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SIMPLICIAL TRIANGULATION OF NONCOMBINATORIAL MANIFOLDS OF DIMENSION LESS THAN 9

BY

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ABSTRACT. Necessary and sufficient conditions are given for the simplicial triangulation of all noncombinatorial manifolds in the dimension range $5 \leq n \leq 7$, for which the integral Bockstein of the combinatorial triangulation obstruction is trivial. A weaker theorem is proven in case $n = 8$.

The appendix contains a proof that a map between PL manifolds which is a TOP fiber bundle can be made a PL fiber bundle.

0. Two of the oldest and most difficult problems arising in manifold theory are the following:

(i) Is every manifold homeomorphic to a simplicial complex?

(ii) Is every simplicial triangulation of a manifold combinatorial (i.e. must the link of every simplex be a sphere)?

Among the consequences of the fundamental breakthrough of Kirby-Siebenmann [9] was that at least one of these questions must be answered negatively, for there are topological manifolds without combinatorial (PL) triangulations.

The existence of a counterexample to the second question is equivalent to the following conjecture: *There is some homology m-sphere K, not* PL *equivalent to S^m such that the p-fold suspension $\Sigma^p K$ is homeomorphic to S^{m+p}.*

Siebenmann shows that if the answer to question (i) is affirmative for manifolds of dimension $n \geq 5$, then the following hypothesis is true for $m = n - 3$:

Hypothesis $H(m)$. There is a PL *homology* 3*-sphere K such that $\Sigma^m K \simeq S^n$, and K bounds a* PL *manifold of index* 8 (mod 16) (*i.e. the Rochlin invariant of K is nontrivial*).

Furthermore, if hypothesis $H(2)$ is true, then all orientable 5-manifolds are simplicially triangulable [15].

The purpose of this paper is to prove

0.1. THEOREM. *Let N^n be a connected closed noncombinatorial manifold of dimension $5 \leq n \leq 8$, and let $k_N \in H^4(N; Z_2)$ be the obstruction to the*

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existence of a PL *structure on N. Suppose the integral Bockstein homomorphism*

$$\beta\colon H^4(N; Z_2) \longrightarrow H^5(N; Z)$$

maps k_N *to zero. Then*

(i) *For* $5 \leqslant n \leqslant 7$ *hypothesis* $H(n-3)$ *is a necessary and sufficient condition for the existence of a simplicial triangulation of N.*

(ii) *For* $n = 8$, $k_N \cup k_N = 0$ *and hypothesis* $H(2)$ *together imply that N is simplicially triangulable.*

REMARK 1. We do not assume that N is orientable. However, the assumption $\beta(k_N) = 0$ is an orientability assumption of sorts. See §1. If N is simply connected, then $\beta(k_N) = 0$ implies that N has the homotopy type of a PL manifold [4]. A pleasant corollary of the theorem is that for $n = 5$, if $w^1(N) \cdot k(N) = 0$, hypothesis $H(2)$ is sufficient for N to have a triangulation.

ADDED IN PROOF (FEBRUARY 1976). R. Edwards has announced that for $m \geqslant 4$, $\Sigma^2 K^m \simeq S^{m+2}$. Using this result, Matumoto and Galewski-Stern have announced necessary and sufficient conditions for the triangulation of all n-manifolds, $n \geqslant 5$.

Operating under the assumptions of Theorem 0.1, the argument proceeds as follows:

§1. It is possible to represent k_N by a codimension 4 smoothable submanifold with smooth orientable normal bundle.

§2. Any such bundle has a simplicial triangulation which restricts to an exotic PL structure on the sphere bundle boundary of the total space. The triangulation is constructed by defining a space X, showing that X is homeomorphic to the bundle space, and simplicially triangulating X in the required manner.

§3. The proof of Theorem 0.1.

I would like to thank the referee for suggestions leading to considerable abbreviation of the original manuscript.

1. **Representing triangulation obstructions by submanifolds.** The following well-known observation is used throughout the proof.

Suppose a connected closed m-manifold M is imbedded in a closed $m + p$ manifold N with normal p-disk bundle $\bar{\nu}(M)$. Let $\dot{\nu}(M)$ be the $p - 1$ sphere bundle boundary of $\nu(M)$, $i\colon M \longrightarrow N$ the inclusion, $e\colon H^*(N, N - M) \longrightarrow H^*(\bar{\nu}(M), \dot{\nu}(M))$ the excision isomorphism and $j\colon (N, \varnothing) \longrightarrow (N, N - M)$ the inclusion of pairs. Let $U \in H^p(\bar{\nu}(M), \dot{\nu}(M); Z_2)$ be the Thom class of $\bar{\nu}(M)$ and $[M] \in H_m(M; Z_2)$ the fundamental class of M.

1.1. LEMMA. *The Poincaré dual of* $i_*[M]$ *in N is the image of U under the composition*

$$H^*(\bar{\nu}(M), \dot{\nu}(M); Z_2) \xrightarrow{e^{-1}} H^*(N, N - M; Z_2) \xrightarrow{j^*} H^*(N; Z_2).$$

NOTATION. For any space X let $r\colon H^*(X; Z) \longrightarrow H^*(X; Z_2)$ be the mod 2 reduction homomorphism. Throughout the paper the notation e and j will always refer to excision homomorphisms and inclusion maps of pairs, respectively.

The following theorem is the central result of this section.

1.2. THEOREM. *Let N^n be a connected closed n-manifold, $5 \leqslant n \leqslant 8$, and let $k_N \in H^4(N; Z_2)$ be the obstruction to the existence of a* PL *structure on N. Then the following conditions are equivalent*:

(i) *There is an imbedding of a connected smooth $n-4$ manifold $M \xrightarrow{i} N$ such that M has an orthogonal oriented (i.e. $SO(4)$) normal bundle in N with $w^4 = 0$ and the image in $H_{m-4}(N; Z_2)$ of the fundamental Z_2 homology class $[M]$ of M is Poincaré dual to k_N.*

(ii) $\beta(k_N) = 0$. *For $n = 8$, $k_N \cup k_N = 0$.*

PROOF OF 1.2. (i) ⇒ (ii). Let $\bar{\nu}(M)$ be the oriented orthogonal normal bundle to M in N with sphere bundle boundary $\dot{\nu}(M)$, and let U be the Thom class in $H^4(\bar{\nu}(M), \dot{\nu}(M); Z)$. By Lemma 1.1, $i^*[M]$ is Poincaré dual to $j^*e^{-1}r(U)$ in N. Hence $k_N = j^*e^{-1}r(U)$. Since the Bockstein is natural, $\beta k_N = j^*e^{-1}\beta r(U) = 0$.

