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YONEDA PRODUCTS IN THE CARTAN-EILENBERG CHANGE OF RINGS SPECTRAL SEQUENCE WITH APPLICATIONS TO $BP_*(BO(n))$

BY

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ABSTRACT. Yoneda product structure is defined on a Cartan-Eilenberg change of rings spectral sequence. Application is made to a factorization theorem for the E_2-term of the Adams spectral sequence for Brown-Peterson homology of the classifying spaces $BO(n)$.

This paper gives an algebraic decomposition of the E_2-term of the Adams spectral sequence of the reduced mod 2 Brown-Peterson homology [3], [4] of the classifying space $BO(n)$.

The first section gives algebraic preliminaries and the statement of the main result. In §2 Yoneda products are introduced in a Cartan-Eilenberg change of rings spectral sequence [5] used to compute the required Ext module. The proof of the main theorem is given in §3.

The main results of §§2 and 3 are contained in the author's doctoral dissertation at the University of Chicago under Arunas Liulevicius, to whom grateful acknowledgement is made for his time and helpful suggestions.

1. **Preliminaries; statement of results.** This section outlines the algebraic constructions needed to construct the spectral sequence of §2 and to introduce the main theorem.

Let F be a field. An algebra A will be a positively graded, augmented, associative F-algebra. Let $\overline{A}$ denote the augmentation ideal of A. Let $B_s(A, A) = A \otimes \overline{A}^s \otimes A$, where $\overline{A}^s$ is the s-fold tensor product of $\overline{A}$. Form the 2-sided bar construction [10] $B(A, A) = \Sigma_{s \geq 0} B_s(A, A)$ and let ∂ denote the standard boundary map. In all that follows the *degree* of an element refers to its total degree.

Let $\overline{B}(A) = F \otimes_A B(A, A) \otimes_A F$ with induced boundary $\overline{\partial}$ and let $\overline{C}(A) = \overline{B}(A)^*$ with coboundary $\overline{\delta} = (\overline{\partial})^*$. Recall that $\overline{C}_s(A) = (\overline{A}^*)^s$ and that $\overline{C}(A)$ is a differential algebra under the cup-product

$$(1.1) \qquad [\alpha_1 | \cdots | \alpha_k][\beta_1 \cdots \beta_l] = [\alpha_1 | \cdots | \alpha_k | \beta_1 | \cdots | \beta_l];$$

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that is, for $\alpha, \beta \in \overline{C}(A)$ we have

$$\overline{\delta}(\alpha\beta) = \overline{\delta}(\alpha)\beta + (-1)^{\deg \alpha}\alpha\overline{\delta}(\beta). \tag{1.2}$$

If M is a positively graded left A-module, let $B(F, M) = F \otimes_A B(A, A) \otimes_A M$ have induced boundary $\overline{\partial}_M$ and let $C(F, M) = B(F, M)^*$ have coboundary $\overline{\delta}_M = \overline{\partial}_M^*$. Then $C(F, M)$ is a differential $\overline{C}(A)$-module under the cup-product, that is

$$\overline{\delta}_M(\alpha \cdot \beta \otimes \lambda) = \overline{\delta}(\alpha) \cdot (\beta \otimes \lambda) + (-1)^{\deg \alpha}\alpha \cdot \overline{\delta}_M(\beta \otimes \lambda) \tag{1.3}$$

where $\alpha, \beta \in \overline{C}(A)$, $\lambda \in M^*$.

The cup-product on $\overline{C}(A)$ induces a product map on $H^{**}(A) = \mathrm{Ext}_A^{**}(F, F) = H_{**}(\overline{C}(A))$ and a structure map

$$H^{s,t}(A) \otimes \mathrm{Ext}_A^{s',t'}(M, F) \longrightarrow \mathrm{Ext}_A^{s+s',t+t'};$$

both structures are called Yoneda products.

Hereafter A denotes the mod 2 Steenrod algebra.

To compute $\widetilde{BP}_*(BO(n))$, we have the Adams spectral sequence

$$E_2 = \mathrm{Ext}_A(H^*(BP; Z_2) \otimes \widetilde{H}^*(BO(n); Z_2), Z_2) \Rightarrow BP_*(BO(n)) \otimes I_2, \tag{1.4}$$

where I_2 denotes the 2-adic integers. We have [4]

$$H^*(BP; Z_2) = A/A(Q_0, Q_1, \ldots) \tag{1.5}$$

where the $Q_i \in A$ are defined by Milnor [11]; recall that $Q_0 = Sq^1$ and $Q_i = [Q_{i-1}, Sq^{2^i}]$. Let $E = \bigwedge(Q_0, Q_1, \ldots)$, where $\bigwedge$ denotes exterior algebra over Z_2, then $H^*(BP; Z_2) = A \otimes_E Z_2$, so by a standard change of rings theorem [7]

$$E_2 \cong \mathrm{Ext}_E(\widetilde{H}^*(BO(n); Z_2), Z_2) = \mathrm{Ext}_{E_*}(Z_2, \widetilde{H}_*(BO(n); Z_2)). \tag{1.6}$$

Here we write E_* for $\mathrm{Hom}(E, Z_2)$ following Milnor's convention since E_* occurs in the context of homology. The second Ext of (1.6) is one of E_*-comodules; see Adams [3].

Since E is a Hopf algebra, so is E_*, which is an exterior algebra over Z_2 on generators $\beta_1, \beta_2, \ldots$ which form a dual basis to $Q_0, Q_1, \ldots$ respectively.

We have [8]

$$\widetilde{H}_*(BO(n); Z_2) = \widetilde{H}_*(MO(n); Z_2) \oplus \widetilde{H}_*(BO(n-1); Z_2)$$

As A_*-comodules. The first summand may be described as follows: let MO denote the Thom spectrum for the orthogonal groups, then

$$H_*(MO; Z_2) = Z_2[b_1, b_2, \ldots]$$

where $b_i \in H_i(MO; Z_2)$ is the image of $x_{i+1} \in H_{i+1}(RP^\infty; Z_2)$ under the composite

$$H_{i+1}(RP^\infty; Z_2) \cong H_{i+1}(MO(1); Z_2) \longrightarrow H_i(MO; Z_2).$$

The subgroup $\widetilde{H}_*(MO(n); Z_2)$ is the span of monomials in the b_i of degree $\leqslant n$. The *coaction map* [8]

(1.7) $$\mu_*: H_*(MO; Z_2) \longrightarrow E_* \otimes H_*(MO; Z_2)$$

is a ring homomorphism given on generators by

(1.8) $$\mu_*(b_{2m}) = 1 \otimes b_{2m}, \quad \mu_*(b_{2m-1}) = \sum_{i>0} \beta_i \otimes b_{2m-2^i}.$$

The cohomology of E, $H^{**}(E)$, is the polynomial algebra $Z_2[q_0, q_1, \ldots]$ where $q_i \in H^{1,2^{i+1}-1}(E)$.

Let $E(m) = \bigwedge(Q_0, \ldots, Q_m)$. Let $M = M(n)$ denote either $\widetilde{H}^*(BO(n); Z_2)$ or $\widetilde{H}^*(MO(n); Z_2)$. The main result states:

THEOREM 1.9. *Under the Yoneda product,* $\mathrm{Ext}_E(M, Z_2)$ *is a free* $Z_2[q_n, q_{n+1}, \ldots]$*-module on* $\mathrm{Ext}_{E(n-1)}(M, Z_2)$. *Hence the* E_2*-term in the* mod 2 *Adams spectral sequence of* $\widetilde{BP}_*(BO(n))$ *is given by*:

$$E_2 \approx Z_2[q_n, q_{n+1}, \ldots] \otimes \mathrm{Ext}_{E(n-1)}(\widetilde{H}^*(BO(n); Z_2), Z_2).$$

2. A change of rings spectral sequence with products. The program for the proof of Theorem 1.9 is to show

(2.1) $$\mathrm{Ext}_{E(r)}(M, Z_2) \cong Z_2[q_n, \ldots, q_r] \otimes \mathrm{Ext}_{E(n-1)}(M, Z_2)$$

for $r \geqslant n$. For this purpose we use a spectral sequence of Cartan and Eilenberg [5] to which we have added the structure of Yoneda products.

