HYPERFINITE EXTENSIONS OF BOUNDED OPERATORS ON A SEPARABLE HILBERT SPACE⁽¹⁾

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ABSTRACT. Let H be a separable Hilbert space and $\hat{H}$ the nonstandard hull of H with respect to an $\aleph_1$ -saturated enlargement. Let S be a *-finite dimensional subspace of *H such that the corresponding hyperfinite dimensional subspace $\hat{S}$ of $\hat{H}$ contains H . If T is a bounded operator on H , then an extension $\hat{A}$ of T to $\hat{S}$ where $\hat{A}$ is obtained from an internal *-linear operator on S is called a *hyperfinite extension* of T . It is shown that T has a compact (selfadjoint) hyperfinite extension if and only if T is compact (selfadjoint). However T has a normal hyperfinite extension if and only if T is subnormal. The spectrum of a hyperfinite extension $\hat{A}$ equals the point spectrum of $\hat{A}$, and if T is quasitriangular, A can be chosen so that the spectrum of $\hat{A}$ equals the spectrum of T . A simple proof of the spectral theorem for bounded selfadjoint operators is given using a hyperfinite extension.

1. **Preliminaries.** The nonstandard hull of a normed space was introduced by Luxemburg in [9]. The properties of these spaces have been investigated by Henson, Cozart, and the author in [6], [7], [2], and [8]. We review the relevant definitions here.

Let $(E, \|\cdot\|)$ be a normed space and let *M be an $\aleph_1$ -saturated enlargement [9] of a set theoretical structure M which contains $(E, \|\cdot\|)$. An element $p \in {}^*E$ is called *finite* if $\|p\|$ is a finite element of ${}^*\mathbb{R}$ and the set of finite elements of *E is denoted by $\text{fin}({}^*E)$. The monad of 0 is defined by

$$\mu(0) = \{p: p \in {}^*E \text{ and } \|p\| \text{ is infinitesimal}\}.$$

Both $\text{fin}({}^*E)$ and $\mu(0)$ are vector spaces over the same field as E . We denote the quotient vector space $\text{fin}({}^*E)/\mu(0)$ by $\hat{E}$ and the canonical quotient mapping of $\text{fin}({}^*E)$ onto $\hat{E}$ by π . A norm may be defined on $\hat{E}$ by letting $\|\pi(p)\|$ be $\text{st } \|p\|$, where st is the standard part operator on ${}^*\mathbb{R}$. The normed space $(\hat{E},$

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$\|\cdot\cdot\cdot\|$) is called the *nonstandard hull* of $(E, \|\cdot\cdot\cdot\|)$ (with respect to *M).

It is shown in [9] that under the assumption of $\aleph_1$ -saturation, $(\hat{E}, \|\cdot\cdot\cdot\|)$ is a Banach space. Further $(E, \|\cdot\cdot\cdot\|)$ may be considered as a subspace of $(\hat{E}, \|\cdot\cdot\cdot\|)$ by identifying $x \in E$ with $\pi(*x)$. In the case that the original space is a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, the nonstandard hull $\hat{H}$ is also a Hilbert space with the extended inner product $\langle \pi(p), \pi(q) \rangle = \text{st}(\langle *p, *q \rangle)$.

If S is a $*$ -finite dimensional subspace of *E , then $\hat{S} = \{\pi(p) : p \in S \cap \text{fin}({}^*E)\}$ is called a *hyperfinite dimensional subspace* of E [7]. We are interested in hyperfinite dimensional subspaces $\hat{S}$ such that $\hat{S} \supseteq E$. It is easy to show that since *M is an enlargement, such hyperfinite dimensional subspaces exist. If H is a separable Hilbert space and $\{e_1, e_2, \dots\}$ is an orthonormal basis, let $S = {}^*\text{-span}\{*e_1, *e_2, \dots, *e_\omega\}$ where ω is an infinite positive integer. It is easy to verify that $\hat{S}$ is a hyperfinite dimensional subspace containing H .

Now let H be a separable Hilbert space. Let C denote the field of complex numbers, Z the set of integers, and N the set of positive integers. Let st denote the standard part operator on *C and if $\alpha, \beta \in {}^*C$ and $\alpha - \beta$ is infinitesimal, we write $\alpha =_1 \beta$. If $p, q \in {}^*H$ and $\|*p - *q\| =_1 0$, we write $p \sim q$. Similarly if A and D are $*$ -bounded operators on *H , we write $A \sim D$ if $\|*A - *D\| =_1 0$. Let B denote the closed unit ball in H . Finally all projections considered on H are assumed to be orthogonal, and in order to avoid confusion, the adjoint of an operator T is denoted by $T^\#$.

Let S be a $*$ -finite dimensional subspace of *H such that $\hat{S} \supseteq H$ and let $\mathfrak{A}(S)$ be the algebra of $*$ -linear operators A on S such that $\|*A\|$ is a finite element of *R . If $A \in \mathfrak{A}(S)$, it is easy to verify that A induces a bounded operator $\hat{A}$ on $\hat{S}$ by setting

$$A(\pi(p)) = \pi(A(p)).$$

The following result is elementary and the proof is left to the reader.