It remains to show that for $n = 8$, $k_N \cup k_N = 0$. Note first that if $\nu(M)$ is the total space of the normal bundle, $\nu(M)$ is smooth. Hence k_N pulled back to $\nu(M)$ is 0, or $i^*(k_N) = 0$. Let x be the Poincaré dual of k_N. Since $n = 8$, $k_N \cup k_N = 0$ if and only if $k_N \cap x = 0$. But $k_N \cap x = k_N \cap i_*[M] = i_*(i^*k_N \cap [M]) = 0$.

(ii) ⇒ (i). Since $\beta(k_N) = 0$ there is an α in $H^4(N; Z)$ such that $r(\alpha) = k_N$. Since $n \leqslant 8$ it follows from [18] that there is a map $N \longrightarrow MSO(4)$ which pulls back the Thom class of $MSO(4)$ to α.

For $n < 8$ f can be made TOP transverse to $BSO(4)$ in $MSO(4)$ [8] so that $f^{-1}(BSO(4)) = M$ is an m-manifold with normal $SO(4)$ bundle in N. M is smoothable and w^4 of the normal bundle is trivial since $m \leqslant 3$, and by standard arguments $i_*[M]$ is Poincaré dual to k_N.

In [13] a codimension 4 TOP transversality theory is developed which here implies that when $n = 8$ we may take $f^{-1}(BSO(4)) = M$ to be a homology manifold equipped with an open neighborhood W such that the inclusion $W - M \longrightarrow W$ has the homotopy type of a 3-spherical fibration over M and the Thom class U in $H^4(W, W - M)$ satisfies $k_N = j^*r(U)$. Since $k_N \cup k_N = 0$ and j^* is an isomorphism in dimension 8 it follows easily that $k_N|W = 0$ and so W is smoothable. Hence by smooth transversality, M can be a smooth manifold, W an $SO(4)$ normal bundle and again $i_*[M]$ is dual to k_N.

To show $w^4 = 0$ it suffices to show $r(U) \cup r(U) = 0$ [10]. Since j^*e^{-1} is an isomorphism in dimension 8, this follows immediately from $k_N \cup k_N = 0$.

It is well known that in all dimensions M may be assumed connected, and the proof is complete.

2. **Simplicial triangulation of oriented 4-disk bundles.** Once the triangulation obstruction is represented by a smooth manifold M with normal bundle ξ, we hope to simplicially triangulate near M in such a way that the simplicial triangulation extends to a PL triangulation away from M.

In case N is non-PL and $n = 8$, there is hope for this procedure only if $w^4(\xi) = 0$. For consider the following portion of the Thom-Gysin cohomology sequence for $\bar{\xi}$ (Z_2 coefficients)

$$0 \longrightarrow H^3(M) \xrightarrow{p^*} H^3(\dot{\xi}) \longrightarrow H^0(M) \xrightarrow{\psi} H^4(M) \longrightarrow H^4(\dot{\xi}).$$

The image of the fundamental class in $H^0(M)$ under ψ is $w^4(\bar{\xi})$ [10, Theorem 12].

If $w^4(\bar{\xi}) \neq 0$, then p^* is an isomorphism and, since p^* is induced by the inclusion $H^3(M) \simeq H^3(\bar{\xi}) \xrightarrow{i^*} H^3(\dot{\xi}), i^*$ is an isomorphism. The classification of PL structures is natural with respect to codimension 0 inclusions [9] so i must induce an isomorphism between PL structures on $\bar{\xi}$ and on a collar neighborhood of $\dot{\xi}$. By the product structure theorem [6] i then induces an isomorphism between PL structures on $\dot{\xi}$ and on $\bar{\xi}$. That is, any PL structure on $\dot{\xi}$ is the restriction of some PL structure on $\bar{\xi}$.

However, if $w^4(\xi) = 0$ then the cokernel of i^* is Z_2, and there are order$(H^3(M; Z_2))$ isotopy classes of PL structures on $\dot{\xi}$ which do not extend to $\bar{\xi}$.

2.1. THEOREM. *Let $\bar{\xi}$ be an oriented smooth closed 4-disk bundle over a connected closed smooth manifold M^m of dimension $m = 2, 3, 4$, and let $\dot{\xi}$ be the sphere bundle boundary of $\bar{\xi}$. Assume $w^4(\bar{\xi}) = 0$ and there is a homology 3-sphere satisfying $H(m + 1)$ for $m = 2, 3$ or $H(2)$ for $m = 4$. Then at least one of the* order $H^3(M; Z_2)$ PL *structures on $\dot{\xi}$ which do not extend to* PL *triangulations of $\bar{\xi}$ does extend to a simplicial triangulation.*

REMARKS. At the end of this section we consider how many of these PL structures on $\dot{\xi}$ extend simplicially.

It is well to recall in the following that hypothesis $H(m)$ for a homology sphere K implies that cone$(K) \times R^{m-1}$ is a manifold [3].

A. *Proof of* 2.1. *Preliminary remarks and notation.* The primary obstruction to trivializing $\bar{\xi}$ lies in $H^2(M; \pi_1(SO(4)))$. There is an $m - 2$ manifold J smoothly imbedded in M such that the inclusion of its fundamental Z_2 homology class $[J]$ represents the Poincaré dual to this obstruction [18]. If the obstruction is trivial, let J be a trivially imbedded $(m - 2)$-sphere for $m = 3, 4$ and a point for $m = 2$. By a well-known argument, we may assume J is connected for $m = 3, 4$. Since, for $m = 2, H^2(M - \text{(point)}) = 0$, we may assume J is connected in this dimension also.

NOTATION (SEE FIGURE 1).

$\xi \equiv \bar{\xi} - \dot{\xi}$.

$K \equiv$ homology sphere satisfying $H(i)$ for relevant i.

$cK \equiv$ cone on K with vertex $*$.

$\mathring{c}K \equiv cK - K \equiv$ open cone on K.

$i\nu(J) \equiv$ normal open tubular neighborhood of J in M of radius $i = 1, 2$ for some fixed Riemannian metric on M.

$i\bar{\nu}(J) \equiv$ closure of $i\nu(J)$ in M.