Let $\varphi: S \longrightarrow A$ be a homomorphism of algebras in the sense of §1, that is φ has degree zero and commutes with the augmentations. The map φ is called (left) *normal* if $\varphi(\overline{S})A$ is a left ideal of A. If φ is normal, $A/\varphi(\overline{S})A \cong F \otimes_S A$ is an algebra.

THEOREM 2.2. *Let* $\varphi: S \longrightarrow A$ *be a left normal homomorphism of algebras such that* A *is projective as a left* S*-module. Let* $T = F \otimes_S A$. *Let* M *be a left* A*-module and* C *a left* T*-module. Then there is a spectral sequence*

(2.2) $$\mathrm{Ext}_T^p(\mathrm{Tor}_q^S(F, M), C) \Rightarrow \mathrm{Ext}_A^{p+q}(M, C).$$

Here A and M are S-modules through φ, and C is an A-module through the projection $\pi: A \longrightarrow A/\varphi(\overline{S})A \cong T$. The left T-operations on $\mathrm{Tor}^S(F, M)$ are induced by left multiplication in T through the isomorphism

(2.3) $$\mathrm{Tor}^S(F, M) \cong \mathrm{Tor}^A(F \otimes_S A, M) = \mathrm{Tor}^A(T, M).$$

The outer Ext may be computed as one of T^*-comodules, so that

(2.4) $$E_2 \cong \mathrm{Ext}_{T^*}^p(C^*, \mathrm{Ext}_S^q(M, F)).$$

Note that $E_2^{0,m}$ may be regarded as the subcomodule of T^*-primitives of $\mathrm{Ext}_S^m(M, F)$, and it will follow from the construction that the edge homomorphism

$$\mathrm{Ext}_A^m(M, F) \longrightarrow E_\infty^{0,m} \hookrightarrow E_2^{0,m} \hookrightarrow \mathrm{Ext}_S^m(M, F)$$

coincides with the induced map $\mathrm{Ext}_\varphi(M, F)$.

To form the products, consider the case $C = F$. Note that $H^*(A)$ acts on $\mathrm{Ext}_S(M, F)$ through $H^*(\varphi)$, and so induces a structure map

$$H^s(A) \otimes E_2^{p,q} \xrightarrow{\mu_1} E_2^{p,q+s}.$$

On the other hand, by (2.4) the Yoneda product gives a structure map

$$H^s(T) \otimes E_2^{p,q} \xrightarrow{\mu_2} E_2^{p+s,q}.$$

In the theorems below, let F^p denote the pth filtration of $\mathrm{Ext}_A(M, F)$ in the spectral sequence and let

$$\rho_p\colon F^p \longrightarrow F^p/F^{p+1} = E_\infty^p$$

be the projection.

THEOREM 2.5. *The spectral sequence of Theorem 2.2 with $C = F$ admits structure maps*

$$H^s(A) \otimes E_r^{p,q} \xrightarrow{\lambda_r} E_r^{p,q+s}, \quad 1 \leqslant r \leqslant \infty,$$

such that:

(1) $d_r\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ *is a left (graded) $H^*(A)$-module homomorphism*;

(2) λ_{r+1} *is induced from λ_r by passing to subquotients*;

(3) $\lambda_2 = \mu_1$;

(4) *the following diagram commutes, where Y denotes the restriction of the Yoneda product map to $H^s(A) \otimes F^p$*:

(2.6)
$$\begin{array}{ccc} H^s(A) \otimes F^p & \xrightarrow{Y} & F^p \\ \downarrow{\scriptstyle 1 \otimes \rho_p} & & \downarrow{\scriptstyle \rho_p} \\ H^s(A) \otimes E_\infty^p & \xrightarrow{\lambda_\infty} & E_\infty^p \end{array}$$

THEOREM 2.7. *The spectral sequence of Theorem 2.2 with $C = F$ admits structure maps*

$$H^s(T) \otimes E_r^{p,q} \xrightarrow{\theta_r} E_r^{p+s,q}, \quad 1 \leqslant r \leqslant \infty,$$

such that:

(1) d_r *is a left (graded) $H^*(T)$-module homomorphism*:

(2) *θ_{r+1} is induced from θ_r by passing to subquotients*;

(3) $\theta_2 = \mu_2$;

(4) *the following diagram commutes, where Y' denotes the restriction of the composite*

$$H^*(T) \otimes \mathrm{Ext}_A(M, F) \xrightarrow{H^*(\pi)\otimes 1} H^*(A) \otimes \mathrm{Ext}_A(M, F) \longrightarrow \mathrm{Ext}_A(M, F)$$

to $H^s(T) \otimes F^p$:

$$\text{(2.8)} \qquad \begin{array}{ccc} H^s(T) \otimes F^p & \xrightarrow{Y'} & F^{p+s} \\ {\scriptstyle 1\otimes\rho_p}\downarrow & & \downarrow{\scriptstyle \rho_{p+s}} \\ H^s(T) \otimes E_\infty^p & \xrightarrow{\theta_\infty} & E_\infty^{p+s} \end{array}$$

Theorem 2.2 is the analog of Theorem 6.1 (1a) of Cartan and Eilenberg [5, p. 349] with Tor^T replaced by Ext_T. If products are not required, the construction in the proof of (2.5) carries over with trivial modifications for $\mathrm{Ext}_A(M, C)$, so a separate proof will be omitted.

For the proof of (2.5), let $X = B(A, M) = B(A, A) \otimes_A M$. Define $\epsilon\colon X \longrightarrow M$ by $\epsilon(a[a_1 | \cdots | a_s]m) = 0$ if $s \geqslant 1$ and $\epsilon(a[\ ]m) = am$. Then X with ϵ is a free A-resolution of M. Let

$$0 \longrightarrow F \xrightarrow{\eta} Y^0 \xrightarrow{\bar{d}} Y^1 \longrightarrow \cdots$$

be a resolution of F by bigraded, injective T-modules. In assigning degrees of maps we follow the usual convention $Y^{s,t} = Y_{-s,-t}$. Form the sum $\mathrm{Hom}_A(X, Y) = \Sigma_{i,j}\mathrm{Hom}_A(X_i, Y^j)$ and define coboundaries

$$\delta\colon \mathrm{Hom}_A(X_i, Y^j) \xrightarrow{\delta} \mathrm{Hom}_A(X_{i+1}, Y^j)$$

$$d\colon \mathrm{Hom}_A(X_i, Y^j) \xrightarrow{\alpha} \mathrm{Hom}_A(X_i, Y^{j+1})$$

by

$$(\delta f)(x) = (-1)^{\deg f+1} f(\partial_M x), \qquad (df)(x) = \bar{d}(f(x))$$

where $\partial_M = \partial \otimes_A 1$ and $\deg f = -(s + s' + t + t')$ if $f\colon X_{s,t} \longrightarrow Y^{s',t'}$.

