(J', H^{-\infty})$,

$$Pu \in C^\infty(J', H^\infty) \Rightarrow u \in C^\infty(J', H^\infty).$$

We say that P is hypoelliptic at a point t_0 of J if there is a neighborhood J' of t_0 in J such that P is hypoelliptic in J' .

DEFINITION 1.2. P is said to be analytic hypoelliptic in J if P is hypoelliptic in J and if, given any open subset J' of J and any distribution $u(t) \in \mathcal{D}'(J', H^{-\infty})$,

$$Pu \in A(J', H^\omega) \Rightarrow u \in A(J', H^\omega).$$

Similarly P is analytic hypoelliptic at a point t_0 of J if there is a neighborhood J' of t_0 in J such that P is analytic hypoelliptic in J' .

We introduce, now, a new scale of Hilbert spaces.

If $s \in \mathbb{R}$ and $s \geq 0$, we denote by E_s the set of all elements $u \in H$ for which there is a $v \in H$ such that $u = e^{-sA}v$ and we put in E_s the norm $\|u\|_{E_s} = \|v\|_0$.

If $s \in \mathbb{R}$ and $s < 0$, E_s is the completion of H with the norm $\|u\|_{E_s} = \|e^{sA}u\|_0$.

For every real s , E_s is a Hilbert space with the norm $\|\cdot\|_{E_s}$ and the following properties are true

- (1.3) $\begin{aligned} \text{(i)} & \text{ For every } s \text{ and } s' \text{ such that } s > s', E_s \text{ is canonically imbedded in } \\ & E_{s'}, \text{ with a norm } \leq 1. \\ \text{(ii)} & \text{ For every } s \text{ and } s' \text{ such that } s > s', A \text{ defines a continuous linear} \\ & \text{map } A: E_s \rightarrow E_{s'}, \text{ with norm } \leq C/(s - s'), \text{ where } C \text{ is a positive} \\ & \text{constant independent on } s \text{ and } s'. \end{aligned}$

For every real s , we denote by $A_s(J, E_s)$ the space of the analytic functions on J with value in E_s .

REMARK 1.1. It is easy to see that

$$(1.4) \quad A(J, H^\omega) = \bigcup_{s>0} A_s(J, E_s).$$

2. "Classes" of cut off functions. Let J denote an open interval centered in the origin; we are going to study the analytic hypoellipticity for "formal" operators of the form

$$(2.1) \quad P = (\partial_t - a(t, A)A)(\partial_t - b(t, A)A) - c(t, A)A = XY - c(t, A)A$$

where

$$(2.1)' \quad X = \partial_t - a(t, A)A; \quad Y = \partial_t - b(t, A)A$$

and $a(t, A), b(t, A), c(t, A) \in Q_A((J))$ which satisfies the following properties:

If $a_0(t)$ and $b_0(t)$ are the leading coefficients of $a(t, A)$ and $b(t, A)$, then

$$(2.2) \quad a_0(0) = b_0(0) = 0$$

$$(2.3) \quad \operatorname{Re} a'_0(0) > 0, \quad \operatorname{Re} b'_0(0) < 0.$$

Furthermore, if we write $a(t, A) = \sum_k a_k(t)A^{-k}$, $b(t, A) = \sum_k b_k(t)A^{-k}$, $c(t, A) = \sum_k c_k(t)A^{-k}$, we suppose that there is a neighborhood U of 0 in C , such that $U \cap \mathbb{R} = J$, on which the C^∞ functions $a_k(t)$, $b_k(t)$, $c_k(t)$ can be extended as holomorphic functions satisfying the inequality

$$(2.4) \quad \{|a_k(z)|, |b_k(z)|, |c_k(z)|\} \leq c^{k+1}k!$$

for every integer $k \geq 0$ and for every $z \in U$.

Therefore, by Cauchy's inequality, after a shrinking of U , we can suppose that there is a constant $M > 0$ such that

$$(2.5) \quad |\partial_t^\alpha a_k(z)|, |\partial_t^\alpha b_k(z)|, |\partial_t^\alpha c_k(z)| \leq c^{k+1}M^\alpha \alpha!k!$$

for every integer $\alpha, k \geq 0$ and for every $z \in U$.

Because A is a selfadjoint operator, using the spectral resolution of A , we have that

$$(2.6) \quad \begin{aligned} A &= \int_{-\infty}^{+\infty} \lambda dE_\lambda = \int_{\sigma(A)} \lambda dE_\lambda, \\ A^{-1} &= \int_{-\infty}^{+\infty} \lambda^{-1} dE_\lambda = \int_{\sigma(A)} \lambda^{-1} dE_\lambda \end{aligned}$$

where $\sigma(A)$ is the spectrum of A .

Below $\{\varphi_k(\lambda)\}$, $k \in \mathbb{N}$, denotes a sequence of continuous real valued functions defined on $\mathbb{R}$.

DEFINITION 2.1. Let c_1 be a number > 0 . We will say that the sequence $\{\varphi_k(\lambda)\}$ belongs to the "class of cut off functions" $[c_1]$ if

- (1) $\varphi_0(\lambda) = 1$ for every $\lambda \in \mathbb{R}$,
- (2) $0 \leq \varphi_k(\lambda) \leq 1$ for every $k \geq 1$ and $\lambda \in \mathbb{R}$ and

$$(2.7) \quad \varphi_k(\lambda) = \begin{cases} 1 & \forall \lambda \geq c_1 k + 1, \\ 0 & \forall \lambda \leq c_1 k. \end{cases}$$

Then for every nonnegative integer k , the operator

$$(2.8) \quad \dot{\varphi}_k(A) = \int \varphi_k(\lambda) dE_\lambda$$

is a bounded selfadjoint linear operator defined on H into H .

We define formally

$$(2.9) \quad P(\{\varphi_k\}) = (\partial_t - a(t, A, \{\varphi_k\})A)(\partial_t - b(t, A, \{\varphi_k\})A) - c(t, A, \{\varphi_k\})A$$

where

$$a(t, A, \varphi_k) = \sum_k a_k(t) \varphi_k(A) A^{-k},$$

$$b(t, A, \varphi_k) = \sum_k b_k(t) \varphi_k(A) A^{-k},$$

$$c(t, A, \varphi_k) = \sum_k c_k(t) \varphi_k(A) A^{-k}.$$

REMARK 2.1. If $c_1 > c$, where c is given by (2.4), for every $\{\varphi_k\} \in [c_1]$ $P(\{\varphi_k\})$ is a "true" operator, that is the coefficients $a(t, A, \{\varphi_k\})$, $b(t, A, \{\varphi_k\})$ and $c(t, A, \{\varphi_k\})$ are C^∞ functions defined on J with value in $L(H, H)$.

Indeed, take an integer α and let $\|\cdot\|$ denote the norm $L(H, H)$; we have, by (2.5),

$$\begin{aligned} & \sup_{t \in J} \left\| \sum_0^h (\partial_t^\alpha a_k)(t) \varphi_k(A)^{-k} \right\| \\ & \leq \sum_{k=0}^h \sup_J |\partial_t^\alpha a_k(t)| \cdot \|\varphi_k(A) A^{-k}\| \\ & \leq \sum_{k=0}^h c^{k+1} M^{\alpha} \alpha! k! \cdot (c_1 k)^{-k} \leq c M^{\alpha} \alpha! \left(\sum_0^\infty \left(\frac{c}{c_1} \right)^k \right), \end{aligned}$$

which proves the result.

Then there is a constant M_1 such that for every $\alpha \in \mathbb{N}$

$$(2.10) \quad \sup_J \{ \|\partial_t^\alpha a(t, A, \{\varphi_k\})\|, \|\partial_t^\alpha b(t, A, \{\varphi_k\})\|, \|\partial_t^\alpha c(t, A, \{\varphi_k\})\| \} = M_1^{\alpha+1} \alpha!$$

Suppose that P_1 and P_2 are operators of the kind

$$P_1 = \sum_{J+k=2} F_{J,k}(t) A^J \partial_t^k, \quad P_2 = \sum_{J+k=2} G_{J,k}(t) A^J \partial_t^k$$

where $F_{J,k}(t)$, $G_{J,k}(t)$ are analytic functions on J with value in $L(H, H)$ which can be extended as holomorphic functions on the complex neighborhood U of 0 ($J = U \cap \mathbb{R}$) with value in $L(H, H)$ and $F_{0,2}(t) = G_{0,2}(t) = I_H$, I_H being the identity mapping of H .