$i\dot{\nu}(J) \equiv i\bar{\nu}(J) - i\nu(J)$.

$i\bar{D} \equiv$ For some Riemannian metric on $K \times (0, 1)$, $i\bar{D}$ is a closed 4-disk PL imbedded in $K \times (0, 1) \subset \mathring{c}K$ of radius $i = 1, 2$.

$p \equiv$ center of $\bar{D}$.

$iD \equiv$ interior of $i\bar{D}$.

$i\dot{D} \equiv i\bar{D} - iD$.

$i\eta \equiv (M - \nu(J)) \times iD \subset (M - \nu(J)) \times cK$, a trivial normal bundle to the imbedding $(M - \nu(J)) \times \{p\} \longrightarrow (M - \nu(J)) \times cK$.

$i\bar{\eta} \equiv (M - \nu(J)) \times i\bar{D}$, the closure of $i\eta$.

$i\dot{\eta} \equiv (M - \nu(J)) \times i\dot{D} \equiv i\bar{\eta} - i\eta$.

$[M] \equiv$ fundamental homology class of $H^m(M; Z_2)$.

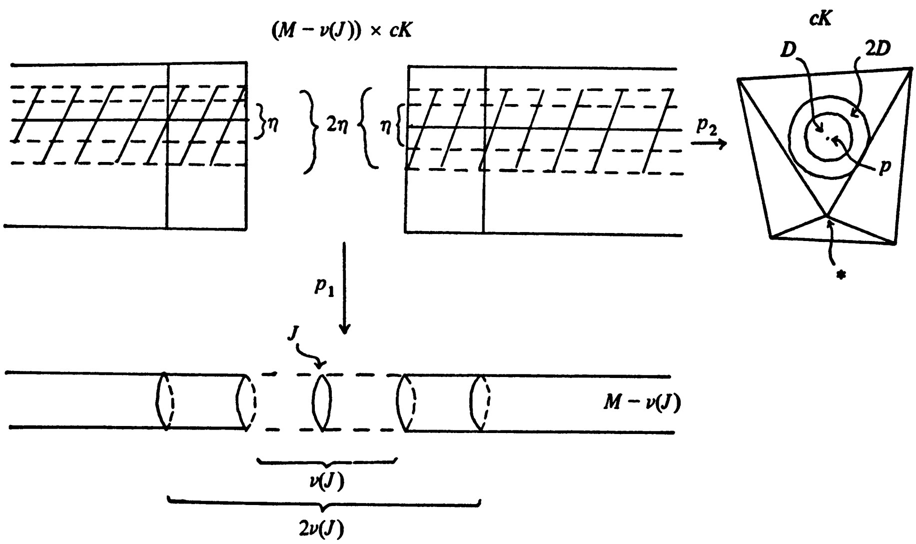

FIGURE 1

B. *Construction of spaces X and X′*. By 1.1 the first obstruction to trivializing ξ is in

$$\text{image}(H^2(M, M - J; Z_2) \xrightarrow{j^*} H^2(M; Z_2)) = \text{kernel}(H^2(M; Z_2) \xrightarrow{i^*} H^2(M - J)),$$

so the first obstruction to trivializing ξ over $M - \nu(J)$ vanishes. The higher obstructions lie in $H^3(M - \nu(J); \pi_2(SO(4)))$ and $H^4(M - \nu(J); \pi_3(SO(4)))$. The first group vanishes because $\pi_2(SO(4)) = 0$ and the second because $H^4(M - \nu(J)) = 0$. Hence $\xi|(M - \nu(J))$ is trivial.

$\bar{\xi}$ is obtained from $\bar{\xi}|(M - \nu(J))$ by adjoining $\bar{\xi}|\bar{\nu}(J)$ along $\bar{\xi}|\dot{\nu}(J)$ by a bundle equivalence. Since $\bar{\xi}|(M - \nu(J))$ and $2\bar{\eta}$ are bundle equivalent (they are both trivial bundles over $M - \nu(J)$) we may attach a bundle, equivalent to $\bar{\xi}|\bar{\nu}(J)$, to $2\bar{\eta}$ along $2\bar{\eta}|\dot{\nu}(J)$ by a bundle equivalence $\bar{\xi}|\dot{\nu}(J) \longrightarrow 2\bar{\eta}|\dot{\nu}(J)$ and thereby extend $2\bar{\eta}$ to a disk bundle over M equivalent to $\bar{\xi}$. Denote this bundle $2\bar{\xi}'$ and *denote by X the space obtained from $(M - \nu(J)) \times cK$ after this attachment.* (Recall that $2\bar{\eta} \subset (M - \nu(J)) \times cK$.) The extension of $2\bar{\eta}$ to the bundle $2\bar{\xi}'$ includes in its interior an extension of $\bar{\eta} \subset 2\bar{\eta}$ to a disk bundle over M equivalent to $\bar{\xi}$. We denote this bundle $\bar{\xi}' \subset 2\bar{\xi}' \subset X$ (see Figure 2).

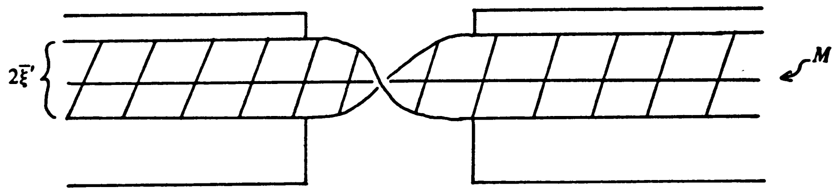

FIGURE 2

Away from $(M - \nu(J)) \times * \subset (M - \nu(J)) \times cK$, X is always a manifold, and points in $(M - \bar{\nu}(J)) \times *$ have neighborhoods homeomorphic to $R^m \times cK$. Therefore, under hypothesis $H(m + 1)$, X is a manifold except possibly near $\dot{\nu}(J) \times * \subset \dot{\nu}(J) \times cK$.

Under hypothesis $H(m)$, X is a manifold with boundary, for points in $\dot{\nu}(J) \times *$ have neighborhoods which are homeomorphic to $R^{m-1} \times cK$. In this case $\partial X = [(M - 2\nu(J)) \times K] \cup Y$, where

$$Y \equiv [(2\bar{\nu}(J) - \nu(J)) \times K] \cup [(\dot{\nu}(J) \times cK) - 2\xi'|\dot{\nu}(J)] \cup 2\dot{\xi}'|\bar{\nu}(J).$$

We will denote $[(M - 2\nu(J)) \times K] \cup Y$ by ∂X, whether or not X is a manifold.