With this definition the squares

$$\begin{array}{ccc} \mathrm{Hom}_A(X_{i+1}, Y^j) & \xrightarrow{d} & \mathrm{Hom}_A(X_{i+1}, Y^{j+1}) \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta} \\ \mathrm{Hom}_A(X_i, Y^j) & \xrightarrow{d} & \mathrm{Hom}_A(X_i, Y^{j+1}) \end{array}$$

anticommute, so that $\mathrm{Hom}_A(X, Y)$ is a bicomplex. The total differential $\Delta = \delta + d$ makes $\mathrm{Hom}_A(X, Y)$ into a cochain complex.

The bicomplex $\mathrm{Hom}_A(X, Y)$ has row and column filtrations,

$$G^p\,\mathrm{Hom}_A(X, Y) = \sum_{i>p,j} \mathrm{Hom}_A(X_i, Y^j),$$

$$F^p\,\mathrm{Hom}_A(X, Y) = \sum_{j>p,i} \mathrm{Hom}_A(X_i, Y^j).$$

The row filtration gives a spectral sequence with

$$E_2^{p,q} = H_p(H_q(\mathrm{Hom}_A(X, Y); d); \delta).$$

Since the X_i are A-free, the rows are acyclic, so that

$$H_q(\mathrm{Hom}_A(X, Y); d) = \begin{cases}\mathrm{Hom}_A(X, F), & q = 0,\\ 0, & q \neq 0,\end{cases}$$

and hence

$$E_2^{p,q} = \begin{cases}\mathrm{Ext}_A^p(M, F), & q = 0,\\ 0 & q \neq 0.\end{cases}$$

Thus the spectral sequence of the row filtration collapses with trivial extensions and identifies $H_*(\mathrm{Hom}_A(X, Y); \Delta)$ with $\mathrm{Ext}_A(M, F)$. We say $x \in F^p\mathrm{Ext}_A(M, F)$ if it has a representing cocycle in $F^p\,\mathrm{Hom}_A(X, Y)$.

The spectral sequence of Theorem 2.2. is the one corresponding to the column filtration F^p. Its E_2 term will be identified in the form (2.4). Define a map

$$\gamma\colon \mathrm{Hom}_A(X_q, Y^p) \longrightarrow \mathrm{Hom}_{T^*}(Y^{p*}, \mathrm{Hom}_S(X_q, F))$$

by

$$\gamma(f)(y^*) = (-1)^{\deg f \deg y^*} y^* f.$$

We have

$$E_2^{p,q} = H_p(H_q(\mathrm{Hom}_A(X, Y); \delta); d).$$

Note that X with augmentation ϵ is a projective S-resolution of M, by the hypothesis of the S-action on A. Thus the homology of $\mathrm{Hom}_S(X, F)$ with the induced coboundary is $\mathrm{Ext}_S(M, F)$. Since Y^{p*} is a projective T^*-comodule, it is easily verified that γ induces an isomorphism

$$H_q(\mathrm{Hom}_A(X, Y); \delta) \cong \mathrm{Hom}_{T^*}(Y^{p*}, \mathrm{Ext}_S^q(M, F)).$$

Hence $E_2^{p,q} = \mathrm{Ext}_{T^*}^p(F, \mathrm{Ext}_S^q(M, F))$.

For the products, we use an equivalent formulation of the bicomplex. Let α be the composite

$$(\bar{A}^*)^i \otimes M^* \otimes Y^j \xrightarrow{\alpha'} \mathrm{Hom}(\bar{A}^i \otimes M, Y^j)$$
$$\cong \mathrm{Hom}_A(A \otimes \bar{A}^i \otimes M, Y^j) = \mathrm{Hom}_A(X_i, Y^j)$$

where

$$(\alpha'(a^*\otimes m^*\otimes y), a\otimes m) = (-1)^{\mu \deg y}(a^*\otimes m^*, a\otimes m)y,$$
$$\mu = \deg(a^*\otimes m^*),$$

and (,) is the dual pairing.

We have commutative diagrams

(2.9)
$$\begin{array}{ccc} (\bar{A}^*)^{i+1}\otimes M^*\otimes Y^j & \xrightarrow[\cong]{\alpha} & \mathrm{Hom}_A(X_{i+1}, Y^j) \\ {\scriptstyle \bar{\delta}_M\otimes 1}\Big\uparrow & & \Big\uparrow{\scriptstyle \delta} \\ (\bar{A}^*)^i\otimes M^*\otimes Y^j & \xrightarrow[\cong]{\alpha} & \mathrm{Hom}_A(X_i, Y^j) \end{array}$$

(2.10)
$$\begin{array}{ccc} (\bar{A}^*)^{i+1}\otimes M^*\otimes Y^{j+1} & \xrightarrow[\cong]{\alpha} & \mathrm{Hom}_A(X_i, Y^{j+1}) \\ {\scriptstyle 1\otimes \bar{d}}\Big\uparrow & & \Big\uparrow{\scriptstyle d} \\ (\bar{A}^*)\otimes M^*\otimes Y^j & \xrightarrow[\cong]{\alpha} & \mathrm{Hom}_A(X_i, Y^j) \end{array}$$

so that the bicomplex $(\mathrm{Hom}_A(X, Y), \delta, d, \Delta)$ may be replaced by $(C(F, M)\otimes Y, \delta', d', \Delta')$ where $\delta' = \bar{\delta}_M\otimes 1$, $d' = 1\otimes \bar{d}$ and $\Delta' = \delta' + d'$ with the usual sign conventions.

Now $C(F, M)\otimes Y$ is a left differential $\bar{C}(A)$-module with respect to the cup product and each of its differentials; that is, if $a\in \bar{C}(A)$, $b\in C(F, M)\otimes Y$ we have

$$\delta'(ab) = \bar{\delta}(a)b + (-1)^{\deg a}a\delta'(b), \qquad d'(ab) = (-1)^{\deg a}ad'(b)$$

where $\bar{\delta}$ is the coboundary in $\bar{C}(A)$.

In particular the identification of $\mathrm{Ext}_A(M, F)$ with $H_*(\mathrm{Hom}_A(X, Y); \Delta)\cong H_*(C(F, M)\otimes Y; \Delta')$ given by the row spectral sequence is an $H^*(A)$-module homomorphism. Suppose $h\in C(F, M)$ is a cocycle identified with a cocycle $\Sigma\alpha_i$ of $C(F, M)\otimes Y$, then

$$\eta'(h) - \sum\alpha_i = \Delta'(w) \quad \text{for some } w\in C(F, M)\otimes Y,$$

where $\eta'\colon C(F, M)\longrightarrow C(F, M)\otimes Y^0$ corresponds to η_* under the isomorphism α. If $z\in \bar{C}(A)$ is a cocycle, then

$$\Delta'((-1)^{\deg z}zw) = z\Delta'(w) = \eta'(zh) - z\Big(\sum\alpha_i\Big)$$

so that zh is identified with $z(\Sigma\alpha_i)$.

If $z\in \bar{C}(A)$ is a cocycle, its action on $C(F, M)\otimes Y$ commutes in the graded sense with the coboundaries, and it clearly preserves filtration, so the cup-product action induces the structure maps λ_r of Theorem 2.5. In particular the differentials are left graded $H^*(A)$-module homomorphisms.

To identify the product on E_2, suppose $f\colon A\otimes \bar{A}^q\otimes M\longrightarrow F$ represents

$x \in \mathrm{Ext}_S^q(M, F)$ and $g\colon \bar{A}^s \longrightarrow F$ represents $z \in H^s(A)$, then z acts on x through $H^*(\varphi)$, and zx is represented by $g \cdot f$, where

$$g \cdot f(a \otimes a' \otimes a'') = (-1)^{\deg f \deg(a' \otimes a'')} g(a \otimes a') f(a'')$$

for $a' \in \bar{A}^s$, $a'' \in \bar{A}^q$. We have for $z \in \bar{C}(A)$, $w \in C(F, M) \otimes Y$, $\gamma\alpha(zw) = z \cdot \gamma\alpha(w)$ from which it follows that $\lambda_2 = \mu_1$. This completes (2.5).