THEOREM 2.1. *Let us suppose that*

$$(2.11) \quad P_1 = P_2 + R$$

where R is a linear operator which has the following property: there is an $\epsilon > 0$ such that for every real s

$$(2.12) \quad R: A_t(J, E_s) \rightarrow A_t(J, E_{s+\epsilon})$$

is a linear operator of $A_t(J, E_s)$ into $A_t(J, E_{s+\epsilon})$.

Then if P_2 is analytic hypoelliptic at $t = 0$, the same is true of P_1 .

PROOF. For the proof we will use some results of [3], using the scale of Hilbert spaces E_s .

Let u be an element of $\mathcal{D}'(J, H^{-\infty})$ such that $P_1 u = f$ where $f \in A(J, H^\omega)$, then $f \in A_t(J, E_0)$, recalling that $E_0 = H$, then if we shrink enough the complex neighborhood U of 0 we can extend f to a holomorphic function on U with values in E_0 , i.e. $f \in A_t(U, E_0)$.

Now we reduce the second order differential equation in t , $P_1 u = f$, to a first order linear system

$$LU = I\partial_t U - M(t, A)U = F(t)$$

where I is the identity matrix 2×2 , $M(t, A)$ a 2×2 matrix and $F(t)$ a two-vector, holomorphically dependent on t varying in U .

Now for (i) and (ii) of (1.3) the hypothesis of Theorem 2.1 in [3] are verified and by the uniqueness of the solution and by Remark 9.3 in [3], for every $s < 0$, u is the restriction of a holomorphic function $\tilde{u}(t)$ on U (after a shrinking of U) with value in E_s , in particular we have $u \in A_t(J, E_s)$.

Then by property (2.12), if we choose $s = -\epsilon/2$, we have that $Ru \in A_t(J, E_{\epsilon/2})$, then by Remark 1.1 we have that $Ru \in A(J, H^\omega)$.

Therefore

$$P_2 u = P_1 u - Ru \in A(J, H^\omega),$$

then by the analytic hypoellipticity, at $t = 0$, of P_2 we get that also P_1 is analytic hypoelliptic at $t = 0$.

We have now the following consequence

THEOREM 2.2. *Suppose that c_1 and c_2 are two constants such that*

$$(2.13) \quad c_1 > c, \quad c_2 > c,$$

where c is the constant given in (2.4), if $\{\varphi_k\} \in [c_1]$ and $\{\psi_k\} \in [c_2]$, then $P(\{\varphi_k\})$ is analytic hypoelliptic at $t = 0$ if and only if this is so for $P(\{\psi_k\})$.

PROOF. We have

$$\begin{aligned}
P(\{\varphi_k\}) &= P(\{\psi_k\} - \{\psi_k - \varphi_k\}) \\
&= [\partial_t - a(t, A, \{\psi_k\})A + a(t, A, \{\psi_k - \varphi_k\})A] \\
&\quad \cdot [\partial_t - b(t, A, \{\psi_k\})A + b(t, A, \{\psi_k - \varphi_k\})A] \\
&\quad - c(t, A, \{\psi_k\})A + c(t, A, \{\psi_k - \varphi_k\})A \\
&= P(\{\psi_k\}) + a(t, A, \{\psi_k - \varphi_k\})A[\partial_t - b(t, A, \{\psi_k\})A] \\
&\quad + (\partial_t - a(t, A, \{\psi_k\})A + a(t, A, \{\psi_k - \varphi_k\})A)b(t, A, \{\psi_k - \varphi_k\})A \\
&\quad + c(t, A, \{\psi_k - \varphi_k\})A \\
&= P(\{\psi_k\}) + R.
\end{aligned}$$

Now to apply Theorem 2.1 with $P(\{\varphi_k\})$ and $P(\{\psi_k\})$ instead of P_1 and P_2 or vice versa, we have to prove that R verifies property (2.12), and for this it is sufficient to prove that $a(t, A, \{\psi_k - \varphi_k\})$, $b(t, A, \{\psi_k - \varphi_k\})$, $c(t, A, \{\psi_k - \varphi_k\})$ verify property (2.12).

Let us take, for instance,

$$a(t, A, \{\chi_k\}) = \sum_1^{\infty} a_k(t)\chi_k(A)A^{-k},$$

where we have put $\chi_k = \psi_k - \varphi_k$.

Now it is sufficient to prove that there is an $\epsilon > 0$ such that for every $s \in \mathbb{R}$, $a(t, A, \{\chi_k\})$ is a holomorphic function on U with values in the Banach space $L(E_s, E_{s+\epsilon})$ of the bounded linear operators of E_s into $E_{s+\epsilon}$ and because for any given u for which $e^{sA}u$ is defined,

$$a(t, A, \{\chi_k\})e^{sA}u = e^{sA}a(t, A, \{\chi_k\})u,$$

it is enough to consider the case $s = 0$.

Because, by hypothesis, $a(t, A, \{\chi_k\})$ depends holomorphically on t when t varies in U , we have to prove that there is an $\epsilon > 0$, independent on t , such that for every $u \in H$, the series

$$\sum_0^{\infty} \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u$$

converges in H , i.e. that there is a constant M , independent of t and u , such that

$$(2.14) \quad \left\| \sum_0^{\infty} \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u \right\|_0 \leq M \|u\|_0.$$

We need the following

LEMMA 2.1. *There is a constant M_1 , independent on $t \in U$, such that*

$$(2.15) \quad \|A^h a(t, A, \{\chi_k\})u\|_0 \leq M_1^{h+1} h! \|u\|_0$$

for every nonnegative integer h and any $u \in H$.

PROOF. First of all we observe that $\chi_k(\lambda)$, for $\lambda \in \mathbf{R}$, is a real function, such that, $\forall k \geq 1$

$$0 \leq \chi_k(\lambda) \leq 1 \quad \text{and} \quad \text{supp } \chi_k \subseteq [c_1 k, c_2 k + 1]$$

if we suppose $c_1 \leq c_2$.

Then by (2.4),

$$\begin{aligned} \|A^h a(t, A, \{\chi_k\})u\|_0 &= \sum_1^{h-1} \sup_U |a_k(t)| \|A^{h-k} \chi_k(A)u\|_0 \\ &\quad + \sum_0^\infty \sup_U |a_{h+k}(t)| \|A^{-k} \chi_{k+h}(A)u\|_0 \\ &\leq \sum_1^{h-1} c^{k+1} k! (2c_2 k)^{h-k} \|u\|_0 + \sum_0^\infty c^{k+h+1} (k+h)! c_1^{-k} (h+k)^{-k} \|u\|_0 \\ &\leq \left\{ (2c_2 h)^h \sum_0^\infty \left(\frac{c}{c_2} \right)^k + c^{h+1} h! \sum_0^\infty \left(\frac{c}{c_1} \right)^k \right\} \|u\|_0; \end{aligned}$$

then if we take M_1 big enough and use the Stirling formula, we have (2.15).

Now we prove (2.14). For every integer l we have

$$\begin{aligned} \left\| \sum_0^l \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u \right\|_0 &\leq \sum_0^l \frac{\epsilon^h}{h!} \|A^h a(t, A, \{\chi_k\})u\|_0 \\ &\leq \sum_0^l \frac{\epsilon^h}{h!} M_1^{h+1} h! \|u\|_0 \leq M_1 \sum_0^l \epsilon (M_1)^h \|u\|_0. \end{aligned}$$

Then if $\epsilon < 1/M_1$, we have (2.14), with

$$M = M_1 \sum_0^\infty \epsilon (M_1)^h$$

which ends the proof of Theorem 2.2.