LEMMA 1.1. *Let S be a $*$ -finite dimensional subspace of *H such that $\hat{S} \supseteq H$, let $A, D \in \mathfrak{A}(S)$, and $\lambda \in \text{fin}({}^*C)$. Then*

- (i) $\|\hat{A}\| = \text{st}(\|*A\|)$,
- (ii) $\widehat{A + D} = \hat{A} + \hat{D}$,
- (iii) $\widehat{\lambda A} = \text{st}(\lambda)\hat{A}$,
- (iv) $\widehat{AD} = \hat{A}\hat{D}$,
- (v) $(\hat{A})^\# = (\hat{A}^\#)$,
- (vi) *if A is a $*$ -orthogonal projection on S , then $\hat{A}$ is an orthogonal projection on $\hat{S}$.*

Now let T be a bounded operator on H .

DEFINITION 1.2. Let S be a $*$ -finite dimensional subspace of *H such that

$\hat{S} \supseteq H$ and let $A \in \mathfrak{A}(S)$. If $\hat{A}$ restricted to H is T , then $\hat{A}$ is called a *hyperfinite extension* of T .

Let S be a $*$ -finite dimensional subspace of $*H$ such that $\hat{S} \supseteq H$ and let P_S be the $*$ -orthogonal projection of $*H$ onto S . Denote the restriction of $P_S * T$ to S by $*T_S$. Since $\|P_S\| = 1$, we have $\|*T_S\| \leq \|P_S * T\| \leq \|*T\| = \|T\|$, so $*T_S \in \mathfrak{A}(S)$. We denote the operator $\widehat{*T_S}$ on $\hat{S}$ by $\hat{T}_S$.

LEMMA 1.3. *If S is a hyperfinite dimensional subspace of $\hat{H}$ such that $\hat{S} \supseteq H$, then $\hat{T}_S$ is a hyperfinite extension of T and $\|\hat{T}_S\| = \|T\|$.*

PROOF. Let $x \in H$. Then by assumption there exist $p, q \in S$ such that $p \sim *x$ and $q \sim *(Tx)$. Then $*Tp \sim *T(*x) = *(Tx) \sim q$ and $*P_S * T(p) \sim *P_S q = q$. Thus $\hat{T}_S(x) = \hat{T}_S(\pi(p)) = \pi(*T_S p) = \pi(q) = Tx$ and $\hat{T}_S$ extends T . By the comments above $\|\hat{T}_S\| \leq \|T\|$ and since $\hat{T}_S$ extends T we have $\|T\| \leq \|\hat{T}_S\|$. Thus $\|T\| = \|\hat{T}_S\|$.

DEFINITION 1.4. Let S be a hyperfinite dimensional subspace of H such that $\hat{S} \supseteq H$. Then $\hat{T}_S$ is called the *standard hyperfinite extension* of T with respect to S .

2. Selfadjoint, compact, and normal operators.

LEMMA 2.1. $(\hat{T}_S)^\# = \widehat{(T^\#)_S}$ for any bounded operator T and any $*$ -finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$.

PROOF. The operator $*T_S$ is the restriction of $P_S * TP_S$ to S and $(*T_S)^\#$ is the restriction of $P_S (*T)^\# P_S = P_S *(T^\#) P_S$ to S . Thus $(*T_S)^\# = *(T^\#)_S$ and the result follows from Lemma 1.1(v).

LEMMA 2.2. *The following conditions are equivalent:*

- (i) T is selfadjoint,
- (ii) some hyperfinite extension of T is selfadjoint,
- (iii) every standard hyperfinite extension of T is selfadjoint.

PROOF. Clearly (iii) implies (ii) and since the restriction of a selfadjoint operator to an invariant subspace is again selfadjoint, (ii) implies (i). Finally if T is selfadjoint, then Lemma 2.1 implies that every standard hyperfinite extension is again selfadjoint, i.e., (i) implies (iii).

LEMMA 2.3. *The following conditions are equivalent:*

- (i) T is compact,
- (ii) some hyperfinite extension of T is compact,
- (iii) every standard hyperfinite extension of T is compact.

PROOF. Clearly (iii) implies (ii) and since the restriction of a compact operator to an invariant subspace is again compact, (ii) implies (i). To show

that (i) implies (iii) assume T is compact and $\hat{S}$ is a hyperfinite dimensional subspace of $\hat{H}$ containing H . Let $\epsilon > 0$. Then there exists a finite set $\{x_1, x_2, \dots, x_n\}$ in H such that $T(B) \subseteq \bigcup_{i=1}^n \{x \in H: \|x - x_i\| < \epsilon\}$. Hence $*T(*B) \subseteq \bigcup_{i=1}^n \{p \in *H: \|p - *x_i\| < \epsilon\}$ and it follows that $\hat{T}_S(\hat{B}_S) \subseteq \bigcup_{i=1}^n \{x \in \hat{S}: \|x - x_i\| \leq \epsilon\}$ where $\hat{B}_S$ is the closed unit ball in $\hat{S}$. Thus $\hat{T}_S$ is compact.

Let ω be an infinite positive integer, then the (external) cardinality of the set $\{1, 2, \dots, \omega\}$ is always greater than or equal to c . To see this note that for each $r \in [0, 1]$ there exists $k_r \in \{1, 2, \dots, \omega\}$ such that $\text{st}(k_r/\omega) = r$. We use this observation in the proof of

THEOREM 2.4. *Let $\hat{S}$ be a hyperfinite dimensional subspace of $\hat{H}$ containing H .*

- (i) *If $A \in \mathfrak{A}(S)$, then $\hat{A}$ is compact if and only if $\hat{A}(\hat{S})$ is separable.*
- (ii) *T is compact if and only if $\hat{T}_S(\hat{S}) \subseteq H$.*

PROOF. (i) Clearly if $\hat{A}$ is compact then $\hat{A}(\hat{S})$ is separable.