COROLLARY 2.11. *Let $A = S \otimes T$ and let $e\colon T \longrightarrow F$ be the augmentation. Then $H^*(S)$ acts on the spectral sequence by the structure maps $\lambda_r(1 \otimes e)^*$. For $r = 2$ this action is induced on Ext_{T^*} from the Yoneda product.*

We note that the maps λ_r annihilate the image of $H^*(T) \otimes E_r$ in $H^*(A) \otimes E_r$ for $r \geqslant 2$ since $H^*(A)$ acts on $\mathrm{Ext}_S(M, F)$ through $H^*(\varphi)$. Thus the structure maps θ_r are needed to "detect" the action of $H^*(T)$.

For the proof of (2.7), it will be convenient to describe the Yoneda products in terms of anticommutative diagrams. Let N be a T-module and let Y be the injective resolution used in (2.5), then an element $x \in \mathrm{Ext}_T^p(N, F)$ is represented by a cocycle $f\colon N \longrightarrow Y^p$. Let $g\colon F \longrightarrow Y^q$ represent $z \in H^q(T)$. Form the anticommutative diagram:

$$\begin{array}{ccccccccccc} & & & & & & & & & & N \\ & & & & & & & & & & \big\downarrow{\scriptstyle f} \\ 0 & \longrightarrow & F & \longrightarrow & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^p \\ & & & {\scriptstyle g}\searrow & \big\downarrow{\scriptstyle g^0} & & \big\downarrow{\scriptstyle g^1} & & & & \big\downarrow{\scriptstyle g^p} \\ & & & & Y^q & \longrightarrow & Y^{q+1} & \longrightarrow & \cdots & \longrightarrow & Y^{q+p} \end{array} \tag{2.12}$$

Then zx is represented by $(-1)^{\deg f \deg g} g^p f$.

There is an analogous construction on the bicomplex $\mathrm{Hom}_A(X, Y)$.

LEMMA 2.13. *Suppose $\Sigma_{i=0}^{p+q} f_i$, $f_i\colon X_i \longrightarrow Y^{p+q-i}$ is a cocycle representing $x \in \mathrm{Ext}_A^{p+q}(M, F)$. Suppose $g\colon F \longrightarrow Y^r$ represents $z \in H^r(T)$. Then we may construct the top squares in the following anticommutative diagram since Σf_i is a cocycle, and the bottom since the Y^j are injective:*

$$\begin{array}{ccccccccccccc} 0 \leftarrow M \leftarrow & X_0 & \leftarrow \cdots \leftarrow & X_q & \leftarrow & X_{q+1} & \leftarrow \cdots \leftarrow & Y_{q+p} & & \\ & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_q} & & \big\downarrow{\scriptstyle f_{q+1}} & & \big\downarrow{\scriptstyle f_{p+q}} & \nwarrow{\scriptstyle f} & \\ & Y^{p+q} & \leftarrow \cdots \leftarrow & Y^p & \leftarrow & Y^{p-1} & \leftarrow \cdots \leftarrow & Y^0 & \xleftarrow{\eta} & F \leftarrow 0 \\ & \big\downarrow{\scriptstyle g^{p+q}} & & \big\downarrow{\scriptstyle g^p} & & \big\downarrow{\scriptstyle g^{p-1}} & & \big\downarrow{\scriptstyle g^0} & \swarrow{\scriptstyle g} & \\ & Y^{p+q+r} & \leftarrow \cdots \leftarrow & Y^{p+r} & \leftarrow & Y^{p+r-1} & \leftarrow \cdots \leftarrow & Y^r & & \end{array} \tag{2.14}$$

Then zx is represented by $(-1)^{\deg z \deg x} \Sigma g^{p+q-i} f_i$.

PROOF. By the identification of $H_*(\mathrm{Hom}_A(X, Y); \Delta)$ with $\mathrm{Ext}_A(M, F)$

using the row spectral sequence (equivalently, by a chain homotopy argument) there exist $f_j'\colon X_{j-1} \longrightarrow Y^{p+q-j}$ $(j = 1, \ldots, p+q)$, $f\colon X_{p+q} \longrightarrow F$ such that $\Sigma f_i - \eta f = \Delta(\Sigma f_j')$. Then

$$\Delta\Big((-1)^{\deg g} \sum g^{p+q-i} f_i'\Big) = \sum g^{p+q-i} f_i - gf,$$

so $\Sigma g^{p+q-i} f_i$ represents the same class as gf.

Now extend (2.14) to the right as follows: Let $0 \longleftarrow F \longleftarrow Z_0 \longleftarrow Z_1 \longleftarrow \cdots$ be a T-projective resolution of F and construct A-maps $h_i\colon X_{p+q-i} \longrightarrow Z_i$ to make the top squares of (2.15) below anticommute, and by a chain homotopy argument construct $g^{i\prime}$, g' so that $\Delta(\Sigma g^{i\prime}) = g\epsilon_0 - \eta g'$:

(2.15)
$$\begin{array}{ccccccccccc}
 & & & & X_{p+q} & \longleftarrow & X_{p+q+1} & \longleftarrow \cdots \longleftarrow & X_{p+q-r-1} & \longleftarrow & X_{p+q+r} \\
 & & & \swarrow f & \downarrow h_0 & & \downarrow h_1 & & \downarrow h_{r-1} & & \downarrow h_r \\
0 & \longleftarrow & F & \overset{\epsilon_0}{\longleftarrow} & Z_0 & \longleftarrow & Z_1 & \longleftarrow \cdots \longleftarrow & Z_{r-1} & \longleftarrow & Z_r \\
 & & g \searrow & & \downarrow g^{1\prime} & & \downarrow g^{2\prime} & & \downarrow g^{r-1\prime} & & \swarrow g' \\
 & & & Y^r & \longleftarrow Y^{r-1} & \longleftarrow & Y^{r-2} & \longleftarrow \cdots \longleftarrow & Y^0 & \overset{\eta}{\longleftarrow} & F \longleftarrow 0
\end{array}$$

Then $\Delta(\Sigma g^{i\prime} h_{i-1}) = gf - g'h_r$, so gf represents the same class as $g'h_r$. The result follows.

To show that the differentials are left $H^*(T)$-maps, consider the diagram:

(2.16)
$$\begin{array}{ccccc}
H_{p+q-1}(F^{p-r+1}/F^p) & \xrightarrow{D_r^1} & H_{p+q}(F^p/F^{p+1}) & \xrightarrow{D_r^2} & H_{p+q+1}(F^{p+1}/F^{p+r}) \\
 & & & \searrow D_{r+1} & \uparrow j \\
 & & & & H_{p+q+1}(F^{p+1}/F^{p+r+1}) \\
 & & & & \uparrow i \\
 & & & & H_{p+q+1}(F^{p+r}/F^{p+r+1})
\end{array}$$

The maps are those of the exact sequences of the respective triples.