3. Concatenations. Starting from the "formal" operator P given by (2.1) and satisfying the hypotheses (2.1), (2.2) we are going to construct a sequence of formal operators $P^0, P^1, \dots, P^J, \dots$ whose coefficients belong to $\mathcal{Q}_A((J))$, still satisfying the properties (2.1). For this construction, we refer to [1].

The first step is to find an operator $\psi(t, A) = \sum_k \psi_k(t) A^{-k} \in \mathcal{Q}_A((J))$, and a "coefficient" $c^0(A) = \sum_k c_k^0 A^{-k} \in C[[A^{-1}]]$, such that the operator

$$(3.1) \quad P^0 = (X + \psi(t, A))(Y - \psi(t, A)) - c^0(A)A$$

is formally equal to P .

Lemma II.4.1 in [1] shows how to construct $\psi(t, A)$ and $c^0(A)$.

Let us now put

$$\begin{aligned} X^0 &= \partial_t - a^0(t, A)A, & Y^0 &= \partial_t - b^0(t, A)A, \\ a^0(t, A) &= a(t, A) - \psi(t, A)A^{-1} = \sum_k a_k^0(t)A^{-k}, \\ b^0(t, A) &= b(t, A) + \psi(t, A)A^{-1} = \sum_k b_k^0(t)A^{-k}. \end{aligned}$$

Then

$$(3.2) \quad P^0 = X^0 Y^0 - c^0(A)A.$$

REMARK 3.1. The important fact is that the "principal symbol" of P is the same as that of P^0 ; in particular the leading coefficients in the power series $a(t, A)$ and $a^0(t, A)$ (respectively $b(t, A)$ and $b^0(t, A)$) are the same.

REMARK 3.2. If we suppose that property (2.4) is verified, then from the proof of Lemma II.4.1 in [1] it follows easily, after a shrinking of J , that there is a neighborhood V of 0 in C such that $R \cap V = J$, on which $\psi_k(t)$ can be extended as a holomorphic function for every nonnegative integer k , we can suppose $V = U$.

Furthermore there is a constant $c_0 > 0$, such that

$$(3.3) \quad \{|a_k^0(t)|, |b_k^0(t)|, |c_k^0|\} \leq c_0^{k+1} k!, \quad \forall k \in \mathbb{N} \text{ and } \forall t \in U.$$

By property (2.4), to prove (3.3) it is sufficient to see that there is a constant M and $D > 0$, such that

$$(3.4) \quad \{|\psi_k(t)|, |c_k^0|\} \leq DM^k k!$$

$\forall k \in \mathbb{N}, \forall t \in U$ (possibly after a shrinking of U).

From (II.4.3) of [1] we have that $\psi_k(t)$ and c_k^0 have to verify the following equality

$$(3.5) \quad \sum_{h=0}^k \delta_h(t) \psi_{k-h}(t) + c_k(t) - c_k^0 = \sum_{h=0}^{k-1} \psi_h(t) \psi_{k-1-h}(t) + \psi'_{k-1}(t)$$

where $\delta_h(t) = a_h(t) - b_h(t)$.

We reason by induction. We choose D so big that (3.4) is verified for $k = 0$, we suppose (3.4) true up to $k - 1$ and we want to prove it for k .

Because $\delta_0(0) = a_0(0) - b_0(0) = 0$, (3.5), computed for $t = 0$, gives us

$$(3.6) \quad c_k^0 = c_k(0) + \sum_{h=1}^k \delta_h(0) \psi_{k-h}(0) - \sum_{h=1}^{k-1} \psi_h(0) \psi_{k-1-h}(0) + \psi'_{k-1}(0).$$

By (2.4), there is a constant c_1 such that $|\delta_k(t)|, |c_k(t)| \leq Dc_1^k k!$ if we suppose $D > 2c$. Then from (3.6) we have

$$\begin{aligned} |c_k^0| &= \sum_{h=1}^k Dc_1^h h! \cdot DM^{k-h}(k-h)! + Dc_1^k k! \\ &\quad + \sum_1^{k-1} D^2 M^{k-1} h!(k-h-1)! + \frac{DM^{k-1}}{d} (k-1)!. \end{aligned}$$

Indeed, by Cauchy's inequality, if we shrink enough U , we have $|\psi'_{k-1}(t)| \leq (DM^{k-1}/d)(k-1)!, \forall t \in U$, where d is a positive constant independent on k .

Then if α is a constant such that $0 < \alpha \leq 1$,

$$\begin{aligned} |c_k^0| &\leq \frac{\alpha}{6} DM^k k! \sum_1^k \left(\frac{6Dc_1}{\alpha M} \right)^h + \frac{\alpha}{6} D \left(\frac{6}{\alpha} c_1 \right)^k k! \\ &\quad + \frac{\alpha}{6} DM^k k! \left(\frac{6D}{\alpha M} \sum_1^{k-1} \frac{h!(k-1-h)!}{(k-1)!} \frac{1}{k} + \frac{6}{\alpha d k M} \right). \end{aligned}$$

If we choose M so big that $M \geq 12Dc_1/\alpha$ and $M > (6D/\alpha + 6/d\alpha)$ the series $\sum_1^\infty (6Dc_1/\alpha M)^h$ converges. We have

$$(3.7) \quad |c_k^0| \leq \frac{1}{2} \alpha DM^k k! \leq DM^k k!$$

which is the second inequality of (3.4).

From (3.5) we get

$$\begin{aligned} h_k(t) &= \delta_0(t) \psi_k(t) \\ &= c_k^0 - c_k(t) - \sum_1^k \delta_h(t) \psi_{k-h}(t) + \sum_1^{k-1} \psi_h(t) \psi_{k-1-h}(t) + \psi'_{k-1}(t) \end{aligned}$$

for every $t \in U$.

As before, and from (3.7), we have for $\forall t \in U$

$$(3.8) \quad |h_k(t)| \leq \frac{1}{2} \alpha DM^k k! + \frac{1}{2} \alpha DM^k k! \leq \alpha DM^k k!.$$

But $h_k(0) = 0$ and $\delta_0(0) = 0$; then

$$h_k(t) = \int_0^1 h'_k(\tau t) d\tau \cdot t, \quad \delta_0(t) = \int_0^1 \delta'_0(\tau t) d\tau \cdot t.$$

By the Cauchy inequality and from (3.8), possibly after a shrinking of U , there is a positive constant d such that

$$|h'_k(t)| \leq \alpha(D/d)M^k k! \quad \forall k \in N.$$

Now, since $\delta'_0(0) \neq 0$, once more after shrinking U , we can suppose $\delta'_0(t) \neq 0, \forall t \in U$, then

$$\psi_h(t) = \int_0^1 h'_k(\tau t) d\tau / \int_0^1 \delta'_0(\tau t) d\tau, \quad t \in U.$$

Let D_1 be equal to $|\int_0^1 \delta'_0(\tau t) d\tau|$, therefore

$$|\psi_k(t)| \leq (\alpha/dD_1)DM^k k!, \quad \forall t \in U.$$

Now if dD_1 is ≤ 1 we put $\alpha = dD_1$, otherwise we have $\alpha/dD_1 \leq 1$. In any case we get the first inequality of (3.4) which proves the result.

The operator P^0 will be the first element in our sequence attached to P . Suppose we have constructed the j th element $P^j = X^j Y^j - c^j(A)A$ we define

$$\begin{aligned} P^{j+1} &= Y^j X^j - c^j(A)A = X^j Y^j - [X^j, Y^j] - c^j(A)A \\ (3.9) \quad &= X^j Y^j - \{c^j(A) + \delta^j_i(t, A)\}A \end{aligned}$$

where we put $\delta^j_i(t, A) = a^j(t, A) - b^j(t, A), j = 0, 1, \dots$

Then, as before, taking P^{j+1} instead of P , we can write P^{j+1} in the way

$$(3.10) \quad P^{j+1} = X^{j+1} Y^{j+1} - c^{j+1}(A)A$$

and we will have

$$\begin{aligned} X^{j+1} &= \partial_t - a^{j+1}(t, A)A, \quad Y^{j+1} = \partial_t - b^{j+1}(t, A)A, \\ (3.11) \quad a^{j+1}(t, A) &= a(t, A) - \psi^{j+1}(t, A)A^{-1}, \\ b^{j+1}(t, A) &= b(t, A) + \psi^{j+1}(t, A)A^{-1}, \end{aligned}$$

where $\psi^j(t, A) = \sum_k \psi^j_k(t)A^{-k} \in \mathcal{Q}_A((J)), c^j(A) = \sum_k c^j_k A^{-k} \in C[[A^{-1}]]$.