Let U be the interior of an open collar neighborhood of Y in X. That is, if $Y \times (-1, 1]$ parameterizes the collar, let U be the image of $Y \times (-1, 1)$. It is clear from the construction that such a collar exists. Note that under hypothesis $H(m + 1)$, Y contains all the points of ∂X near which ∂X may fail to be Euclidean. We intend, assuming $H(m + 1)$, to split U by a genuine manifold and discard the piece lying between the split and Y. The result will be a manifold X' which can be substituted for X in much of the later argument.

Since K is a homology sphere, $(X, \partial X)$ is a homology manifold pair regardless of any suspension assumptions made about K. In particular, $(X, \partial X)$ is always a Poincaré pair.

The following observation of H. King allows us to split U. Let CAT be a manifold category (DIFF, TOP or PL):

2.2. LEMMA. *Let $(C, \partial C)$ be a compact pair of topological spaces such that ∂C is a* CAT *manifold. If $C \times R$ is a* CAT *manifold of dimension ≥ 6 and $\partial(C \times R) \simeq \partial C \times R$ then there is a* CAT *manifold N and a* CAT *homeomorphism $N \times R \simeq C \times R$.*

PROOF OF 2.2. See [5] for proof. Here King observes that the portion P of $C \times R$ lying between $C \times \{0\}$ and an $n - 3$ neighborhood of an end of $C \times R$, when crossed with I, is a compact manifold. Therefore P has the homotopy type of a finite complex, and so the obstruction in $K_0(\pi_1(C))$ to splitting $C \times R$ vanishes. See [16].

U is homeomorphic to $Y \times (-1, 1)$ and $\partial Y = 2\dot{\nu}(J) \times K$ is a manifold, so *under assumption* $H(m + 1)$ there is a manifold N such that U is homeomorphic to $N \times (-1, 1)$. Let $\mathring{N}$ denote N with a small closed collar of ∂N removed. Modify X by removing $\mathring{N} \times (0, 1)$. Call the resulting manifold with boundary X'. The portion of ∂X lying "over" $2\bar{\nu}(J)$ has been changed from the nonmanifold Y to the manifold obtained from N by attaching to ∂N the cobordism between ∂N and ∂Y given by removing $\partial N \times (-1, 0)$ from $\partial N \times (-1, 1) \simeq \partial Y \times (-1, 1) \hookrightarrow \partial Y \times (-1, 1]$. Revert to N as notation for this manifold. Note that:

(i) $\partial N \simeq \partial Y \simeq 2\dot{\nu}(J) \times K$,

(ii) X' is a manifold with boundary.

The following lemma explains our interest in X.

2.3. LEMMA. *$X - \xi'$ is a homotopy product.*(1)

2.4. COROLLARY. *Under assumption $H(m)$ [resp. $H(m + 1)$] the manifold $X - \xi'$ [resp. $X' - \xi'$] is an h-cobordism.*

PROOF OF 2.3.

$$X - \xi' = [(M - \nu(J)) \times (cK - D)] \cup_\rho [2\bar{\xi}'|\bar{\nu}(J) - \xi'|\bar{\nu}(J)],$$

where ρ is an $S^3 \times I$ fiber bundle equivalence

$$\dot{\nu}(J) \times (2\bar{D} - D) \xrightarrow{\rho} 2\bar{\xi}'|\dot{\nu}(J) - \xi'|\dot{\nu}(J).$$

Now $2\bar{\xi}'|\bar{\nu}(J) - \xi'|\bar{\nu}(J)$ is homeomorphic to the actual product $\dot{\xi}'|\bar{\nu}(J) \times I$, so in order to verify 2.3 it suffices to show that $W \equiv (M - \nu(J)) \times (cK - D)$ is a homotopy product from $W_1 \equiv (M - \nu(J)) \times \dot{D}$ to $W_2 \equiv [(M - \nu(J)) \times K] \cup [\dot{\nu}(J) \times (cK - 2D)]$ rel $\dot{\nu}(J) \times (2\bar{D} - D) \simeq \dot{\nu}(J) \times \dot{D} \times I$. That is, we must show

(1) A Poincaré triple $(C; Y_1, Y_2)$ is a homotopy product if the inclusions $Y_i \hookrightarrow C$ are homotopy equivalences.

that the inclusions $W_i \longrightarrow W$ are homotopy equivalences. This will follow from two facts (see [19])

(a) $H_*(W, W_i; Z[\pi_1(W)]) = 0$.

(b) The inclusions induce isomorphisms $\pi_1(W_i) \longrightarrow \pi_1(W)$ with base points in $\dot{\nu}(J) \times (2\bar{D} - D)$.

PROOF OF (a). By general position $\pi_1(cK - D) \simeq \pi_1(cK - \{\text{point}\}) \simeq \pi_1(cK) = 0$. Therefore $\pi_1(W) \simeq \pi_1(M - \nu(J))$.

Since the inclusion $D \longrightarrow cK$ is a homotopy equivalence, $H_*(cK - D, \partial D; Z) = 0$ by excision. Hence $H_*(W, W_1; Z[\pi_1(M - \nu(J)]) \simeq H_*(W, W_1; Z[\pi_1(W)]) = 0$. By Poincaré duality $H^*(W, W_2; Z[\pi_1(W)]) = 0$, so $H_*(W, W_2; Z[\pi_1(W)]) = 0$. This proves (a).

PROOF OF (b). Let x, q_1 and q_2 be points in $\dot{\nu}(J)$, ∂D and K respectively. Since $\pi_1(cK - D) = 0$, the inclusion induces an isomorphism $\pi_1(\partial D, q_1) \xrightarrow{\simeq} \pi_1(cK - D, q_1)$. Hence the inclusion induces an isomorphism $\pi_1(W_1, (x, q_1)) \xrightarrow{\simeq} \pi_1(W, (x, q_1))$.

A path in $\{x\} \times (cK - D)$ from (x, q_1) to (x, q_2) provides a natural isomorphism $\pi_1(W, (x, q_1)) \simeq \pi_1(W, (x, q_2))$. It therefore suffices to show that $\pi_1(W_2, (x, q_2)) \longrightarrow \pi_1(W, (x, q_2))$ is an isomorphism.