By a standard construction of the spectral sequence of a filtered chain complex [9], we have $E_r^{p,q} = \operatorname{Ker} D_r^2/\operatorname{Im} D_r^1$. If $x \in \operatorname{Ker} D_r^2$ represents $\bar{x} \in E_r^{p,q}$, then $d_r(\bar{x}) = \bar{y}$ where $i(y) = D_{r+1}(x)$. In turn x is represented by a map f_q: $X_q \longrightarrow Y^p$ with $\delta f = 0$, and $D_{r+1}(x)$ is represented by $df_q\colon X_q \longrightarrow Y^{p+1}$. By exactness $D_{r+1}(x) \in \operatorname{Im} i$, so there exist $f_{q-j}'\colon X_{q-j} \longrightarrow Y^{p+j}$ $(j = 0, \ldots, r)$, $f_{q-r+1}\colon X_{q-r+1} \longrightarrow Y^{p+r}$ with $\Delta(\Sigma f_{q-j}') = f_{q-r+1} - df_q$. Let $z \in H^s(T)$ be represented by $g, g^0, \ldots, g^p$ as in (2.13), then zx is represented by

$$(-1)^{\deg z \deg x} g^p f_q.$$

But

$$\Delta\Big(\sum g^{p+j} f'_{q-j}\Big) = (-1)^{\deg z} g^{p+r} f_{q-r+1} - d(g^p f_q),$$

so $d_r(zx) = (-1)^{\deg z} z d_r(x)$.

This completes (1) of (2.7). Now (2) and (3) are clear, and (4) follows from (2.13) since $\rho_p([\Sigma f_i]) = [f_q]$.

The following result, needed for §3, may be established by a simple diagram.

PROPOSITION 2.17. *Let μ denote the T^*-coaction on $\operatorname{Ext}_S(M, F)$ in the change of rings spectral sequence, and let Y denote Yoneda product and t the twist map. The following diagram commutes*:

$$\begin{array}{ccc}
 & \xrightarrow{1 \otimes \mu} & H^*(S) \otimes T^* \otimes \operatorname{Ext}_S(M, F) \\
H^*(S) \otimes \operatorname{Ext}_S(M, F) & & \downarrow t \otimes 1 \\
\downarrow Y & & T^* \otimes H^*(S) \otimes \operatorname{Ext}_S(M, F) \\
\operatorname{Ext}_S(M, F) & & \downarrow 1 \otimes Y \\
 & \xrightarrow{\mu} & T^* \otimes \operatorname{Ext}_S(M, F)
\end{array}$$

3. **The factorization theorem.** This section is devoted to the proof of Theorem 1.9. For purposes of induction we prove the more general form below.

For s, t nonnegative integers define $E(s, t) = \bigwedge(Q_s, \dots, Q_t)$ if $t \geqslant s$ and $E(s, t) = Z_2$ for $t < s$. Let $E(s, \infty) = \bigcup_{t \geqslant s} E(s, t)$. Let $M = M(n)$ be as in (1.9).

THEOREM 3.1. *With the notation as above, there is an isomorphism*

$$\operatorname{Ext}_{E(s,\infty)}(M, Z_2) \cong Z_2[q_{n+s}, q_{n+s+1}, \dots] \otimes \operatorname{Ext}_{E(s,n+s-1)}(M, Z_2)$$

of $Z_2[q_{n+s}, q_{n+s+1} \cdots]$-modules, where the action on $\operatorname{Ext}_{E(s,n+s-1)}$ is trivial.

The proof of (3.1) relies on the spectral sequence of §2, taking as φ the inclusion $E(s, t) \subset E(s, t + 1)$. Thus

(3.2) $$E_2^{p,q} = \operatorname{Ext}^p_{\wedge(\beta_{t+2})}(Z_2, \operatorname{Ext}^q_{E(s,t)}(M, Z_2)).$$

The argument will show that the coaction

(3.3) $$\operatorname{Ext}_{E(s,t)}(M, Z_2) \longrightarrow \bigwedge(\beta_{t+2}) \otimes \operatorname{Ext}_{E(s,t)}(M, Z_2)$$

is trivial for $t \geqslant n + s - 1$; it follows that

$$\begin{aligned}
E_2 &\cong H^*(\bigwedge(Q_{t+1})) \otimes \operatorname{Ext}_{E(s,t)}(M, Z_2) \\
&= Z_2[q_{t+1}] \otimes \operatorname{Ext}_{E(s,t)}(M, Z_2) \quad \text{for } t \geqslant n + s - 1.
\end{aligned}$$

It will follow from the succeeding arguments that the spectral sequence collapses. The abelian group extensions are trivial since the Ext groups are vector spaces, and the action of q_{t+1} corresponds on both sides, so the theorem follows.

The required triviality of (3.3) will be established in two steps.

LEMMA 3.4. *The restriction of the coaction* (3.3) *to* $\mathrm{Ext}^0_{E(s,t)}(M, Z_2)$ *is trivial for* $t \geqslant n + s - 1$.

LEMMA 3.5. $\mathrm{Ext}_{E(s,t)}(M, Z_2)$ *is decomposable as an* $H^*(E(s, t))$*-module in terms of* $\mathrm{Ext}^0_{E(s,t)}(M, Z_2)$.

The triviality of (3.3) follows by (2.17).

In what follows, all coefficients in the homology or cohomology of a space are in Z_2.

We prove (3.4) and (3.5) for $M(n) = \widetilde{H}^*(BO(n))$; the results for $\widetilde{H}^*(MO(n))$ follow since the latter is a direct summand of $\widetilde{H}^*(BO(n))$ as $E(s, t)$-modules.

Turning to the proof of (3.4), consider the identifications

$$(3.6) \qquad H^*(MO(n)) \subset H^*(BO(n)) \overset{(Bi)^*}{\subset} H^*(BD(n)),$$

where $D(n)$ is the set of diagonal matrices in $O(n)$. The first inclusion is the Thom isomorphism which identifies $H^*(MO(n))$ with the ideal generated by w_n in $H^*(BO(n)) = Z_2[w_1, \dots, w_n]$. The second is induced by the inclusion i: $D(n) \subset O(n)$. Recall that $(Bi)^*$ is a monomorphism whose image is the algebra of symmetric polynomials in $H^*(BD(n)) = Z_2[V_1, \dots, V_n]$, and $(Bi)^*(w_j) = \sigma_j(V_1, \dots, V_n)$. Define the "exterior degree" of a monomial in the V_i to be the number of V_i which occur to an odd power. Similarly define the exterior degree of a polynomial if all of its monomial terms have a common exterior degree.

Dually, $H_*(BO(n))$ has a basis $x_{i_1} \cdots x_{i_n}$ for $i_j \geqslant 0$ (where $x_0 = 1$ in the multiplication on $H_*(BO)$) or equivalently $x_{i_1} \cdots x_{i_k}$, $1 \leqslant k \leqslant n$, $i_j \geqslant 1$, where $0 \neq x_j \in H_j(RP^\infty)$. Define the exterior degree of $x_{i_1} \cdots x_{i_k}$ as the number of i_j which are odd. Also note that the image of $H_*(BO(n-1))$ in $H_*(BO(n))$ is the span of monomials $x_{i_1} \cdots x_{i_k}$ for $k \leqslant n - 1$.

Write M_* for $\mathrm{Hom}(M, Z_2)$ and define D_i: $M_* \to M_*$ to be the coefficient of β_i in the coaction over E_*. Note D_i is a derivation with respect to the multiplication on $H_*(BO)$.

The triviality of the coaction (3.3) for $t \geqslant n + s - 1$ may now be restated: if $x \in M(n)_*$ and $D_{s+1}(x) = 0, \dots, D_{t+1}(x) = 0$, then $D_{t+2}(x) = 0$. Since $M(n)_* \subset M(n+1)_*$ it will be sufficient to show

LEMMA 3.7. *If* $x \in M(n)_*$ *and* $D_{s+1}(x) = 0, \dots, D_{s+n}(x) = 0$, *then* $D_{s+n+1}(x) = 0$.