REMARK 3.3. P^{j+1} will have the same "principal symbol" as P , and therefore verifies (2.2) and (2.3). By Remark 3.2, like P^0 it will verify a property analogous to (3.3), possibly after a shrinking of U , with $a^0_k(t), b^0_k(t), c^0_k$ and c_0 by $a^{j+1}_k(t), b^{j+1}_k(t), c^{j+1}_k, c_{j+1}$.

We will suppose that the constant c_j has the property

$$(3.12) \quad c_{j+1} > c_j > c, \quad j \geq 0,$$

where c is the constant (2.4).

From (3.9) follows immediately the

PROPOSITION 3.1. We have, for $j = 0, 1, \dots$,

$$(3.13) \quad Y^j P^j = P^{j+1} Y^j,$$

$$(3.14) \quad X^j P^{j+1} = P^j X^j.$$

DEFINITION 3.1. The sequence of formal operators $\{P^j\}_{j \geq 0}$ will be called the *XY-concatenation* attached to P .

PROPOSITION 3.2. Let $\{P^j\}$ be the *XY-concatenation* attached to P . Then the leading coefficient c_0^j of $c^j(A)$ is given by

$$(3.15) \quad c_0^j = c_0(0) + j\delta_0'(0),$$

where $\delta_0(t)$ is the leading coefficient in $\delta^0(t, A)$.

See Proposition II.4.2 of [1].

THEOREM 3.1. Let us suppose that $\{\varphi_k\} \in [\tilde{c}_0]$ and $\tilde{c}_0 > 2c_0$, where c_0 is given by (3.3). Then $P(\{\varphi_k\})$ and $P^0(\{\varphi_k\})$ are true operators and $P(\{\varphi_k\})$ is analytic hypoelliptic at $t = 0$ if and only if so is $P^0(\{\varphi_k\})$.

PROOF. The proof is similar to that of Theorem 2.2. We observe that

$$\begin{aligned} & A^{-1}(P^0(\{\varphi_k\}) - P(\{\varphi_k\})) \\ &= \delta(t, A, \{\varphi_k\})\psi(t, A, \{\varphi_k\}) + c(t, A, \{\varphi_k\}) - c_0(A, \{\varphi_k\}) \\ &\quad - A^{-1}(\psi^2 + \psi_t)(t, A, \{\varphi_k\}) \\ &= \sum_0^\infty k \left(\sum_0^k \delta_h(t) \psi_{k-h}(t) \varphi_h(A) \varphi_{k-h}(A) \right) A^{-k} + \sum_0^\infty k (c_k(t) - c_k^0) \varphi_k(A) A^{-k} \\ &\quad - \sum_0^\infty k \left(\sum_0^k \psi_h(t) \psi_{k-h}(t) \varphi_h(A) \varphi_{k-h}(A) \right) A^{-k-1} - \sum_0^\infty k \psi'_k(t) \varphi_k(A) A^{-k-1} \\ &= \left\{ \sum_0^\infty k \left(\sum_0^k \delta_h(t) \psi_{k-h}(t) \right) \varphi_k(A) A^{-k} + \sum_0^\infty k (c_k(t) - c_k^0) \varphi_k(A) A^{-k} \right. \\ &\quad \left. - \sum_0^\infty k \left[\left(\sum_0^k \psi_h(t) \psi_{k-h}(t) \right) - \psi'_k(t) \right] \varphi_{k+1}(A) A^{-k-1} \right\} \\ &\quad - \sum_0^\infty k \sum_0^k \psi_{k-h}(t) \delta_h(t) (\varphi_k - \varphi_h \varphi_{k-h})(A) A^{-k} \\ &\quad + \sum_0^\infty k \sum_0^k \psi_h(t) \psi_{k-h}(t) (\varphi_{k+1} - \varphi_h \varphi_{k-h})(A) A^{-k-1} \\ &\quad + \sum_0^\infty k \psi'_k(t) (\varphi_{k+1} - \varphi_k)(A) A^{-k-1}. \end{aligned}$$

For how P^0 has been formally constructed, we have that the sum in the big brackets is zero, then

$$\begin{aligned}
 & A^{-1}(P^0(\{\varphi_k\}) - P(\{\varphi_k\})) \\
 &= - \sum_0^\infty k \sum_0^k \delta_h(t) \psi_{k-h}(t) (\varphi_k - \varphi_h \varphi_{k-h})(A) A^{-k} \\
 (3.16) \quad &+ \sum_0^\infty k \sum_0^k \psi_h(t) \psi_{k-h}(t) (\varphi_{k+1} - \varphi_h \varphi_{k-h})(A) A^{-k-1} \\
 &+ \sum_0^\infty k \psi'_k(t) (\varphi_{k+1} - \varphi_k)(A) A^{-k-1} \\
 &= \tilde{R}.
 \end{aligned}$$

Then

$$(3.17) \quad P^0(\{\varphi_k\}) - P(\{\varphi_k\}) = A\tilde{R} = R.$$

Now to apply Theorem 2.1, we need to prove that R or $\tilde{R}$ verifies property (2.12).

Then as in the proof of Theorem 2.2, we reduce ourselves to prove

LEMMA 3.1. *There is a constant M_1 independent on $t \in U$, such that*

$$(3.18) \quad \|A^h \tilde{R} u\|_0 \leq M_1^{h+1} h! \|u\|_0$$

for every nonnegative integer h and any $u \in H$.

PROOF. First of all we observe that: $\forall h, 0 \leq h \leq k, k \geq 1$

$$\varphi_h \varphi_{k-h}(\lambda) = \begin{cases} 0, & \lambda \leq \tilde{c}_0 k/2, \\ 1, & \lambda \leq \tilde{c}_0 k + 1. \end{cases}$$

Therefore

$$\begin{aligned}
 (\varphi_k - \varphi_h \varphi_{k-h})(\lambda) &= \begin{cases} 0, & \lambda \leq \tilde{c}_0 k/2, \lambda \geq \tilde{c}_0 k + 1, \\ \leq 1, & \text{otherwise.} \end{cases} \\
 (3.19) \quad (\varphi_{k+1} - \varphi_h \varphi_{k-h})(\lambda) &= \begin{cases} 0, & \lambda \leq \tilde{c}_0 k/2, \lambda \geq \tilde{c}_0 (k+1) + 1, \\ \leq 1, & \text{otherwise.} \end{cases} \\
 (\varphi_{k+1} - \varphi_k)(\lambda) &= \begin{cases} 0, & \lambda \leq \tilde{c}_0 k, \lambda \geq \tilde{c}_0 (k+1) + 1, \\ \leq 1, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Now using the first of (3.19) we prove that (3.18) is verified from the sum

$$(3.20) \quad \sum_0^\infty k \sum_0^k \delta_l(t) \psi_{k-l}(t) (\varphi_k - \varphi_l \varphi_{k-l})(A) A^{-k}.$$