On the other hand if $\hat{A}$ is not compact then for some $\epsilon > 0$ there exists a sequence $\{x_n\}$ in $\hat{S}$ such that $\|x_n\| \leq 1$ for all n and $\|\hat{A}x_n - \hat{A}x_m\| > \epsilon$ for all $n \neq m$. For each n pick $p_n \in S \cap *B$ such that $\pi(p_n) = x_n$ and using the $\aleph_1$ -saturation extend this to an internal sequence defined on $*N$. Now there exists an infinite positive integer ω such that $p_k \in *B \cap S$ for all $k \leq \omega$ and $\|Ap_k - Ap_j\| > \epsilon$ for all $1 \leq j < k \leq \omega$. Thus $\{\hat{A}(\pi(p_k)): 1 \leq k \leq \omega\}$ is a set in $\hat{A}(\hat{S})$ with cardinality the same as the cardinality of $\{1, 2, \dots, \omega\}$ and such that every pair of elements is at least ϵ apart. Since $\{1, 2, \dots, \omega\}$ is not countable, this implies $\hat{A}(\hat{S})$ is not separable.

(ii) If $\hat{T}_S(\hat{S}) \subseteq H$, then $\hat{T}_S$ is compact by (i) above and T is compact by Lemma 2.3. On the other hand if T is compact, then $p \in \text{fin}(*H)$ implies that for some $x \in H$ we have $*Tp \sim *x$ [10]. It follows easily that $\hat{T}_S(\hat{S}) \subseteq H$.

The situation for normal operators is more complicated. We recall a few definitions.

DEFINITION 2.5. (i) T is said to be *quasidiagonal* if there is an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \rightarrow 1$ (strongly) and $\|TE_n - E_nT\| \rightarrow 0$.

(ii) T is said to be *quasitriangular* if there is an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \rightarrow 1$ (strongly) and $\|TE_n - E_nTE_n\| \rightarrow 0$.

Both definitions are due to Halmos ([4] and [5]). Note that every quasidiagonal operator is quasitriangular. Recall that a closed subspace of H is said to be reducing for T if it is invariant for both T and $T^\#$. Thus T is quasidiagonal (quasitriangular) if and only if there are sufficiently many "almost reducing" ("almost invariant") finite dimensional subspaces. It is easy to show [5] that

compact operators and normal operators are quasidiagonal.

The following theorem is proved by Halmos [4] in the quasitriangular case. The proof in the quasidiagonal case is analogous.

THEOREM 2.6 (HALMOS). (i) T is quasidiagonal if and only if there is a sequence $\{E_n\}$ of projections of finite rank such that $E_n \rightarrow 1$ (strongly) and $\|TE_n - E_nT\| \rightarrow 0$.

(ii) T is quasitriangular if and only if there is a sequence $\{E_n\}$ of projections of finite rank such that $E_n \rightarrow 1$ (strongly) and $\|TE_n - E_nTE_n\| \rightarrow 0$.

In other words both definitions in 2.4 are unchanged if the requirement that the sequence of projections be increasing is dropped. The following lemma is a simple consequence of Theorem 2.5.

LEMMA 2.7. (i) T is quasidiagonal if and only if there exists a *-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*T$.

(ii) T is quasitriangular if and only if there exists a *-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*TP_S$.

LEMMA 2.8. T has a normal standard hyperfinite extension if and only if T is normal.

PROOF. Assume that $\hat{T}_S$ is a normal standard hyperfinite extension. Since $(\hat{T}_S)^\# = (\hat{T}^\#)_S$ both $\hat{T}_S$ and $(\hat{T}_S)^\#$ leave H invariant. Thus H is a reducing subspace for $\hat{T}_S$ and since T is the restriction of $\hat{T}_S$ to H , T is also normal.

Assume T is normal. Then T is quasidiagonal and by Lemma 2.7 there exists a *-finite dimensional subspace S of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S*T$. Now $(\hat{T}_S)^\# = (\hat{T}^\#)_S$ by Lemma 2.1, so $\hat{T}_S(\hat{T}_S)^\# = \hat{T}_S(\hat{T}^\#)_S = (*T_S*(T^\#)_S)^\#$. Similarly $(\hat{T}_S)^\# \hat{T}_S = (*T^\#)_S(*T_S)^\#$. Thus in order to show that $\hat{T}_S$ is normal, it is sufficient to show that $(P_S*TP_S)(P_S*(T^\#)_S) \sim (P_S*(T^\#)_S)(P_S*TP_S)$. But since $*TP_S \sim P_S*T$, we have $(P_S*TP_S)(P_S*(T^\#)_S) \sim P_S*T*(T^\#)_S = P_S*(TT^\#)_S = P_S*(T^\#T)_S \sim (P_S*(T^\#)_S)(P_S*TP_S)$.