For use in (3.5) we state a more general version.

LEMMA 3.8. *Suppose* $x \in M(n)_*$ *is in the span of monomials of exterior* *degree* $\leq k - 1$, *and suppose* $D_{s+1}(x) = 0, \ldots, D_{s+k}(x) = 0$. *Then* $D_{s+k+1}(x) = 0$.

Note that (3.8) implies (3.7) by taking $k = n$, since monomials of exterior degree n in $M(n)_*$ are in fact primitive.

Lemma 3.8 will be proved by dualizing to take advantage of the ring structure of $H^*(BD(n))$. We have

LEMMA 3.9. *If* $y \in \widetilde{H}^*(BO(n)) \subset H^*(BD(n))$ *is spanned by monomials of exterior degree* $\leq k$, *then* $Q_{s+k}y = \Sigma_{i=s}^{s+k-1} Q_i y_i$, *where* $y_i \in \widetilde{H}^*(BO(n))$.

Next recall that the Q_i are primitive in the Steenrod algebra [11] and hence act as (mod 2) derivations, so they commute with squared elements. Now $H^*(BD(n))$ is a free module over its subring $Z_2[V_1^2, \ldots, V_n^2]$ by multiplication, and the indecomposables of the form $V_{i_1} \cdots V_{i_p}$ may be identified, after renumbering, with the Thom class $u_p \in H^p(MO(p))$. We have

PROPOSITION 3.10. *To establish* (3.9) *it suffices to consider the case where* $y = u_k \in \widetilde{H}^k(MO(k))$ *and* $y_s, \ldots, y_{s+k-1} \in \widetilde{H}^*(MO(k))$ *have exterior degree* k.

To prove (3.10), let $\epsilon_m: S^1 \wedge MO(m) \longrightarrow MO(m+1)$ denote the mth structure map of the Thom spectrum MO, and let Σ be the cohomology suspension. Then the composite

$$\widetilde{H}^{q+1}(MO(m+1)) \xrightarrow{m^*} \widetilde{H}^{q+1}(S^1 \wedge MO(m)) \underset{\cong}{\xleftarrow{\Sigma}} \widetilde{H}^q(MO(m))$$

is an E-homomorphism, maps u_{m+1} to u_m, and reduces the exterior degree of each monomial by 1. Thus (3.9) for k implies its analog for each $p < k$.

Next note that Q_i reduces the exterior degree of each monomial by 1, so $Q_{s+k}u_k$ has exterior degree $k-1$, and the y_i may be assumed to have exterior degree k.

Since y is symmetric and spanned by monomials of exterior degree $\leq k$ it may be written

$$y = \sum_{p=1}^{k} \sum_{\sigma(1)<\cdots<\sigma(p)} f_{p\sigma} V_{\sigma(1)} \cdots V_{\sigma(p)}$$

where $\sigma \in S_n$ and $f_{j\sigma} = f_j(V_{\sigma(1)}^2, \ldots, V_{\sigma(n)}^2)$. By the first part of the proof, we may write

$$Q_{s+k} V_1 \cdots V_p = \sum_{i=1}^{s+k-1} Q_i y_i^p(V_1, \ldots, V_p);$$

let $y_{i\sigma}^p = y_i^p(V_{\sigma(1)}, \ldots, V_{\sigma(p)})$. By the above remarks,

$$Q_{s+k} y = \sum_{i=1}^{s+k-1} Q_i \left(\sum_{p=1}^{k} \sum_{\sigma(1)<\cdots<\sigma(p)} f_{p\sigma} y_{i\sigma}^p \right).$$

It remains to show the polynomials in parentheses are symmetric so that they represent elements $y_i \in \widetilde{H}^*(BO(n))$.

By the symmetry of y each sum $\Sigma_\sigma f_{p\sigma} V_{\sigma(1)} \cdots V_{\sigma(p)}$ for fixed p is symmetric and so consists of the sum of the terms in the orbit of $f_{pe} V_1 \cdots V_p$ where $e \in S_n$ is the identity. Now $y_{ie}^p \in \widetilde{H}^*(MO(p))$ has exterior degree p so may be written $y_{ie}^p = g_i^p(x_1^2, \ldots, x_p^2) V_1 \cdots V_p$ where g_i^p is symmetric in its p indeterminates. Thus if σ fixes $f_{pe} V_1 \cdots V_p$ it also fixes $f_{pe} y_{ie}^p$, so (writing C for stabilizer) $C_{f_{pe} V_1 \cdots V_p} \subset C_{f_{pe} y_{ie}^p}$ hence

$$|S_n : C_{f_{pe} y_{ie}^p}| \leqslant |S_n : C_{f_{pe} V_1 \cdots V_p}| = \binom{n}{p}.$$

On the other hand $f_{pe} y_{ie}^p$ has at least the $\binom{n}{p}$ elements in its orbit corresponding to the combinations of indeterminates. Thus $\Sigma_\sigma f_{p\sigma} y_{i\sigma}^p$ is symmetric, completing (3.10).

To prove the condition of (3.10) it will be convenient to use the notation of §1, where $\widetilde{H}_*(MO(k))$ is the span of monomials $b^E = b_1^{e_1} b_2^{e_2} \cdots$ of degree $\leqslant k$. If $E = (e_1, e_2, \ldots)$ let $|E| = \Sigma e_i$ and grade $b^E = \Sigma i e_i$, that is b^E is assigned its "stable grade." Note that $1 \in H_0(MO)$ is the unique element dual to each Thom class u_k. If we form a minimal resolution

$$0 \longrightarrow H_*(MO(k)) \longrightarrow E(s, s+k)_* \otimes V \longrightarrow \cdots$$

then the existence of the relation of (3.10) is equivalent to the statement $\beta_{s+k+1} \otimes 1_V \notin \operatorname{Im} \epsilon$, where $\epsilon(1) = 1 \otimes 1_V$. In particular it is not necessary to exhibit the elements y_i.

Denote by $H \subset V$ the span of images of monomials in $b_2, b_4, \ldots, b_2r, \ldots$ and identify elements of H by their polynomial pre-images. Let $h\colon V \longrightarrow H$ be the projection induced through ϵ from the one in $\widetilde{H}_*(MO(k))$ which preserves monomials in the b_2r and annihilates all others. Let $\epsilon' = (1 \otimes h)\epsilon$, then by the coaction formula (1.8) it will suffice to show $\beta_{s+k+1} \otimes 1 \notin \operatorname{Im} \epsilon'|_W$, where $W \subset \widetilde{H}_*(MO(k))$ (with stable grading) is the span of monomials containing only one factor b_i of odd degree.

PROPOSITION 3.11. *Let $y = b_{2m-1} b^E \in W_{2^{s+k+1}-1}$, then $\epsilon'(y)$ is either zero or the sum of precisely two "monomials," i.e. terms of the form $\beta_j \otimes z$ where z is a monomial of H.*

For the proof, suppose $\epsilon'(y) \neq 0$, then $b^E \in H$. Let M denote the term in $\epsilon'(y)$ containing the β_j of highest grade; it has the form $\beta_i \otimes b^E$ or $\beta_i \otimes b_2 r b^E$.

If $M = \beta_i \otimes b^E$, then $D_i(b_{2m-1}) = 1$, so $2m - 1 = 2^i - 1$. Since $D_i(b_{2m-1}) \in H$, $2m - 2^q$ must be zero or a positive power of 2; i.e. $q = i$ or $q = i - 1$. Thus for $i \geqslant s+2$, $s+1 \leqslant i-1$, $i \leqslant s+k+1$ and (3.11) holds.