We proceed in the same way for the other series in R . By (2.4), (3.3) and (3.12), $\forall t \in U$

$$\begin{aligned} & \left\| A^h \sum_0^\infty k \sum_0^k l (\delta_l \psi_{k-l})(t) (\varphi_k - \varphi_l \varphi_{k-l})(A) A^{-k} u \right\|_0 \\ & \leq \sum_1^{h-1} k \sum_0^k l |(\delta_l \psi_{k-l})(t)| \|(\varphi_k - \varphi_l \varphi_{k-l})(A) A^{h-k} u\|_0 \\ & \quad + \sum_0^\infty k \sum_0^{k+h} l |(\delta_l \psi_{k+h-l})(t)| \|(\varphi_{k+h} - \varphi_l \varphi_{k+h-l})(A) A^{-k} u\|_0 \\ & \leq \sum_1^{h-1} k \sum_0^k 4c_0^{k+2} l! (k-l)! (\tilde{c}_0 k + 1)^{h-k} \cdot \|u\|_0 \\ & \quad + \sum_0^\infty k \sum_0^{k+h} 4c_0^{k+h+2} l! (k+h-l)! \left[\frac{\tilde{c}_0}{2} (k+h) \right]^{-k} \cdot \|u\|_0 \\ & \leq 4c_0^2 \sum_1^{h-1} k k c_0^k! (2\tilde{c}_0 k)^{h-k} \cdot \|u\|_0 \\ & \quad + 4c_0^2 \sum_0^\infty k (k+h) c_0^{k+h} (k+h)! \frac{2^k}{c_0^k (k+h)^{-k}} \cdot \|u\|_0. \end{aligned}$$

Then if M is a constant big enough we have

$$\leq M(2\tilde{c}_0)^h h^h \sum_0^\infty k \left(\frac{c_0}{\tilde{c}_0} \right)^k \cdot \|u\|_0 + M c_0^h h! \sum_0^\infty k \left(\frac{(2c_0)}{\tilde{c}_0} \right)^k \cdot \|u\|_0.$$

Then because $\tilde{c}_0 > 2c_0$, by Stirling formula, we have the (3.18).

With this lemma, we end the proof as in Theorem 2.2.

Let j be a positive integer and let $\mathbf{P}^0, \mathbf{P}^1, \dots, \mathbf{P}^j$ be the first $j+1$ operators in the XY -concatenation attached to $\mathbf{P}$.

Let $\tilde{c}_j$ be a constant bigger than $2c_j$, where c_j is the constant given by (3.12) and $\{\rho_k\} \in [\tilde{c}_j]$.

Then all $\mathbf{P}^0(\{\rho_k\}), \mathbf{P}^1(\{\rho_k\}), \dots, \mathbf{P}^j(\{\rho_k\})$ are true operators.

COROLLARY 3.1. *In the previous hypothesis, if for some integer h , $0 \leq h \leq j-1$, $\mathbf{P}^{h+1}(\{\rho_k\})$ is analytic hypoelliptic at $t=0$ and we suppose that the following property is verified:*

there is an open neighborhood J' of 0, relatively compact contained in J , such that if $u \in \mathcal{D}'(J', H^{-\infty})$ and

$$(3.21) \quad Y^h(\{\rho_k\})u \in A(J', H^\omega), \quad P^h(\{\rho_k\})u \in A(J', H^\omega)$$

then $u \in A(J', H^\omega)$

then also $P^h(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

PROOF. From (3.9), we have

$$\begin{aligned} Y^h(\{\rho_k\})P^h(\{\rho_k\}) \\ &= [X^h(\{\rho_k\})Y^h(\{\rho_k\}) - \{c^h(A, \{\rho_k\}) + \delta_t^h(t, A, \{\rho_k\})\}A]Y^h(\{\rho_k\}) \\ &= \tilde{P}^h(\{\rho_k\})Y^h(\{\rho_k\}) \end{aligned}$$

where we put $\tilde{P}^h = X^h Y^h - \{c^h(A) + \delta_t^h(t, A)\}A$.

Now if we recall as P^{h+1} has been constructed, from Theorem 3.1 we have that $\tilde{P}^h(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$ if and only if so is $P^{h+1}(\{\rho_k\})$.

Then by hypothesis, there is a neighborhood J'' of 0 that we can suppose equal to J' , on which $\tilde{P}^h(\{\rho_k\})$ is hypoelliptic.

Then let $u \in \mathcal{D}'(J', H^{-\infty})$ such that $P^h(\{\rho_k\})u \in A(J', H^\omega)$, then

$$\tilde{P}^h(\{\rho_k\})Y^h(\{\rho_k\})u = Y^h(\{\rho_k\})P^h(\{\rho_k\})u \in A(J', H^\omega);$$

therefore $Y^h(\{\rho_k\})u \in A(J', H^\omega)$ then by (3.21) $u \in A(J', H^\omega)$, therefore $P^h(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

4. Statement of the main theorem. Let P be the formal operator given by (2.1) and satisfying the hypothesis (2.2), (2.3) and (2.4) and let c be as usual, the constant given by (2.4).

We recall that for any sequence $\{\rho_k\} \in [c_1]$, where $c_1 > c$, $P(\{\rho_k\})$ is a true operator.

DEFINITION 4.1. The formal operator P is said to be analytic hypoelliptic at a point $t \in J$ if for any constant $c_1 \geq c$ and for any sequence $\{\rho_k\} \in [c_1]$, the true operator $P(\{\rho_k\})$ is analytic hypoelliptic at t .

REMARK 4.1. By Theorem 2.2, for P to be analytic hypoelliptic at a point $t \in J$, it is necessary and sufficient that there is a constant c_1 bigger than c and a sequence $\{\rho_k\} \in [c_1]$ for which $P(\{\rho_k\})$ is analytic hypoelliptic at t .

We can now state the main theorem of this paper.

THEOREM 4.1. Let P be the formal operator given by (2.1) and satisfying the hypotheses (2.2), (2.3) and (2.4); let $P^j = X^j Y^j - c^j(A)A$, $j = 0, 1, \dots$, denote the successive elements in the XY -concatenation attached to P .

The following are equivalent:

$$(4.1) \quad P \text{ is hypoelliptic at } t = 0;$$

(4.2) P is analytic hypoelliptic at $t = 0$;

(4.3) For no integer $j \geq 0$, the formal power series in A^{-1} , $c^j(A)$, is identically zero.

The equivalences (4.1) $\iff$ (4.3) have been already proved in [1]. We have to prove that (4.3) implies (4.2) (by Definition 1.2, (4.2) $\Rightarrow$ (4.1)).

For the proof of Theorem 4.1 we need the following theorem which will be proved in §6.

THEOREM 4.2. *Let I be an open interval centered in the origin and $\bar{I}$ its closure. Suppose that Q is a (true) operator given by*

$$(4.4) \quad Q = (\partial_t - \alpha(t)tA - \tilde{\alpha}(t, A))(\partial_t - \beta(t)tA - \tilde{\beta}(t, A)) - \gamma(t)A - \tilde{\gamma}(t, A)$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are complex valued C^∞ functions on $\bar{I}$ and $\tilde{\alpha}(t, A)$, $\tilde{\beta}(t, A)$, $\tilde{\gamma}(t, A)$ are C^∞ functions on $\bar{I}$ with values in $L(H, H)$, for which the following hypotheses are verified:

$$(4.5) \quad \operatorname{Re} \alpha(0) > 0, \quad \operatorname{Re} \beta(0) < 0;$$

$$(4.6) \quad |\alpha(0) - \beta(0)|^2 \leq 2 \operatorname{Re} \{ \gamma(0) [\overline{\alpha(0) - \beta(0)}] \}.$$

Suppose furthermore that there is a constant $M > 0$, such that for every $t \in I$ and any nonnegative integer h

$$(4.7) \quad \{ |\partial_t^h \alpha(t)|, |\partial_t^h \beta(t)|, |\partial_t^h \gamma(t)|, \|\partial_t^h \tilde{\alpha}(t, A)\|, \|\partial_t^h \tilde{\beta}(t, A)\|, \|\partial_t^h \tilde{\gamma}(t, A)\| \} \leq M^{h+1} h!$$

where $\| \cdot \|$ is the norm in $L(H, H)$.

Under these hypotheses P is analytic hypoelliptic at $t = 0$.

5. Proof of Theorem 4.1.

PROOF OF (4.3) $\Rightarrow$ (4.2). Let P be the operator given in Theorem 4.1 and j and integer such that

$$(5.1) \quad j \geq \frac{1}{2} - \operatorname{Re}(c(0)/\delta'_0(0)).$$

Let $\tilde{c}_j$ be a constant larger than $2c_j$, where c_j is given by (3.12), and $\{\rho_k\}$ a sequence belonging to $[\tilde{c}_j]$; then $P(\{\rho_k\})$, $P^0(\{\rho_k\})$, $\dots$, $P^j(\{\rho_k\})$ are true operators.