By Van Kampen's theorem $\pi_1(W_2, (x, q_2))$ is the push-out of the following diagram.

$$\begin{array}{ccc} \pi_1(\dot{\nu}(J) \times K, (x, q_2)) & \longrightarrow & \pi_1((M - \nu(J)) \times K, (x, q_2)) \\ \downarrow & & \downarrow \\ \pi_1(\dot{\nu}(J) \times (cK - D), (x, q_2)) & \longrightarrow & \pi_1(W_2, (x, q_2)) \end{array}$$

A copy of $M - \nu(J)$ is imbedded in $(M - \nu(J)) \times K$ as $(M - \nu(J)) \times \{q_2\}$.

A standard argument now shows

$\pi_1((M - \nu(J)) \times \{q_2\}, (x, q_2)) \longrightarrow \pi_1(W_2, (x, q_2))$ is an isomorphism.

Since $\pi_1(cK - D, q_2) = 0$, the inclusion also induces an isomorphism

$\pi_1((M - \nu(J)) \times \{q_2\}, (x, q_2)) \longrightarrow \pi_1(W, (x, q_2))$.

This proves (b) and hence 2.3.

2.5. PROPOSITION. (i) *Assuming $H(2)$ then X is a manifold with boundary and there is a* PL *triangulation of ∂X which extends to a simplicial triangulation of X but does not extend to a* PL *triangulation of X.*

(ii) *Assuming $H(m + 1)$ and $m = 2, 3$ then X' is a manifold with boundary and there is a* PL *triangulation of $\partial X'$ which extends to a simplicial triangulation of X' but does not extend to a* PL *triangulation of X'.*

We delay the proof of 2.5 briefly.

PROOF OF THEOREM 2.1 ASSUMING PROPOSITION 2.5. PL adjoin to the PL triangulated ∂X (resp. $\partial X'$) a PL triangulated h-cobordism H with Whitehead torsion the negative of the torsion of the h-cobordism from ∂X (resp. $\partial X'$) to $\dot{\xi}'$ (see [11]). By 2.4 the resultant manifold will be ξ' with a topological s-cobordism adjointed to $\dot{\xi}'$. By the TOP s-cobordism theorem, this manifold will be homeomorphic to ξ. We so identify it. ξ is then simplicially triangulated by a triangulation which is PL near $\dot{\xi}$.

Since H is an h-cobordism between ∂X (resp. $\partial X'$) and $\dot{\xi}$ there are 1-1 correspondences between PL structures on ∂X (resp. $\partial X'$), $\dot{\xi}$, and H. Thus a PL extension of the triangulation near $\dot{\xi}$ to all of ξ would induce a PL extension of the PL triangulation of ∂X (resp. $\partial X'$) to all of X. This would contradict 2.5, and so verifies 2.1 assuming 2.5.

C. *Triangulating X and X': A proof of* 2.5. It was observed in §2.B that $H(m)$ (respectively $H(m + 1)$) implies that X (resp. X') is a manifold with boundary. Clearly $H(2)$ implies $H(m)$ for m greater than 2. This proves the first assertion of both parts of 2.5. The second assertion is the heart of this paper.

The following lemma shows that if ∂X (resp. $\partial X'$) is a manifold, ∂X (resp. $\partial X'$) is a PL manifold. First note that the manifold $\partial U \simeq \partial Y \times (-1, 1) = 2\dot{\nu}(J) \times K \times (-1, 1)$ has a natural PL-structure—the product of the Whitehead structure on $\dot{\nu}(J)$ and the unique structures on K and $(-1, 1)$ [20].

2.6. LEMMA. *If U is a manifold (e.g. under hypothesis $H(m + 1)$) then the natural* PL-*structure near ∂U extends to all of U.*

PROOF OF 2.6. By construction the natural structure extends to all of U, except possibly across that portion of U which is mapped by the collar projection $U \xrightarrow{p} Y$ to $\dot{\nu}(J) \times * \subset Y$. By an argument similar to that in 1.1 it follows that the dual in $H_m^{\inf}(U; Z_2)$ of the obstruction in $H^4(U, \partial U; Z_2)$ to extending the natural structure on ∂U to all of U is represented by $p^{-1}(\dot{\nu}(J) \times *)$ and so is carried by p to the element α in $H_{m-1}(Y; Z_2)$ represented by the manifold $\dot{\nu}(J) \times * \subset \dot{\nu}(J) \times (cK - D)$. In order to verify 2.6, it suffices therefore to show that α is null-homologous in Y. Let L be a line in $cK - 2D$ connecting $*$ to some $\{q\}$ in $2\dot{D}$. Then $\dot{\nu}(J) \times L$ is a homotopy in Y between $\dot{\nu}(J) \times *$ and the cross-section $\dot{\nu}(J) \times \{q\}$ of the bundle $2\dot{\xi}'|\dot{\nu}(J)$. Thus $\dot{\nu}(J) \times \{q\}$ also represents α.

From obstruction theory we know that any two cross-sections of $2\dot{\xi}'|\dot{\nu}(J)$ can be homotoped together except possibly over a 3-cell of $\dot{\nu}(J)$, where the difference in cross-sections defines an element of $\pi_3(\text{fiber of } 2\dot{\xi}' \simeq S^3) \simeq Z$. If this obstruction is trivial mod 2 the cross-sections represent the same homology class in $H_{m-1}(2\dot{\xi}'|\dot{\nu}(J); Z_2)$. Thus α is represented by any cross-section of $2\dot{\xi}'|\dot{\nu}(J)$ for which the obstruction in $H^3(\dot{\nu}(J); Z)$ to the existence of a homotopy of this cross-section to $\dot{\nu}(J) \times \{q\}$ is trivial mod 2.