Suppose $i = s + 1$, then $|E| = (2^{s+k+1} - 1) - (2^{s+1} - 1)$ so $\alpha(|E|) \geqslant k$, where α denotes the number of 1's in the dyadic expansion. Thus deg $b^E \geqslant k$, deg $y \geqslant k + 1$, a contradiction since $y \in \widetilde{H}_*(MO(k))$.

If $M = \beta_i \otimes b_{2^r} b^E$, then $r \neq i$, otherwise the term $\beta_{i+1} \otimes b^E$ would also appear, since $i \neq s + k + 1$ by the grade of y. Now $r \leqslant s + k + 1$ by the grade of y, so $\epsilon'(y) = \beta_i \otimes b_{2^r} b^E + \beta_r \otimes b_{2^i} b^E$ and (3.11) holds, provided $r \geqslant s + 1$.

Assume $r \leqslant s$. Note $i \leqslant s + k$ by the grade of y. Then

$$\begin{aligned} 2^{s+k+1} - 1 > |E| &= (2^{s+k+1} - 1) - (2m - 1) \\ &= 2^{s+k+1} - 2^i - 2^r \geqslant 2^{s+k+1} - 2^{s+k} - 2^r \\ &= 2^{s+k} - 2^r \geqslant 2^{s+k} - 2^s, \end{aligned}$$

so $\alpha(|E|) \geqslant \alpha(2^{s+k} - 2^s) = k$. Thus deg $b^E \geqslant k$, so deg $y \geqslant k + 1$, a contradiction.

COROLLARY 3.12. *Every element of* Im $\epsilon'|_W$ *is the sum of an even number of "monomials." In particular* $\beta_{s+k+1} \otimes 1 \notin$ Im ϵ'.

This completes (3.11), (3.9) and (3.4).

The proof of (3.5) is by induction on t. For $t < s$ we have $E(s, t) = Z_2$ so the result holds trivially.

Assume (3.5) for t and all $s \geqslant 0$; this requires only finitely many steps. Apply the spectral sequence (3.2). By (2.5) and (2.12) it has a module structure over $H^*(E(s, t)) = Z_2[q_s, \ldots, q_t]$ which is induced from the Yoneda product on $\mathrm{Ext}_{E(s,t)}(M, Z_2)$ in E_2. It also has the structure $\{\theta_r\}$ over $Z_2[q_{t+1}]$. By naturality these structures commute and induce a structure over $H^*(E(s, t + 1)) = Z_2[q_s, \ldots, q_{t+1}]$ which restricts to the given ones. We show E_2 is decomposable over $H^*(E(s, t + 1))$.

Let $T = \bigwedge(\beta_{t+2})$. By the cobar construction, $\mathrm{Ext}_{T^*}(Z_2, N)$ is decomposable over $H^*(T)$ in terms of $\mathrm{Ext}^0_{T^*}$ for any T^*-comodule N. Thus E_2 is decomposable over $H^*(T)$ in terms of

$$\mathrm{Ext}^0_{T^*}(Z_2, \mathrm{Ext}_{E(s,t)}(M, Z_2)) \subset \mathrm{Ext}_{E(s,t)}(M, Z_2).$$

By the inductive hypothesis, any $x \in \mathrm{Ext}_{E(s,t)}$ is decomposable over $H^*(E(s, t))$ in terms of $x_i \in \mathrm{Ext}^0_{E(s,t)}$. The x_i, however, may not lie in

$$\mathrm{Ext}^0_{T^*}(Z_2, \mathrm{Ext}^0_{E(s,t)}(M, Z_2)) = E_2^{0,0}.$$

LEMMA 3.13. *Let* $z \in \mathrm{Ext}^0_{E(s,t)}(M, Z_2)$. *Then z may be written* $z = z_1 + z_2$ *where* $z_1 \in \mathrm{Ext}^0_{E(s,t)}$, $z_2 \in \mathrm{Ext}^0_{E(s,t+1)} \subset \mathrm{Ext}^0_{E(s,t)}$ *and* $q_s z_1 = 0, \ldots, q_t z_1 = 0$.

Using (3.13) we can replace the x_i by $x_i' \in \mathrm{Ext}^0_{E(s,t+1)}$ since the terms corresponding to z_1 are annihilated by $\overline{H^*(E(s, t))}$. Note that the x_i' are permanent cycles. The differentials commute (in the graded sense) with the composite action of $H^*(E(s, t + 1))$ on E_r so the spectral sequence collapses. The abelian group extensions are trivial and the spectral sequence preserves products in the sense of (2.5) and (2.7), so $\mathrm{Ext}_{E(s,t+1)}$ is decomposable over $H^*(E(s, t + 1))$ by induction over the filtration.

To prove (3.13), write $z \in M(n)_* = \widetilde{H}_*(BO(n))$ in terms of the basis $\{x_{i_1} \cdots x_{i_p}\}$, $1 \leq p \leq n$, $i_p \geq 1$, and let z_2 denote the sum of terms of exterior degree $\leq t - s$. By (3.8) (with $k = t - s + 1$) we have $D_{t+2}(z_2) = 0$, so $z_2 \in \mathrm{Ext}^0_{E(s,t+1)}$. Let $z_1 = z - z_2$.

LEMMA 3.14. *Suppose $x \in \mathrm{Ext}^0_{E(s,t)}(M(n), Z_2)$ is spanned by monomials $x_{i_1} \cdots x_{i_p}$ of exterior degree $\geq t - s + 1$. Then $q_s x = 0, \ldots, q_t x = 0$ in $\mathrm{Ext}_{E(s,t)}(M(n), Z_2)$.*

The proof will be by induction on n.

REMARK 3.15. Let $s \leq i \leq t$ and let N be an $E(s, t)$-module. Examination of a resolution shows that $q_i x = 0$ in $\mathrm{Ext}_{E(s,t)}(N, Z_2)$ if and only if there exists $y \in N_*$ with $D_{i+1}(y) = x$, $D_j(y) = 0$ for $j = s + 1, \ldots, t + 1, j \neq i + 1$.

As in (3.8) the proof will exploit ring structure, in this case the multiplication of $H^*(BO(n)) = Z_2[w_1, \ldots, w_n]$. Let $U = Z_2[w_n^2]$. The Q_i commute with the multiplicative action of U, so $H^*(BO(n))$ and $M(n) = \widetilde{H}^*(BO(n))$ are $E(s, t) \otimes U$-modules. The following description of $\mathrm{Ext}^0_U(M(n), Z_2)$ will be used in a spectral sequence argument.

PROPOSITION 3.16. *There is an isomorphism*

$$f\colon \mathrm{Ext}^0_U(M(n), Z_2) \longrightarrow M(n-1)_* \oplus \widetilde{H}_*(MO(n-1))$$

of $E(s, t)_*$-comodules.