First of all we observe that $P^j(\{\rho_k\})$ verifies the hypotheses of Theorem 4.2.

Because the leading coefficients of $a^j(t, A)$, $b^j(t, A)$ are the same as those of $a(t, A)$, $b(t, A)$ by (2.2) and (2.3), we have that $P^j(\{\rho_k\})$ can be written in the form (4.4); and (4.5) will be verified.

Furthermore by (3.15) and (5.1), since $\gamma(0) = c_0^j$, (4.6) is verified; finally by Remark 3.3 and by Cauchy's inequality used in the same way as in Remark

2.1, we see that also (4.7) is verified, therefore we may conclude that $P^j(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

Now we want to prove that under the hypothesis (4.3), starting from the analytic hypoellipticity at $t = 0$ of $P^j(\{\rho_k\})$, all the $P^h(\{\rho_k\})$, $h = 0, 1, \dots, j-1$, are analytic hypoelliptic at $t = 0$, which will imply by Theorem 3.1 that $P(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$. By Remark 4.1 this will prove Theorem 4.1. We shall use Corollary 3.1, and we will have to prove that (3.21) is verified.

We will do the first step; for the others we proceed in the same way.

LEMMA 5.1. *Under the previous hypotheses (in particular, that $P^j(\{\rho_k\})$ is hypoelliptic analytic in J), there is an open neighborhood J' of 0, relatively compact, contained in J , such that if $u \in \mathcal{D}'(J', H^{-\infty})$ and*

$$(5.2) \quad Y^{j-1}(\{\rho_k\})u \in A(J', H^\omega), \quad P^{j-1}(\{\rho_k\})u \in A(J', H^\omega),$$

then $u \in A(J', H^\omega)$.

PROOF. We set $Y^{j-1}(\{\rho_k\})u = g$, $P^{j-1}(\{\rho_k\})u = h$; then by (5.2), g and $h \in A(J', H^\omega)$; therefore

$$(5.3) \quad c^{j-1}(A, \{\rho_k\})Au = X^{j-1}(\{\rho_k\})g - h \in A(J', H^\omega).$$

Let m_{j-1} be the smallest integer such that $c_{m_{j-1}}^{j-1} \neq 0$; such an integer exists, by (4.3).

Therefore we can find a number $\rho > 0$, so large that

$$(5.4) \quad \rho |c_{m_{j-1}}^{j-1}|^2 > \sup_{t \in J} |b_0(t)|.$$

Let

$$\begin{aligned} \gamma(A, \{\rho_k\}) &= \overline{c_{m_{j-1}}^{j-1}} \cdot c^{j-1}(A, \{\rho_k\})A^{m_{j-1}} \\ &= |c_{m_{j-1}}^{j-1}|^2 \rho_{m_{j-1}}(A) + \sum_{k=m_{j-1}+1}^{\infty} \overline{c_{m_{j-1}}^{j-1}} c_k^{j-1} \rho_k(A) A^{m_{j-1}-k}. \end{aligned}$$

Then

$$\begin{aligned} \lambda(A, \{\rho_k\}) &= |c_{m_{j-1}}^{j-1}|^2 + \sum_{k=m_{j-1}+1}^{\infty} \overline{c_{m_{j-1}}^{j-1}} c_k^{j-1} \rho_k(A) A^{m_{j-1}-k} \\ &= \gamma(A, \{\rho_k\}) + |c_{m_{j-1}}^{j-1}|^2 (1 - \rho_{m_{j-1}}(A)) = \gamma(A, \{\rho_k\}) + R, \end{aligned}$$

where $R = |c_{m_{j-1}}^{j-1}|^2 (1 - \rho_{m_{j-1}}(A))$.

Now it is easy to see R verifies a property analogous to (3.18), then it will satisfy property (2.12) and by the fact that $P^{j-1}(\{\rho_k\})u \in A(J', H^\omega)$, we will

get, as in the proof of Theorem 2.1, that $Ru \in A(J', H^\omega)$.

Then, by (5.3), $\rho\lambda(A, \{\rho_k\})Au \in A(J', H^\omega)$. Therefore

$$Z^{j-1}u = Y^{j-1}(\{\rho_k\})u - \rho\lambda(A, \{\rho_k\})Au \in A(J', H^\omega)$$

and by virtue of the ellipticity of Z^{j-1} , due to (5.4), we conclude that $u \in A(J', H^\omega)$, which proves the lemma.

As a consequence, by Corollary 3.1 we can conclude that $P^{j-1}(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

By repeating these arguments $j - 1$ times we reach the conclusion that $P^0(\{\rho_k\})$ and (see Theorem 3.1) $P(\{\rho_k\})$ are analytic hypoelliptic at $t = 0$.

This ends the proof of Theorem 4.1, by Remark 4.1.

6. Proof of Theorem 4.2. We follow closely the argument in [2]. Let us consider the operator

$$Q = (\partial_t - \alpha(t)tA - \tilde{\alpha}(t, a))(\partial_t - \beta(t)tA - \tilde{\beta}(t, A)) - \gamma(t)A - \tilde{\gamma}(t, A).$$

If we redefine the coefficients appropriately, we can write Q in the form

$$(6.1) \quad Q = \partial_t^2 + \alpha(t)t\partial_t A + \beta(t)t^2 A^2 + \gamma(t, A)\partial_t + \lambda(t, A)A + \mu(t, A)$$

where $\alpha(t)$, $\beta(t)$ are complex valued C^∞ functions and $\gamma(t, A)$, $\lambda(t, A)$, $\mu(t, A)$ are C^∞ functions, defined in I , with values in $L(H, H)$. There is a constant $M > 0$, such that, for every $t \in I$,

$$(6.2) \quad \{|\partial_t^h \alpha(t)|, |\partial_t^h \beta(t)|, \|\partial_t^h \gamma(t, A)\|, \|\partial_t^h \lambda(t, A)\|, \|\partial_t^h \mu(t, A)\|\} \leq M^{h+1} h!$$

where $\|\cdot\|$ is the norm in $L(H, H)$.

Because of the hypotheses (4.5) and (4.6), by Corollary II.2.1 of [1], if we shrink enough I , there is a constant $c > 0$, such that

$$(6.3) \quad \int (\|\partial_t v\|_0^2 + \|tAv\|_0^2) dt \leq c \left| \int (Qv, v)_0 dt \right|, \quad \forall v \in C_c^\infty(I, H^\infty),$$

where $(\cdot, \cdot)_0$ is the inner product of H .

If we use the norm $\|\cdot\|_{s,k}$ introduced in [4], the inequality (6.3) can be written

$$\|v\|_{1,1}^2 \leq c \left| \int (Qv, v)_0 dt \right|,$$

whence, by Schwarz' inequality,

$$\|v\|_{1,1}^2 \leq c \|Qv\|_{-1,-1} \|v\|_{1,1},$$

which implies

$$(6.4) \quad \|v\|_{1,1} \leq c \|Qv\|_{-1,-1} \quad \forall v \in C_c^\infty(I, H^\infty).$$

Therefore, as shown in [4], (6.3) implies

$$(6.5) \quad \|v\|_{2,2} \leq c \|Qv\|_{0,0} = c \left\{ \int \|Qv\|_0^2 dt \right\}^{1/2},$$

$\forall v \in C_c^\infty(I, H^\infty)$. If we use the notation $\|f\|^2 = \int \|f\|_0^2 dt$, a norm equivalent to $\| \cdot \|_{2,2}$ is given by

$$N(v)^2 = \|\partial_t^2 v\|^2 + \|t \partial_t A v\|^2 + \|t^2 A^2 v\|^2 + \|\partial_t v\|^2 + \|A v\|^2 + \|v\|^2.$$

When the integration with respect to t is performed over some interval w we write $N_w(v)$.

Let us put

$$(6.6) \quad \Gamma_p = \{(h, k) \in \mathbb{N}^2 \mid h \leq 4, h + k \leq p - h\} \\ \cup \{(h, k) \in \mathbb{N}^2 \mid h \geq 4, h + k \leq p - 4\}.$$

Note that Γ_p contains all the pairs (h, k) such that $h + k \leq p - 4$.