EXAMPLE 2.9. Even if T is normal it does not follow that every standard hyperfinite extension of T is normal. For example let T be the two-sided shift, i.e., T is defined on $l_2(Z)$ and $Te_k = e_{k+1}$ where $\{e_k: k \in Z\}$ is the usual orthonormal basis on $l_2(Z)$. Pick ω an infinite positive integer and let $S = *$ -span of $\{e_k: -\omega \leq k \leq \omega\}$. Then $*T(e_\omega) = e_{\omega+1}$, so $*T_S(e_\omega) = 0$ and $(T^\#)_S*T_S(e_\omega) = 0$. But $*T_S*(T^\#)_S(e_\omega) = *T_S(e_{\omega-1}) = e_\omega$. Thus $(\hat{T}_S)^\# \hat{T}_S(\pi(e_\omega)) \neq 0$ and $\hat{T}_S(\hat{T}_S)^\#(\pi(e_\omega)) = 0$, hence $\hat{T}_S$ is not normal.

Recall that an operator T is said to be *subnormal* if it has a normal extension. It is easy to verify that an operator on a separable Hilbert space is subnormal if and only if it has a normal extension to a separable Hilbert space.

THEOREM 2.10. *T has a normal hyperfinite extension if and only if T is subnormal.*

PROOF. Assume T is subnormal. Then we may assume H is a subspace of a separable Hilbert space K and that T has a normal extension Q on K . Further since K is separable we may assume $K \in \mathcal{M}$. Since K is normal it is quasidiagonal, so there is a *-finite dimensional subspace V of $*K$ such that $\hat{V} \supseteq K$ and $*QP_V \sim P_V*Q$.

Since both H and K are separable, there is an isometry Φ of K onto H . Let S be the image of V under $*\Phi$. Then S is a *-finite dimensional subspace of $*H$. We assert that $\hat{S} \supseteq H$. To see this let $x \in H$ and let $z = \Phi^{-1}x$. Since $\hat{V} \supseteq K$ there exists $q \in V$ such that $q \sim *z$. It follows that $\Phi q \sim \Phi *z = *(\Phi z) = *x$, and so $\hat{S} \supseteq H$.

Next let $\{e_k: k \in N\}$ be a fixed orthonormal basis in H . For each $k \in N$, pick $p_k \in S$ and $q_k \in V$ such that $p_k \sim *e_k$ and $q_k \sim *e_k$. For $k, j \in N$ with $k \neq j$, we have $*\langle p_k, p_j \rangle =_1 *\langle *e_k, *e_j \rangle = 0$, thus by applying the Gram-Schmidt process to $\{p_k: k \in N\}$ we may assume $*\langle p_k, p_j \rangle = 0$ for $k \neq j$. Similarly we may assume $*\langle q_k, q_j \rangle = 0$ for $k, j \in N$ and $k \neq j$.

Now for each $n \in N$ let A_n be the set of all *-linear mappings θ of V onto S such that $\theta q_k = p_k$ for $k = 1, 2, \dots, n$ and $1 - 1/n \leq *\|\theta\| \leq 1 + 1/n$. Clearly each set A_n is nonempty and internal. Hence by $\aleph_1$ -saturation there exists $\Psi \in \bigcap_{n \in N} A_n$; so Ψ is a *-linear mapping of V onto S such that $\Psi(q_k) = p_k$ for all $k \in N$ and $*\|\Psi\| =_1 1$. Note that if q and $q' \in V \cap \text{fin}(*K)$ then it follows from the parallelogram identity that $*\langle q, q' \rangle =_1 *\langle \Psi q, \Psi q' \rangle$.

We define a mapping A of S into S by $Ap = \Psi P_V *Q \Psi^{-1} p$. Clearly A is *-linear and $*\|A\| \leq *\|Q\| + \epsilon$ for any real positive ϵ . Thus $A \in \mathfrak{A}(S)$. In order to show that $\hat{A}$ extends T we have to show that if $x \in H$ and $q \in V$ with $q \sim *x$, then $\Psi q \sim *x$. So assume $x = \sum_{k \in N} x_k e_k \in H$ and $q \in V$ such that $*x \sim q$. Then for every $\epsilon > 0$ there exists n such that $\|x - \sum_{k=1}^n x_k e_k\| = \alpha < \epsilon$. Hence $*\|*x - \sum_{k=1}^n x_k *e_k\| = \alpha$ and since $*e_k \sim q_k$ for each k and $q \sim *x$, we have $*\|q - \sum_{k=1}^n x_k q_k\| =_1 \alpha$, so $*\|\Psi q - \sum_{k=1}^n x_k p_k\| =_1 \alpha$. Then $*\|\Psi q - \sum_{k=1}^n x_k e_k\| =_1 \alpha$ and it follows that $*\|\Psi q - *x\| < 2\epsilon$ for every real positive ϵ . Hence $\Psi q \sim *x$.

Now we show that $\hat{A}$ extends T . In order to do this it is sufficient to show that $\hat{A}e_k = Te_k$ for all $k \in N$. If $k \in N$, then $\hat{A}e_k = \hat{A}\pi(p_k) = \pi(Ap_k)$. But $\Psi^{-1}(p_k) = q_k$ and $*Q(q_k) \sim *Q(*e_k) = *(Te_k)$, so $P_V *Q q_k \sim *(Te_k)$. By the observation above $Ap_k = \Psi P_V *Q q_k \sim *(Te_k)$ and $\hat{A}(e_k) = Te_k$.