Recall we are assuming $w^4(\xi') = 0$. $w^4(\xi')$ is the Z_2 reduction of the obstruction in $H^4(M; Z)$ to constructing a cross-section to $\dot{\xi}'$ [10]. But there is a cross-section of $2\dot{\xi}'$ (viz. $(M - \nu(J)) \times \{q\}$) over $M - \nu(J)$ which restricts to $\dot{\nu}(J) \times \{q\}$. Therefore the obstruction to extending this cross-section over $\nu(J)$, reduced mod 2, lies in the pre-image of $w^4(\xi)$ under the homomorphism

$$H^4(\bar{\nu}(J), \dot{\nu}(J); Z_2) \xrightarrow{e^{-1}} H^4(M, M - \nu(J); Z_2) \xrightarrow{j^*} H^4(M; Z_2).$$

Since M and J are connected, and $m \leqslant 4$, this is an isomorphism, and the Z_2 reduction of this obstruction is therefore trivial. Some cross-section of $\dot{\xi}'|\dot{\nu}(J)$ therefore does extend over $\nu(J)$, a cross-section for which the obstruction in $H^3(\dot{\nu}(J); Z)$ to homotoping to $\dot{\nu}(J) \times \{q\}$ has trivial Z_2 reduction and therefore still represents α. The extension of the cross-section over $\nu(J)$ then provides a null-homology in Y of the class α. This completes the proof of 2.6.

REMARK. By the product structure theorem, 2.6 is true for N. If Y is a manifold, it is true for Y also.

Now we construct the triangulation of X. Choose a Whitehead triangulation of M, which we denote $|M|$, so that $M - 2\nu(J)$ is a PL subcomplex $|M - 2\nu(J)|$, and choose a Whitehead triangulation $|K|$ of K. $|M - 2\nu(J)| \times c|K|$ is a cell complex. Let $|M - 2\nu(J)| \times c|K|$ here denote a simplicial complex obtained from this cell-complex by subdivision in which no new vertices are introduced (see [12, Chapter 2]).

The following series of assertions show that under assumption $H(2)$ some subdivision of this triangulation may be extended to all of X. Later the assertions are modified to apply to X' under assumption $H(m + 1)$, $m = 2, 3$.

Let p_ν, $p_{\xi'}$ represent the bundle projections in $2\bar{\nu}(J)$ and $2\bar{\xi}'$ respectively. Denote $X - ((M - 2\bar{\nu}(J)) \times cK)$ by $X|2\bar{\nu}(J)$ and $X - ((M - 2\nu(J)) \times cK)$ by $X|2\nu(J)$.

ASSERTION 1. *The natural fibering*

$$2\dot{\nu}(J) \times cK \xrightarrow{p_1} 2\dot{\nu}(J) \xrightarrow{p_\nu} J$$

of $2\dot{\nu}(J) \times cK$ *over* J *extends to a fibering* $X|2\bar{\nu}(J) \xrightarrow{\tilde{f}} J$ *with contractible fiber* F.

PROOF OF ASSERTION 1. $X|2\bar{\nu}(J) = (2\bar{\nu}(J) - \nu(J)) \times cK \cup_\rho \bar{\xi}'|\bar{\nu}(J)$ for some bundle equivalence ρ: $\bar{\xi}'|\dot{\nu}(J) \longrightarrow \dot{\nu}(J) \times 2\bar{D}$. Since ρ is a bundle equivalence, the projections

$$(2\bar{\nu}(J) - \nu(J)) \times cK \xrightarrow{p_1} 2\bar{\nu}(J) - \nu(J) \xrightarrow{p_\nu} J$$

and

$$\bar{\xi}'|\nu(J) \xrightarrow{p_{\xi'}} \nu(J) \xrightarrow{p_\nu} J$$

coincide on $\bar{\xi}'|\dot{\nu}(J) \simeq_\rho \dot{\nu}(J) \times 2\bar{D}$. These maps define the required fibering $\tilde{f}$. Since the fiber of $\dot{\nu}(J)$ is S^1, the fiber F of this projection is homeomorphic to

$((S^1 \times I) \times cK) \cup_{\bar{\rho}} (D^2 \times D^4)$. Here $\bar{\rho}$ is a homeomorphism $(S^1, 1, 2\bar{D}) \longrightarrow (\partial D^2, D^4)$. An easy calculation shows that F is contractible, proving Assertion 1.

The fiber bundle projection $\tilde{f}$ restricts to a fiber bundle projection $Y \cup (2\dot{\nu}(J) \times cK) \longrightarrow J$. We denote this restriction by f.

ASSERTION 2. *Under assumption $H(m + 1)$, the interior of the mapping cylinder $Z(f)$ is a manifold.*

PROOF OF ASSERTION 2. Away from J the interior of the mapping cylinder is just $(Y \cup (2\dot{\nu}(J) \times cK)) \times R$. Since dimension $(2\dot{\nu}(J) \times R)$ is m, all of these points have Euclidean neighborhoods under assumption $H(m + 1)$.

Near a point on J the mapping cylinder is homeomorphic to cone $(\partial F) \times R^{m-2}$, where F is defined in Assertion 1. The five lemma and Van Kampen's theorem show that ∂F is simply connected and has the homology of S^5. A homotopy equivalence $\partial F \longrightarrow S^5$ may therefore be defined by collapsing the complement of a 5-cell in ∂F to a point.

Under assumption $H(m + 1)$ $\partial F \times R^{m-1}$ is a manifold, for in ∂F any vertex $*$ lies in the cross-product of $*$ with an S^1 fiber of $\dot{\nu}(J)$ or $2\dot{\nu}(J)$, hence has neighborhood homeomorphic to $cK \times R^m$. By [3, Corollary 2], $\Sigma^{m-1}(\partial F) \simeq S^7$, so cone$(\partial F) \times R^{m-2}$ is locally Euclidean. This proves Assertion 2.

ASSERTION 3. *Under assumption $H(2)$ the manifold $\bar{X}$ obtained by attaching the mapping cylinder $Z(f)$ to $X - X|2\nu(J) = (M - 2\bar{\nu}(J)) \times cK$ along $2\dot{\nu}(J) \times cK$ is homeomorphic to X.*

PROOF OF ASSERTION 3. According to Assertion 1, $X|2\bar{\nu}(J)$ fibers over J with fiber F. But under assumption $H(2)$ both ∂F and F are manifolds. We have shown that ∂F is a homotopy 5-sphere and F is contractible. By the Poincaré theorem, $F \simeq D^6 \simeq$ cone(∂F). The group of the D^6 bundle $X|2\bar{\nu}(J)$ is $\mathrm{Aut}_{\mathrm{TOP}}(B^6)$. There is a natural imbedding of $\mathrm{Aut}_{\mathrm{TOP}}(\partial B^6)$ in $\mathrm{Aut}_{\mathrm{TOP}}(B^6)$ induced by coning. The topological version of the Alexander trick shows that the quotient group $\mathrm{Aut}_{\mathrm{TOP}}(B^6)/\mathrm{Aut}_{\mathrm{TOP}}(\partial B^6)$ is contractible. Hence $X|2\bar{\nu}(J)$ is equivalent to the disk bundle got by coning on the fibers of its sphere bundle boundary. This space is clearly $Z(f)$. This proves Assertion 3.