Now let:

$$(6.7) \quad N_\omega^p(v) = \sup_{(h,k) \in \Gamma_p} N_\omega(\partial_t^h A^k v)$$

and let $u \in \mathcal{D}'(I, H^{-\infty})$ be such that

$$(6.8) \quad Qu = f \in A(I, H^\omega).$$

We want to prove that there is an open subinterval J of I centered in 0, independent of u , such that

$$(6.9) \quad u \in A(J, H^\omega).$$

First of all (6.8) implies that

$$(6.10) \quad u \in C^\infty(I, H^\infty),$$

since Q is hypoelliptic in a suitable interval J (cf. [1, Corollary II.3.2])

Therefore, in order to prove (6.9) it suffices to show that we can choose J and a constant B such that

$$(6.11) \quad N_J^p(u) \leq B^{p+1} p! \quad \forall p \in \mathbb{N}, p \geq p_0,$$

where p_0 is a fixed integer. By the standard embedding theorems, (6.11) will imply

$$\sup_J \|\partial_t^h A^k u\|_0 \leq C B_1^{h+k} (h+k)!$$

where B and C are suitable constants.

Possibly after shrinking I , we may suppose that, for some $c > 0$,

$$(6.12) \quad \sup_I \|\partial_t^h A^k Qu\|_0 \leq C^{h+k+1} (h+k)!.$$

Let $\omega =]-2, 2[$, we can always suppose that ω is contained in I and put $\omega_\delta =]-2 + \delta, 2 - \delta[$ with $0 \leq \delta < 2$.

Let ϵ and ϵ_1 be two real numbers, $0 < \epsilon, \epsilon_1 < 1$, and $\rho(t) \in C_c^\infty(\omega_{\epsilon_1})$ such that $\rho(t) = 1, \forall t \in \omega_{\epsilon+\epsilon_1}$, and $|\partial_t^\alpha \rho| \leq C_1 \epsilon^{-\alpha}, 0 \leq \alpha \leq 2$. By (6.5), if $p \geq \alpha$ we have that

$$\begin{aligned} N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) &\leq N(\rho \partial_t^h A^k u) \leq C \|Q \rho \partial_t^h A^k u\| \\ &\leq C \{ \|\rho \partial_t^h A^k Q u\| + \|[Q, \rho] \partial_t^h A^k u\| + \|\rho [Q, \partial_t^h A^k] u\| \}, \end{aligned}$$

$$\forall (h, k) \in \Gamma_p.$$

From now on, C will denote a generic constant independent of h, k, p, ϵ and ϵ_1 .

Now, by (6.12), we have

$$N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) \leq C \{ C^p p! + \|[Q, \rho] \partial_t^h A^k u\| + \|\rho [Q, \partial_t^h A^k] u\| \}.$$

LEMMA 6.1. *Under the preceding hypotheses,*

$$\|[Q, \rho] \partial_t^h A^k u\| \leq C \{ \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u) + \epsilon^{-2} N_{\omega_{\epsilon_1}}^{p-2}(u) \}.$$

PROOF.

$$\begin{aligned} \|[Q, \rho] \partial_t^h A^k u\| &\leq 2 \|\rho' \partial_t \partial_t^h A^k u\| + \|\rho'' \partial_t^h A^k u\| \\ &\quad + \|\alpha(t) \rho' t \partial_t^h A^{k+1} u\| + \|\gamma(t, A) \rho' \partial_t^h A^k u\|. \end{aligned}$$

Let us estimate $\|\rho' \partial_t \partial_t^h A^k u\|$. Suppose $k \geq 1$; since:

$$(6.13) \quad |\rho'/t| \leq C \epsilon^{-1},$$

we have

$$\begin{aligned} \|\rho' \partial_t \partial_t^h A^k u\| &\leq C \epsilon^{-1} \|t \partial_t \partial_t^h A^k u\|_{\omega_{\epsilon_1}} \\ &\leq C \epsilon^{-1} \|t \partial_t A(\partial_t^h A^{k-1} u)\|_{\omega_{\epsilon_1}} \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}(\partial_t^h A^{k-1} u). \end{aligned}$$

If $h \geq 4, h+k \leq p-4 \Rightarrow h+k-1 \leq p-1-4 \Rightarrow (h, k-1) \in \Gamma_{p-1}$; analogously, if $h \leq 4, (h, k-1) \in \Gamma_{p-1}$, then $\|\rho' \partial_t \partial_t^h A^k u\| \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u)$.

Suppose $k = 0$. Then we can suppose $h \geq 1$ and

$$\|\rho' \partial_t \partial_t^h u\| \leq \|\rho' \partial_t \partial_t^{h-1} u\| \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}(\partial_t^{h-1} u) \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u),$$

because, as before, we have $(h-1, 0) \in \Gamma_{p-1}$. Then in general we have $\|\rho' \partial_t \partial_t^h A^k u\| \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u)$.

Let us estimate $\|\alpha(t) \rho' t \partial_t^h A^{k+1} u\|$. Suppose $k \geq 1$; by (6.13) we have

$$\begin{aligned}\|\rho' t \partial_t^h A^{k+1} u\| &\leq C \epsilon^{-1} \|t^2 A^2 (\partial_t^h A^{k+1} u)\|_{\omega_{\epsilon_1}} \\ &\leq C \epsilon^{-1} N_{\omega_{\epsilon_1}} (\partial_t^h A^{k-1} u) \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u).\end{aligned}$$

If $k = 0$ we can suppose $h \geq 1$; then

$$\begin{aligned}\|\rho' t \partial_t^h A u\| &\leq C \epsilon^{-1} \|t \partial_t A (\partial_t^{h-1} u)\|_{\omega_{\epsilon_1}} \\ &\leq C \epsilon^{-1} N_{\omega_{\epsilon_1}} (\partial_t^{h-1} u) \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u).\end{aligned}$$

Then in general $\|\alpha(t) \rho' t \partial_t^h A^{k+1} u\| \leq C \epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u)$.

If we consider $\|\rho'' \partial_t^h A^k u\|$, $\|\gamma(t, A) \rho' \partial_t^h A^k u\|$, we have that both of them are $\leq C \epsilon^{-2} \|\partial_t^h A^k u\|_{\omega_{\epsilon_1}}$.

Now if $h \geq 2$,

$$\|\partial_t^h A^k u\|_{\omega_{\epsilon_1}} = \|\partial_t^2 (\partial_t^{h-2} A^k u)\|_{\omega_{\epsilon_1}} \leq N_{\omega_{\epsilon_1}} (\partial_t^{h-2} A^k u) \leq N_{\omega_{\epsilon_1}}^{p-2}(u);$$

if $h = 1$,

$$\|\partial_t A^k u\|_{\omega_{\epsilon_1}} \leq N_{\omega_{\epsilon_1}} (A^k u) \leq N_{\omega_{\epsilon_1}}^{p-2}(u),$$

because $1 + k \leq p - 1 \Rightarrow k \leq p - 2 \Rightarrow (0, k) \in \Gamma_{p-2}$.

If $h = 0$, we can suppose $k \geq 2$; then, since $|\rho''/t^2| \leq c \epsilon^{-2}$, and by (6.13),

$$\begin{aligned}\|\rho'' A^k u\|, \|\gamma(t, A) \rho' A^k u\| &\leq C \epsilon^{-2} \|t^2 A^k u\|_{\omega_{\epsilon_1}} \\ &\leq c \epsilon^{-2} N_{\omega_{\epsilon_1}} (A^{k-2} u) \leq c \epsilon^{-2} N_{\omega_{\epsilon_1}}^{p-2}(u)\end{aligned}$$

then in general

$$\|\rho'' \partial_t^h A^k u\|, \|\gamma(t, A) \rho' \partial_t^h A^k u\| \leq c \epsilon^{-2} N_{\omega_{\epsilon_1}}^{p-2}(u),$$

which proves the lemma.