It remains to show that $\hat{A}$ is normal. Define D on S by $Dp = \Psi P_V *Q^{\#} \Psi^{-1} p$. We assert that $D \sim A^{\#}$. To see this let $p, p' \in S$ such that $*\|p\| = *\|p'\| = 1$. Then

$$\begin{aligned}
*\langle Ap, p' \rangle &= *\langle \Psi P_V *Q \Psi^{-1} p, p' \rangle = {}_1 * \langle P_V *Q \Psi^{-1} p, \Psi^{-1} p' \rangle \\
&= *\langle *Q \Psi^{-1} p, \Psi^{-1} p' \rangle = *\langle \Psi^{-1} p, *Q^{\#} \Psi^{-1} p' \rangle \\
&= *\langle \Psi^{-1} p, P_V *Q^{\#} \Psi^{-1} p' \rangle = {}_1 * \langle p, \Psi P_V *Q^{\#} \Psi^{-1} p' \rangle = *\langle p, Dp' \rangle.
\end{aligned}$$

Hence $D \sim A^{\#}$.

Finally we show that $\hat{A}$ is normal. Since $\hat{A}^{\#} = \widehat{A^{\#}} = \hat{D}$, it is sufficient to show that for $p \in \text{fin}(*H) \cap S$ we have $ADp \sim DAp$. If $p \in \text{fin}(*H) \cap S$, then $ADp = (\Psi P_V *Q \Psi^{-1})(\Psi P_V *Q^{\#} \Psi^{-1} p) = \Psi P_V *Q P_V *Q^{\#} \Psi^{-1} p$. Since $P_V *Q \sim *Q P_V$ we have $ADp \sim \Psi P_V *Q *Q^{\#} \Psi^{-1} p$ and since Q is normal $ADp \sim \Psi P_V *Q^{\#} *Q \Psi^{-1} p$. But $\Psi^{-1} p = P_V \Psi^{-1} p$ and $P_V *Q \sim *Q P_V$, hence

$$\begin{aligned}
ADp &\sim \Psi P_V *Q^{\#} P_V *Q \Psi^{-1} p \\
&= \Psi P_V *Q^{\#} \Psi^{-1} \Psi P_V *Q \Psi^{-1} p = DAp.
\end{aligned}$$

This ends the proof of Theorem 2.10.

3. Spectrum of hyperfinite operators. We introduce the following notation. If Q is an operator on a Hilbert space H , then $\Lambda(Q)$ is the *spectrum* of Q , $\Pi_0(Q)$ is the *point spectrum* of Q (eigenvalues of Q), and $\Pi(Q)$ is the *approximate point spectrum* of Q ($\lambda \in \Pi(Q)$ if for every $\epsilon > 0$ there exists $x \in H$ such that $\|x\| = 1$ and $\|Qx - \lambda x\| < \epsilon$). Note that if $\lambda \in \Lambda(Q) \setminus \Pi(Q)$ then the range of $Q - \lambda I$ is a proper closed subspace of H .

THEOREM 3.1. *Let S be a *-finite dimensional subspace of $*H$ such that $\hat{S} \supseteq H$ and let $A \in \mathfrak{A}(S)$. Then $\Lambda(\hat{A}) = \Pi_0(\hat{A})$.*

PROOF. First assume $\lambda \in \Pi(\hat{A})$. Then for every $n \in N$ there exists $p_n \in S$ such that $*\|p_n\| = 1$ and $*\|Ap_n - \lambda p_n\| < 1/n$. It follows by $\aleph_1$ -saturation that there exists $p \in S$ such that $*\|p\| = 1$ and $Ap \sim \lambda p$. Hence $\hat{A}(\pi(p)) = \lambda \pi(p)$ and $\lambda \in \Pi_0(\hat{A})$.

Now suppose there exists $\lambda \in \Lambda(\hat{A}) \setminus \Pi(\hat{A})$. Then the range of $\hat{A} - \lambda I$ is a proper closed subspace of $\hat{S}$ so there exists $z \in \hat{S}$ such that $\|z\| = 1$ and $\langle (\hat{A} - \lambda I)x, z \rangle = 0$ for all $x \in \hat{S}$. Denote $A - \lambda I$ by A_{λ} . Pick $w \in S$ such that $*\|w\| = 1$ and $\pi(w) = z$. Then $\langle A_{\lambda}(p), w \rangle = {}_1 0$ for all $p \in \text{fin}(*H) \cap S$. If A_{λ} is not 1-1 on S , then since S is *-finite and A_{λ} is *-linear, there exists $p \in S$ such that $*\|p\| = 1$ and $A_{\lambda}p = 0$. But then $\hat{A}(\pi(p)) = \lambda \pi(p)$ and $\lambda \in \Pi_0(\hat{A}) \subseteq \Pi(\hat{A})$ which is a contradiction. Hence A_{λ} is 1-1 on S and so for some $q \in S$ we have $A_{\lambda}q = w$. Now if $q \in \text{fin}(*H)$, then $1 = *\|w\|^2 = \langle A_{\lambda}q, w \rangle = {}_1 0$ which is impossible. But if $*\|q\|$ is infinite, then $A_{\lambda}[(1/*\|q\|)q] = (1/*\|q\|)w \sim 0$ and $(\hat{A} - \lambda I)\pi[(1/*\|q\|)q] = 0$. Again $\lambda \in \Pi_0(\hat{A})$ which is a contradiction. Thus $\Lambda(\hat{A}) = \Pi(\hat{A}) = \Pi_0(\hat{A})$.