For the proof, consider $E(s, t)_*$-maps $p_1\colon M(n)_* \longrightarrow M(n - 1)_*$, $p_2\colon M(n)_* \longrightarrow \widetilde{H}_*(MO(n))$ defined by $p_1(x_{i_1} \cdots x_{i_k}) = x_{i_1} \cdots x_{i_k}$ for $k < n$, $p_1(x_{i_1} \cdots x_{i_n}) = 0$; $p_2(x_{i_1} \cdots x_{i_n}) = x_{i_1} \cdots x_{i_n}$, $p_2(x_{i_1} \cdots x_{i_k}) = 0$ for $k < n$ ($i_j \geq 1$ throughout). Let $\Delta_n\colon M(n)_* \longrightarrow M(n)_*$ be the map dual to multiplication by w_n. Then $\mathrm{Ker}\, \Delta_n = M(n - 1)_*$ and $\mathrm{Ker}\, \Delta_n^2 = \mathrm{Ext}^0_U(M(n), Z_2)$. Thus if $x \in \mathrm{Ext}^0_U$, $\Delta_n^2 p_2(x) = 0$, so $\Delta_n p_2(x) \in M(n - 1)_*$. Since $\Delta_n(x_{i_1} \ldots x_{i_n}) = x_{i_1 - 1} \cdots x_{i_n - 1}$ ($x_0 = 1$), $p_2(x)$ has the form $x_1 h + u$ where $h \in \widetilde{H}_*(MO(n-1))$, $u \in \mathrm{Ker}\, \Delta_n$. By definition of p_2, u is a sum of terms $x_{i_1} \cdots x_{i_n}$, $i_j \geq 1$, so $u \in \widetilde{H}_*(MO(n))$. But also $u \in M(n - 1)_*$, so $u = 0$, $p_2(x) = x_1 h$. Multiplication by x_1 is an $E(s, t)_*$-monomorphism, so define $f(x) = (p_1(x), h)$. The inverse map is given by $(y, z) \longmapsto y + x_1 z$.

For the proof of (3.14), note that D_i reduces the exterior degree of each term $x_{i_1} \cdots x_{i_k}$ by 1 so we may assume x of (3.14) has homogeneous exterior degree $q \geqslant t - s + 1$. For $n = 1$ the highest exterior degree that can occur is 1, so $t = 0$. For $s > 0$, $E(s, 0) = Z_2$ so the result is trivial. For $s = 0$, if $D_1(x) = 0$, $x \in \widetilde{H}_*(BO(1)) = \widetilde{H}_*(RP^\infty)$ has exterior degree 1, so $x = x_{2r-1}$ for some r, and $x = D_1(x_{2r})$. Thus $q_0 x = 0$ by (3.15).

Assume (3.14) for $M(n-1)$ and suppose $x \in \mathrm{Ext}^0_{E(s,t)}(M(n), Z_2)$ has exterior degree $q \geqslant t - s + 1$ and $q_i x \neq 0$ for some i, $s \leqslant i \leqslant t$. Let m be the largest integer r such that $\Delta_n^{2r}(q_i x) \neq 0$, then

$$u = \Delta_n^{2m}(q_i x) \in \mathrm{Ext}^0_{U^*}(Z_2, \mathrm{Ext}^1_{E(s,t)}(M(n), Z_2)),$$

which is $E_2^{0,1}$ in the change of rings spectral sequence converging to $\mathrm{Ext}_{E(s,t)\otimes U}$. No differentials can hit u, and the only possible differential on $E_2^{0,1}$ is

$$d_2\colon E_2^{0,1} \longrightarrow E_2^{2,0} = \mathrm{Ext}^2_U(\mathrm{Tor}_0^{E(s,t)}(Z_2, M), Z_2)$$

and $E_2^{2,0} = 0$ since U is a PID. Let $0 \neq u' \in \mathrm{Ext}^1_{E(s,t)\otimes U}$ represent the class of u in E_∞.

CLAIM. $\mathrm{Ext}_{E(s,t)\otimes U}(M(n), Z_2))$ is decomposable by Yoneda products over $H^*(E(s, t))$ (acting by the homomorphism induced from the projection $E(s, t) \otimes U \longrightarrow E(s, t)$) in terms of $\mathrm{Ext}^0_{E(s,t)\otimes U}$. We have the spectral sequence

$$\mathrm{Ext}^a_{E(s,t)_*}(Z_2, \mathrm{Ext}^b_U(M(n), Z_2)) \Rightarrow \mathrm{Ext}^{a+b}_{E(s,t)\otimes U}(M(n), Z_2)$$

with the product structure $\{\theta_r\}$. Since U acts freely on $M(n)$, $E_2^{a,b} = 0$ for $b \neq 0$ and the spectral sequence collapses with trivial extensions. By the inductive hypothesis, (3.14) holds for $\widetilde{H}^*(BO(n-1))$ and thus for $\widetilde{H}^*(MO(n-1))$, so by (3.16) $E_2 = E_\infty$ is decomposable, and the claim follows.

Thus $u' = \Sigma_{r=s}^t q_r z_r$, $z_r \in \mathrm{Ext}^0_{E(s,t)\otimes U}$, and by the coaction formula (1.8) we may assume each z_r has exterior degree q. Let $j\colon E(s, t) \longrightarrow E(s, t) \otimes U$ be the inclusion, then $0 \neq u = j^*(u') = \Sigma_{r=s}^t q_r j^*(z_r)$, so for some z_v, $q_v j^*(z_v) \neq 0$ in $\mathrm{Ext}^1_{E(s,t)}(M, Z_2)$, where $w = j^*(z_v)$ has exterior degree q and $\Delta_n^2(w) = 0$. This will be shown to be impossible if $q \geqslant t - s + 1$.

Let $y = p_1(w)$, $z = p_2(w)$ where p_1, p_2 are as in (3.16). Then $y \in M(n-1)_*$ has exterior degree q, so by induction $q_s y = 0, \ldots, q_t y = 0$ in $\mathrm{Ext}_{E(s,t)}(M(n-1), Z_2)$. By naturality the images of the $q_j y$ are zero in $\mathrm{Ext}_{E(s,t)}(M(n), Z_2)$.

As in (3.16), $z = x_1 h$, where $h \in \widetilde{H}_*(MO(n-1))$ has exterior degree $q - 1 \geqslant (t - s + 1) - 1 = t - (s + 1) + 1$. By induction on n, we have $q_{s+1} h = 0, \ldots, q_t h = 0$ in where the inclusion is induced by the projection of $M(n-1)$ onto the direct summand $\widetilde{H}^*(MO(n-1))$. Thus by (3.15) there exist $z_{s+1}, \ldots, z_t \in \widetilde{H}_*(MO(n-1))$ with $D_{i+1}(z_i) = h$, $D_j(z_i) = 0$ for $j = s+2, \ldots, t+1, j \neq i+1$. Let

$$z'_{x+1} = x_{2^{s+1}} h \in M(n)_*,$$

$$z'_k = x_1 z_k + x_{2^{s+1}} D_{s+1}(z_k) \in M(n)_*, \quad k = s+2, \ldots, t+1.$$

Recall that the D_i are derivations satisfying $D_i D_j = D_j D_i$, $D_i^2 = 0$. By an easy calculation,

$$D_{i+1}(z'_i) = z, \quad i = s, \ldots, t,$$

$$D_j(z'_i) = 0, \quad j = s+1, \ldots, t+1, j \neq i+1.$$

Thus $q_i w = q_i y + q_i z = 0$ in $\mathrm{Ext}_{E(s,t)}(M(n), Z_2)$, contradicting $q_v w \neq 0$. This completes (3.14), (3.5) and (3.1).

Note. The last calculation relies on the fact that $D_i(x_{2^{s+1}}) = 0$ for $i > s+1$, which requires the inductive step to treat the lowest dimensional generator, q_s. This is the reason for the generality of (3.1).

We note in closing that Theorem 3.1 is the best possible result of its type, i.e. $n+s-1$ cannot be replaced by $n+s-2$. Let

$$x = D_{s+1} D_{s+2} \cdots D_{s+n-1}(x_{2^{s+1}} \cdots x_{2^{s+n}}).$$

Then $x \in \mathrm{Ext}^0_{E(s,n+s-2)}(\widetilde{H}^*(BO(n); Z_2), Z_2)$, but $D_{s+n}(x) = x_1^n \neq 0$, so $x \notin \mathrm{Ext}^0_{E(s,n+s-1)}$.

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