LEMMA 6.2. *Same hypotheses as in Lemma 6.1.*

$$(6.14) \quad \|\rho[Q, \partial_t^h A^k] u\| \leq C \sum_{k=1}^p C^k \frac{p!}{(p-k)!} N_{\omega_{\epsilon_1}}^{p-k}(u).$$

PROOF.

$$\begin{aligned}(6.15) \quad \|\rho[Q, \partial_t^h A^k] u\| &= \|\rho[Q, \partial_t^h] A^k u\| \\ &\leq \|\rho[\alpha(t) t, \partial_t^h] \partial_t A^{k+1} u\| \\ &\quad + \|\rho[\beta(t) t^2 \partial_t^h] A^{k+2} u\| + \|\rho[\gamma(t, A), \partial_t^h] \partial_t A^k u\| \\ &\quad + \|\rho[\lambda(t, A), \partial_t^h] A^{k+1} u\| + \|\rho[\mu(t, A), \partial_t^h] A^k u\|.\end{aligned}$$

We may suppose $h \geq 1$. We have, first,

$$\begin{aligned} & \|\rho[\alpha(t)t, \partial_t^h] \partial_t A^{k+1} u\| \\ & \leq \sum_{\lambda+\nu=h; \nu < \alpha} \frac{h!}{\lambda! \nu!} \|(\partial_t^\lambda \alpha) \rho t \partial_t A (\partial_t^\nu A^k u)\| \\ & \quad + \sum_{\lambda+\nu+1=h} \frac{h!}{\lambda! \nu!} \|(\partial_t^\lambda \alpha) \rho (\partial_t^{\nu+1} A^{k+1} u)\|. \end{aligned}$$

By (6.2), it suffices to prove that

$$\begin{aligned} & \|\rho t \partial_t A (\partial_t A^k u)\| \leq CN_{\omega_{\epsilon_1}}^{p-\lambda}(u), \\ (6.16) \quad & \|\rho \partial_t^{\nu+1} A^{k+1} u\| \leq CN_{\omega_{\epsilon_1}}^{p-\lambda-1}(u). \end{aligned}$$

Suppose $h \geq 4$. For the first inequality (6.16), with $(h, k) \in \Gamma_p$, we have $\lambda + \nu + k \leq p - 4 \Rightarrow p - \lambda - 4 \Rightarrow (\nu, k) \in \Gamma_{p-\lambda}$. Consequently,

$$\|\rho t \partial_t A (\partial_t^\nu A^k u)\| \leq CN_{\omega_{\epsilon_1}} (\partial_t^\nu A^k u) \leq CN_{\omega_{\epsilon_1}}^{p-\lambda}(u).$$

As for the second inequality (6.16) we have (1) if $\nu \geq 1$, $\lambda + \nu + 1 + k \leq p - 4 \Rightarrow \nu - 1 + k + 1 \leq (p - \lambda - 1) - 4 \Rightarrow (\nu - 1, k + 1) \in \Gamma_{p-\lambda-1}$, and therefore

$$\|\rho (\partial_t^{\nu+1} A^{k+1} u)\| \leq CN_{\omega_{\epsilon_1}} (\partial_t^{\nu-1} A^{k+1} u) \leq CN_{\omega_{\epsilon_1}}^{p-\lambda-1}(u);$$

(2) if $\nu = 0$, $\lambda + 1 + k \leq p - 4 \Rightarrow 1 + k \leq p - \lambda - 4 \leq (p - \lambda - 1) - 4 \Rightarrow (0, k + 1) \in \Gamma_{p-\lambda-1}$, and

$$\|\rho \partial_t A^{k+1} u\| \leq CN_{\omega_{\epsilon_1}} (A^{k+1} u) \leq CN_{\omega_{\epsilon_1}}^{p-\lambda-1}(u).$$

Similar argument when $h \leq 4$. As a consequence of (6.16) we see that $\|\rho[\alpha(t)t, \partial_t^h] \partial_t A u\|$ verifies the estimate (6.14).

We operate in the same way with $\|\rho[\beta(t)t^2, \partial_t^h] A^{k+2} u\|$.

Next we see, by (6.2), that

$$\begin{aligned} & \|\rho[\gamma(t, A), \partial_t^h] \partial_t A^k u\| \leq \sum_{\nu < h} \binom{h}{\nu} \|\rho \partial_t^{h-\nu} \gamma(t, A) \cdot \partial_t^{\nu+1} A^k u\| \\ & \leq \sum_{\nu < h} \binom{h}{\nu} c^{h-\nu+1} (h-\nu)! \|\rho \partial_t (\partial_t^\nu A^k u)\| \\ & \leq \sum_{\nu < h} c^{h-\nu+1} \frac{h!}{\nu!} N_{\omega_{\epsilon_1}} (\partial_t^\nu A^k u). \end{aligned}$$

But if $h \geq 4$, $h + k \leq p - 4 \Rightarrow \nu + k \leq p - (h - \nu) - 4 \Rightarrow (\nu, k) \in \Gamma_{p-(h-\nu)}$.
If $h \leq 4$, $h + k \leq p - h \Rightarrow \nu + k \leq p - (h - \nu) - h \leq p - (h - \nu) - \nu \Rightarrow$

$(\nu, k) \in \Gamma_{p-(h-\nu)}$. Therefore

$$\|\rho[\gamma(t, A), \partial_t^h] \partial_t A^k u\| \leq \sum_{\nu < h} c^{h-\nu+1} \frac{h!}{\nu!} N_{\omega_{\epsilon_1}}^{p-(h-\nu)}(u).$$

We obtain a similar estimate for $\|\rho[\lambda(t, A), \partial_t^h] A^{k-1} u\|$ and $\|\rho[\mu(t, A), \partial_t^h] A^k u\|$ and the lemma is thus proved.

By Lemmas 6.1 and 6.2, we get that there is a constant $c > 0$ such that, $\forall p \in N, p \geq 4, \forall (h, k) \in \Gamma_p$,

$$N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) \leq C \left\{ C^p p! + \sum_{k=1}^p C^k \frac{p!}{(p-k)!} N_{\omega_{\epsilon_1}}^{p-k}(u) + \sum_{k=1}^2 \epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u) \right\},$$

whence:

$$(6.17) \quad N_{\omega_{\epsilon+\epsilon_1}}^p(u) \leq C \left\{ C^p p! + \sum_{k=1}^p C^k \frac{\epsilon^k p!}{(p-k)!} \epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u) + \sum_{k=1}^2 \epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u) \right\}.$$

Let l be an integer such that $l \geq p+4, \epsilon = 1/l, \epsilon_1 = p/l$ and put

$$d_p = N_{\omega_{(p+1)/l}}^p(u) l^{-p-4}.$$

The inequality (6.17) can be rewritten

$$d_p \leq C \left\{ C^p + \sum_{k=1}^p C^k d_{p-k} + \sum_{k=1}^2 d_{p-k} \right\}$$

for every integer $p \geq 4, p \leq l-4$, which easily implies, for some $B > 0$,

$$(6.18) \quad d_p \leq B^{p+1} \quad \forall p \leq l-4.$$

In turn (6.18) implies, for all $l \geq 4$,

$$N_{\omega_1}^{l-4}(u) \leq B^{l-3} l^l.$$

Finally, after an increase of the constant B we get (6.11) with $J = \omega_1$.

REFERENCES

1. F. Trèves, *Concatenations of second order evolution equation applied to local solvability and hypoellipticity*, Comm. Pure Appl. Math. **26** (1973), 201–250.
2. V. V. Grušin, *On a class of elliptic pseudodifferential operators that are degenerate*

on a submanifold, Mat. Sb. 84 (126) (1971), 163–195 = Math USSR Sb. 13 (1971), 155–186. MR 44 #860.

3. F. Trèves, *Ovcyannicov theorem and hyperdifferential operators*, Notas de Matemática, no. 46, Instituto de Matematica Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro, 1968. MR 44 #7386.

4. L. Boutet De Monvel and F. Trèves, *On a class of pseudodifferential operators with double characteristics*, Invent. Math. 24 (1974), 1–34.

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