ASSERTION 4. *Under assumption $H(2)$ the mapping cylinder $Z(f)$ has a simplicial triangulation which restricts to a subdivision of $|2\dot{\nu}(J)| \times c|K|$ on $2\dot{\nu}(J) \times cK$ and which is a* PL *triangulation on Y.*

PROOF OF ASSERTION 4. The projection $|2\dot{\nu}(J)| \times c|K| \longrightarrow |2\dot{\nu}(J)|$ is certainly a piecewise linear map of polyhedra. So is the PL manifold fiber bundle projection $|2\dot{\nu}(J)| \longrightarrow J$.

The PL structure on $2\dot{\nu}(J) \times K$ given by the triangulation $|2\dot{\nu}(J)| \times |K|$ extends to a PL structure of Y by the remark following Lemma 2.6, and such an extension may be chosen so that the TOP bundle $f_Y \equiv f|Y$: $Y \longrightarrow J$ is a PL bundle. This follows from the bundle straightening theorem of the Appendix. We have the commutative diagram of piecewise linear maps of polyhedral:

$$\begin{array}{ccc} |2\dot{\nu}(J)| \times |K| & \xrightarrow{\text{inclusion}} & Y \\ {\scriptstyle \text{inclusion}}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ |2\dot{\nu}(J)| \times c|K| & \xrightarrow{\quad f \quad} & J \end{array}$$

By [12, Theorem 2.15] Y and J may be triangulated and $|2\dot{\nu}(J)| \times c|K|$ subdivided so that all maps are simplicial. The mapping cylinder of a simplicial map is simplicial [17, p. 151], and so the assertion is proven.

We now embark on proofs of versions of these four assertions for X' when $m = 2, 3$.

For $m = 3$, J is a circle. Fundamental use is made of the following theorem which Browder and Levine [1] originally proved in the smooth category. It may also be verified in PL and TOP by use of PL and TOP transversality and handlebody theory.

2.7. THEOREM (BROWDER-LEVINE). *Let W be a compact connected* CAT *manifold of dimension $n \geq 6$ and f: $W \longrightarrow S^1$ a map such that*

(i) *$f|\partial W \longrightarrow S^1$ is a* CAT *fiber bundle*;

(ii) *$f_{\#}$: $\pi_1(W) \longrightarrow \pi_1(S^1)$ is an isomorphism.*

Then the universal cover $\widetilde{W}$ of W has the homotopy type of a finite complex if and only if f is homotopic rel ∂W *to a* CAT *fiber bundle map.*

REMARK. If f is homotopic rel ∂W to a fiber bundle map with fiber F then $\widetilde{W} \simeq F \times R$.

Denote $X' - ((M - 2\overline{\nu}(J)) \times cK)$ by $X'|2\overline{\nu}(J)$ and $X' - ((M - 2\nu(J)) \times cK)$ by $X'|2\nu(J)$.

ASSERTION 1′. *For $m = 2, 3$ the natural fibering*

$$2\dot{\nu}(J) \times cK \xrightarrow{p_1} 2\dot{\nu}(J) \xrightarrow{p} J$$

extends to a fibering $N \xrightarrow{f_N} J$.

PROOF OF ASSERTION 1′. The assertion is obvious for $m = 2$, since then J consists of a single point.

For $m = 3$, $J \simeq S^1$. By Assertion 1, $p_\nu p_1$ extends to a fibering $Y \xrightarrow{f_Y} J$. The map f_Y induces an isomorphism on fundamental groups since the fiber F_Y is simply connected. [$F_Y \simeq (S^1 \times (cK - 2D)) \cup_{\overline{\rho}} (D^2 \times S^3)$, where $\overline{\rho}$ is a homeomorphism $(S^1, 2\dot{D}) \xrightarrow{\overline{\rho}} (\partial D^2, S^3)$. See proof of Assertion 1.] By definition $Y \times R \simeq N \times R$, so the universal covers of Y and N have the same homotopy type. The composition

$$N \xrightarrow{\text{inclusion}} Y \times R \xrightarrow{p_1} Y \xrightarrow{f_Y} J$$

then satisfies the hypothesis of 2.7, and Assertion $1'$ follows from the conclusion of 2.7.

We denote by f' the bundle map $N \cup (2\dot{\nu}(J) \times cK) \longrightarrow J$ which is defined to be equal to f_N on N and $p_\nu p_1$ on $2\dot{\nu}(J) \times cK$.

ASSERTION $2'$. *Under assumption $H(m + 1)$ and $m = 2, 3$, the interior of the mapping cylinder $Z(f')$ is a manifold.*

PROOF OF ASSERTION $2'$. Once we know that a fiber of f' has the homotopy type of S^5, the proof follows exactly as did the proof of Assertion 2.

Let F_N and F_Y be the fibers of f_N and f_Y respectively. The universal covers $\widetilde{N}$ and $\widetilde{Y}$ are then $F_N \times R$ and $F_Y \times R$ respectively. By definition there is a homeomorphism h: $N \times (-1, 1) \longrightarrow Y \times (-1, 1)$ such that near $\partial N \times (-1, 1)$ $p_2 = hp_2$: $\partial N \times (-1, 1) \longrightarrow (-1, 1)$ and $f_N p_1 = f_Y p_1 h$: $N \times (-1, 1) \longrightarrow J$. Hence $F_N \times R \times (-1, 1)$ is homeomorphic to $F_Y \times R \times (-1, 1)$ by a homeomorphism which respects projection to $R \times (-1, 1)$ near $\partial F_N \times R \times (-1, 1)$.

Thus there is a homotopy equivalence g: $F_N \longrightarrow F_Y$ which is a homeomorphism near ∂F_N.

Now ∂F of Assertion 2 was obtained by adjoining $cK \times S^1$ to F_Y along their boundaries by a homeomorphism we will denote ζ: $K \times S^1 \longrightarrow \partial F_Y$. ∂F was there shown to have the homotopy type of S^5. The fiber of f' is obtained by adjoining $cK \times S^1$ to F_N by a homeomorphism ζ': $K \times S^1 \longrightarrow \partial F_N$ such that $\zeta = g\zeta'$. It follows from the 5-lemma and Van Kampen's theorem that the fiber of f' has the homotopy type of S^5. This proves Assertion $2'$.