Note that if $\hat{A}$ is a hyperfinite extension of T , then $\Pi(T) \subseteq \Pi(\hat{A}) = \Lambda(\hat{A}) = \Pi_0(\hat{A})$. For standard hyperfinite extensions we can say more.

THEOREM 3.2. *If $\hat{T}_S$ is a standard hyperfinite extension of T , then $\Lambda(T) \subseteq \Lambda(\hat{T}_S) = \Pi_0(\hat{T}_S)$.*

PROOF. By Theorem 3.1 we have $\Lambda(\hat{T}_S) = \Pi_0(T_S)$ and by the observation above $\Pi(T) \subseteq \Lambda(\hat{T}_S)$. Suppose $\lambda \in \Lambda(T) \setminus \Pi(T)$. Then there exists $z \in H$ such that $\|z\| = 1$ and $\langle z, (T - \lambda I)x \rangle = 0$ for all $x \in H$. Now if $p \in \text{fin}(*H) \cap S$, then $\langle *z, (*T - \lambda I)p \rangle = 0$ so $\langle P_S *z, P_S(*T - \lambda I)p \rangle = 0$. Thus if $w = P_S *z$, then $\langle w, (*T_S - \lambda I)p \rangle = 0$ for all $p \in \text{fin}(*H) \cap S$. Using the argument given above in the last part of the proof of Theorem 3.1, it follows that $\lambda \in \Pi_0(T)$.

EXAMPLES 3.3. (i) We give an example of a hyperfinite extension $\hat{A}$ of an operator T such that $\Lambda(T) \not\subseteq \Lambda(\hat{A})$. Let T be the shift operator on $l_2(N)$ ($Te_k = e_{k+1}$ for all $k \in N$). Then T is subnormal, indeed T has an extension to the shift operator Q on $l_2(Z)$. Since Q is a unitary operator, it is easy to verify that in this case the hyperfinite extension $\hat{A}$ constructed in the proof of Theorem 2.10 is also a unitary operator. Thus $\Lambda(\hat{A}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, but $\Lambda(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

We give an example of an operator T and a standard hyperfinite extension $\hat{T}_S$ such that $\Lambda(T) \neq \Lambda(\hat{T}_S)$. Let T be the shift operator on $l_2(Z)$ and let $\hat{T}_S$ be the standard hyperfinite extension constructed in Example 2.9. Then $0 \in \Lambda(\hat{T}_S) \setminus \Lambda(T)$.

THEOREM 3.4. *If T is quasitriangular then there exists a standard hyperfinite extension $\hat{T}_S$ such that $\Lambda(T) = \Lambda(\hat{T}_S)$.*

PROOF. Let S be a *-finite dimensional subspace of $*H$ such that $\hat{S} \supseteq H$ and $*TP_S \sim P_S *TP_S$. If $\lambda \in \Lambda(\hat{T}_S)$ then by Theorem 3.1 there exists $p \in S$ such that $\|p\| = 1$ and $P_S *Tp \sim \lambda p$. But $*Tp \sim P_S *Tp$, so $*Tp \sim \lambda p$. It follows by a simple application of the transfer principle that $\lambda \in \Pi(T)$. Thus $\Lambda(\hat{T}_S) \subseteq \Lambda(T)$ and by Theorem 3.2 we have $\Lambda(\hat{T}_S) = \Lambda(T)$.

Note that the proof of Theorem 3.4 shows that if T is quasitriangular then $\Lambda(T) = \Pi(T)$. This was first shown by Deddens [3]. It is an interesting question whether for an arbitrary bounded operator T there exists a standard hyperfinite extension $\hat{T}_S$ such that $\Lambda(T) = \Lambda(\hat{T}_S)$.

The following result is a partial answer to a question posed to the author by W. A. J. Luxemburg. If S is a *-finite dimensional subspace of $*H$ and $A \in \mathfrak{A}(S)$, let $E(A) = \{\lambda \in \mathbb{C} : \text{there exists } p \in S \text{ with } \|p\| = 1 \text{ and } Ap = \lambda p\}$, i.e. $E(A)$ is the set of *-eigenvalues of A .

THEOREM 3.5. *If T is quasitriangular there exists a *-finite dimensional subspace S with $\hat{S} \supseteq H$ and $A \in \mathfrak{A}(S)$ such that $A \sim *T_S$ and $\text{st}(E(A)) = \Lambda(T)$.*

PROOF. By Theorem 3.4 there exists a standard hyperfinite dimensional extension $\hat{T}_S$ of T such that $\Pi_0(\hat{T}_S) = \Lambda(\hat{T}_S) = \Lambda(T)$. Assume $\Lambda(T)$ is infinite and pick a countable dense subset $\{\lambda_k: k \in N\}$ of $\Lambda(T)$.

We show first that for every $n \in N$ there exists $A_n \in \mathfrak{A}(S)$ such that $A_n \sim {}^*T_S$ and $E(A_n) \supseteq \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Since $\Pi_0(\hat{T}_S) = \Lambda(\hat{T}_S) = \Lambda(T)$, for each $k \in N$ there exists $x_k \in \hat{S}$ such that $\|x_k\| = 1$ and $\hat{T}_S(x_k) = \lambda_k x_k$. For each k pick $p_k \in S$ such that $\|p_k\| = 1$ and $\pi(p_k) = x_k$. Thus ${}^*T_S p_k \sim \lambda_k p_k$ for each $k \in N$. Now let $n \in N$. Since $\{x_1, x_2, \dots, x_n\}$ is linearly independent, it follows that $\{p_1, p_2, \dots, p_n\}$ is $*$ -linearly independent (Theorem 1.8 of [6]). Let $S_n = *$ -span of $\{p_1, p_2, \dots, p_n\}$ and let P_n be the $*$ -projection of S onto S_n .

Let $p \in \text{fin}({}^*H) \cap S_n$. Then $p = \sum_{k=1}^n a_k p_k$ and each $a_k \in \text{fin}({}^*R)$. If to the contrary some a_k is infinite, let

$$b = \max \{|a_1|, |a_2|, \dots, |a_n|\}.$$

Then $\sum_{k=1}^n (a_k/b)p_k = (1/b)p \sim 0$ and so $\sum_{k=1}^n \text{st}(a_k/b)x_k = 0$ which contradicts the independence of the set $\{x_1, x_2, \dots, x_n\}$, since for some k , $|\text{st}(a_k/b)| = 1$. Define the $*$ -linear mapping Q_n on S_n by $Q_n(\sum_{k=1}^n a_k p_k) = \sum_{k=1}^n a_k \lambda_k p_k$ for all $a_1, a_2, \dots, a_n \in {}^*C$ and set $A_n = Q_n P_n + {}^*T_S(I - P_n)$. Now if $p \in S$ and $\|p\| = 1$, then $A_n p - {}^*T_S p = (Q_n - {}^*T_S)P_n p$. But $\|P_n p\| \leq 1$, so $P_n p = \sum_{k=1}^n a_k p_k$ where a_k is finite for each k . Thus

$$(Q_n - {}^*T_S)p = \sum_{k=1}^n a_k (\lambda_k p_k - {}^*T_S p_k) \sim 0$$

since $\lambda_k p_k \sim {}^*T_S p_k$ for each k . It follows that $A_n \sim {}^*T_S$ and clearly $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \in E(A_n)$.

Hence for each $n \in N$ there exists $A_n \in \mathfrak{A}(S)$ such that $A_n(p_k) = \lambda_k p_k$ for $k = 1, 2, \dots, n$ and $\|{}^*T_S - A_n\| < n^{-1}$. It follows by $\aleph_1$ -saturation that there exists $A \in \mathfrak{A}(S)$ such that $A \sim {}^*T_S$ and $A p_k = \lambda_k p_k$ for all $k \in N$. Thus $E(A) \supseteq \{\lambda_k: k \in N\}$ and so $\text{st}(E(A)) \supseteq \{\lambda_k: k \in N\}$. But since $E(A)$ is an internal set, $\text{st}(E(A))$ is closed. Thus $\text{st}(E(A)) \supseteq \overline{\{\lambda_k: k \in N\}} = \Lambda(T)$. On the other hand let $\lambda \in E(A)$. Since $|\lambda| \leq \|A\| \leq \|{}^*T_S\| + 1$, we have that λ is finite. Pick $p \in S$ such that $\|p\| = 1$ and $A p = \lambda p$. Then $\hat{A}\pi(p) = \hat{T}_S\pi(p) = \text{st}(\lambda)\pi(p)$ and $\text{st}(\lambda) \in \Pi_0(\hat{T}_S)$. Since $\Pi_0(\hat{T}_S) = \Lambda(T)$, we have that $\text{st}(\lambda)$ is in $\Lambda(T)$. Hence $\text{st}(E(A)) \subseteq \Lambda(T)$ and thus $\text{st}(E(A)) = \Lambda(T)$.

Finally note that if $\Lambda(T)$ is finite, say $\Lambda(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then we may take $A = A_n$ above.

4. Spectral theorem for a bounded self adjoint operator. There have been previous nonstandard proofs of the spectral theorem for a bounded selfadjoint operator. In particular, there is a proof by A. Bernstein [1]. The interest of

the proof presented here is that it is short and makes essential use of the external projection of $\hat{S}$ onto H .

Let T be a bounded selfadjoint operator on H and let S be any *-finite dimensional subspace of *H such that $\hat{S} \supseteq H$. Then *T_S is *-selfadjoint on S and $\hat{T}_S$ is a selfadjoint extension of T to $\hat{S}$. Now if V is a finite dimensional inner product space and Q is a selfadjoint linear transformation of V into itself, there exist an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of V and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subseteq R$ such that

$$(i) \quad \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \text{ and}$$

$$(ii) \quad Q(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i \lambda_i e_i \text{ for all choices of } a_1, a_2, \dots, a_n \text{ in } C.$$

Hence there exists a *-orthonormal basis $\{\psi_1, \psi_2, \dots, \psi_\omega\}$ for S and a *-finite set $\{\lambda_1, \lambda_2, \dots, \lambda_\omega\} \subseteq {}^*R$ such that

$$(i) \quad \lambda_i \leq \lambda_{i+1} \text{ for } i = 1, 2, \dots, \omega - 1 \text{ and}$$

$$(ii) \quad {}^*T_S(\sum_{i=1}^\omega a_i \psi_i) = \sum_{i=1}^\omega a_i \lambda_i \psi_i \text{ for any internal *-finite sequence } \{a_1, a_2, \dots, a_\omega\} \subseteq {}^*C.$$

In particular ${}^*T_S \psi_i = \lambda_i \psi_i$ and since $\|{}^*T_S\| \leq \|T\|$ it follows that λ_i is finite for $i = 1, 2, \dots, \omega$. For each real μ and $n \in N$ define $S(\mu, n)$ to be the *-span of $\{\psi_k: \lambda_k \leq \mu + n^{-1}\}$ and $F(\mu, n)$ to be the *-projection of S onto $S(\mu, n)$. Now $F(\mu, n) = 0$ if $\lambda_k > \mu + n^{-1}$ for all k , and for each μ the sequence $\{\hat{F}(\mu, n)\}$ is a monotone decreasing sequence of projections on $\hat{S}$. Define $E(\mu)$ to be the strong limit of $\hat{F}(\mu, n)$. Then $\{E(\mu): \mu \in R\}$ is the spectral resolution for $\hat{T}_S$. More precisely:

THEOREM 4.1. (i) $E(\mu) = 0$ if $\mu < \|\hat{T}\|$ and $E(\mu) = I$ if $\|\hat{T}\| < \mu$.

(ii) $E(\mu)E(\alpha) = E(\alpha)E(\mu) = E(\min(\alpha, \mu))$ for all $\mu, \alpha \in R$.

(iii) $E(\mu)\hat{T}_S = \hat{T}_S E(\mu)$ all $\mu \in R$,

(iv) $\alpha(E(\beta) - E(\alpha)) \leq \hat{T}_S(E(\beta) - E(\alpha)) \leq \beta(E(\beta) - E(\alpha))$ if $\alpha, \beta \in R$ with $\alpha < \beta$.

(v) $\lim_{\mu \rightarrow \alpha^+} E(\mu) = E(\alpha)$.

PROOF. (i) This is immediate since $\text{st}(\|{}^*T_S\|) = \|\hat{T}_S\| = \|T\|$.

(ii) Let $\mu, \alpha \in R$ and $n \in N$, then

$$F(\mu, n)F(\alpha, n) = F(\alpha, n)F(\mu, n) = F(\min(\mu, \alpha), n).$$

Hence $\hat{F}(\mu, n)\hat{F}(\alpha, n) = \hat{F}(\alpha, n)\hat{F}(\mu, n) = \hat{F}(\min(\mu, \alpha), n)$. Now taking limits we obtain (ii).

(iii) Let $\mu \in R$ and $n \in N$. Then $F(\mu, n){}^*T_S = {}^*T_S F(\mu, n)$, hence $\hat{F}(\mu, n)\hat{T}_S = \hat{T}_S \hat{F}(\mu, n)$ and taking limits we obtain (iii).

(iv) Let $\alpha, \beta \in R$ with $\alpha < \beta$ and $n \in N$. Then

$$\begin{aligned} (\alpha + n^{-1})(F(\beta, n) - F(\alpha, n)) &\leq {}^*T_S(F(\beta, n) - F(\alpha, n)) \\ &\leq (\beta + n^{-1})(F(\beta, n) - F(\alpha, n)). \end{aligned}$$

Again (iv) follows by applying the $\hat{}$ operation and taking limits.

(v) Finally if $\alpha \in R$ then $\{\hat{F}(\mu, n): \mu \in R, \mu > \alpha \text{ and } n \in N\}$ is cofinal with $\{\hat{F}(\alpha, n): n \in N\}$. Hence (v) holds.

Now let P be the (external) projection of $\hat{S}$ onto H . Since $\hat{T}_S$ leaves H invariant, it is easy to show that P commutes with each projection $E(\mu)$. Hence if $G(\mu)$ is the restriction of $PE(\mu)$ to H for each $\mu \in R$, then $\{G(\mu): \mu \in R\}$ is the spectral resolution for T .

REFERENCES

1. A. R. Bernstein, *The spectral theorem—a nonstandard approach*, Z. Math. Logik Grundlagen Math. 18 (1972), 419–434. MR 47 #4048.
2. D. Cozart and L. C. Moore, Jr., *The nonstandard hull of a normed Riesz space*, Duke Math. J. 41 (1974), 263–275.
3. J. A. Deddens, *A necessary condition for quasitriangularity*, Proc. Amer. Math. Soc. 32 (1972), 630–631. MR 44 #5810.
4. P. R. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) 29 (1968), 283–293. MR 38 #2627.
5. ———, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. 76 (1970), 877–933. MR 42 #5066.
6. C. Ward Henson and L. C. Moore, Jr., *The nonstandard theory of topological vector spaces*, Trans. Amer. Math. Soc. 172 (1972), 405–435; Erratum, ibid 184 (1973), 509. MR 46 #7836; 48 # 2708.
7. ———, *Subspaces of the nonstandard hull of a normed space*, Trans. Amer. Math. Soc. 197 (1974), 131–143.
8. ———, *Nonstandard hulls of the classical Banach spaces*, Duke Math. J. 41 (1974), 277–284.
9. W. A. J. Luxemburg, *A general theory of monads*, Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967), Holt, Rinehart and Winston, New York, 1969, pp. 18–86. MR 39 #6244.
10. A. Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966. MR 34 #5680.

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