ASSERTION $3'$. *Under assumption $H(m + 1)$ and for $m = 2, 3$ the manifold $\overline{X}'$ obtained by attaching in the natural way the mapping cylinder $Z(f')$ to $X' - X'|2\nu(J) = (M - 2\bar{\nu}(J)) \times cK$ along $2\dot{\nu}(J) \times cK$ is homeomorphic to X'.*

PROOF OF ASSERTION $3'$. We will show that $X'|2\bar{\nu}(J)$ and $Z(f')$ are s-cobordant rel boundary; that is, there is a manifold C such that $\partial C = X'|2\bar{\nu}(J) \cup_h Z(f')$, where h is the natural identification of the boundaries $\partial(X'|2\bar{\nu}(J)) = N \cup (2\dot{\nu}(J) \times cK) \simeq \partial Z(f)$, and such that the inclusions $X'|2\bar{\nu}(J) \longrightarrow C$ and $Z(f') \longrightarrow C$ are simple homotopy equivalences. Note that whereas $X'|2\bar{\nu}(J)$ and $Z(f')$ may not be manifolds along their boundaries, ∂C will be a manifold because $2\dot{\nu}(J) \times cK$ is bicollared in ∂C.

Observe that $X'|2\bar{\nu}(J)$ has the homotopy type of $X|2\bar{\nu}(J)$ which, by Assertion 1, has the homotopy type of J. Similarly $Z(f')$ has the homotopy type of J (indeed collapses to J). Furthermore the inclusion of $N \cup (2\dot{\nu}(J) \times cK)$ in $X'|2\bar{\nu}(J)$ or $Z(f')$ clearly induces an isomorphism on fundamental groups. Hence $\pi_1(\partial C) \simeq Z$ for $m = 3$ and $\pi_1(\partial C) = 0$ for $m = 2$.

Therefore the universal covers of both $X'|2\bar{\nu}(J)$ and $Z(f')$ are contractible, and the universal cover of ∂C is obtained by adjoining these in the natural way along the universal cover of $N' \cup (2\dot{\nu}(J) \times cK)$. By the proof of Assertion 2′ the universal cover of $N' \cup (2\dot{\nu}(J) \times cK)$ has the homotopy type of S^5. It follows by Van Kampen's theorem and the 5-lemma that the universal cover of ∂C has the homotopy type of S^6.

By Theorem 2.7 when $m = 3$ and trivially when $m = 2$, ∂C fibers over J with fiber a homotopy S^6, hence an actual S^6. The induced D^7 bundle is then the required s-cobordism C. (Recall there is no Whitehead torsion for $\pi_1(C) = 0$ or Z.)

Extend the s-cobordism C by the product s-cobordism over $X' - X'|2\nu(J) = \overline{X}' - Z(f')$. The result is an s-cobordism between X' and $\overline{X}'$ which is a product cobordism between $\partial X'$ and $\partial \overline{X}' = \partial X'$. The assertion then follows from the TOP s-cobordism theorem.

ASSERTION 4′. *Under assumption* $H(m + 1)$ *and for* $m = 2, 3$ *the mapping cylinder* $Z(f')$ *has a simplicial triangulation which restricts to a subdivision of* $|2\dot{\nu}(J)| \times c|K|$ *on* $2\dot{\nu}(J) \times cK$ *and which is a* PL *triangulation on* N.

PROOF OF ASSERTION 4′. The proof is exactly that of Assertion 4, except in case $m = 3$ the PL version of 2.7 is used instead of the bundle straightening theorem of the Appendix to deduce that $f_N: N \longrightarrow J$ of Assertion 1′ may be assumed PL.

In order to conclude the proof of 2.5 it suffices to show that the PL structure on ∂X (resp. $\partial X'$) which we have defined does not extend to a PL structure over all of X (resp. X').

Let R^m be an open m-disk in $M - 2\nu(J)$. If the PL structure on ∂X (resp. $\partial X'$) did extend over all of X (resp. X') then the codimension zero open imbedded submanifold $R^m \times cK \subset ((M - 2\nu(J)) \times cK) \subset X' \subset X$ would inherit a PL structure extending the natural PL structure on $R^m \times K$. Siebenmann shows that this is impossible [15, Theorem 2, Assertion 2]. This completes the proof of 2.5 and so of 2.1.

Theorem 2.1 suffers a weakness which must be surmounted to obtain any triangulation results on 8-manifolds. For $m = 3, 4$ there may be PL structures on $\dot{\xi}$ which do not extend to simplicial structures on $\overline{\xi}$. Theorem 2.1 has shown only that at least one which does not PL extend does extend simplicially. There is a trick presented later which allows the triangulation of the 7-manifolds of Theorem 0.1 anyway, but the situation is more serious for 8-manifolds, when $m = 4$. We show here that the problem reduces to an existence problem for s-cobordisms; later sufficient s-cobordisms will be created.

Let p be the bundle projection $\overline{\xi}' \longrightarrow M$ and i: $\partial X \cup \dot{\xi}' \longrightarrow X - \xi'$ the inclusion. There is a natural injective map $H^3(M; Z_2) \xrightarrow{\rho} H^3(\partial X; Z_2)$ defined as the composition of the injective map $H^3(M; Z_2) \xrightarrow{p^*} H^3(\dot{\xi}'; Z_2)$ and the isomorphism

$$H^3(\dot{\xi}'; Z_2) \xrightarrow{(i^*)^{-1}} H^3(X - \xi'; Z_2) \xrightarrow{i^*} H^3(\partial X; Z_2).$$

By an s-cobordism between manifolds M and M' with boundary we will mean an s-cobordism which restricts to a product cobordism between ∂M and $\partial M'$.

Let W be a topological s-cobordism W from $M - \nu(J)$ to a smooth manifold $\overline{M}$. Consider the composition d of the maps

$$H^4(W, \partial W) \xrightarrow[(j^*)^{-1}]{} H^3(\partial W, \overline{M}) \xrightarrow{\cong} H^3(M, \overline{\nu}(J)) \xrightarrow[j^*]{} H^3(M).$$

2.8. Lemma. *Let α in $H^4(W, \partial W)$ be the obstruction to extending the smooth structure on ∂W to all of W. Then, assuming $H(2)$, the* PL *triangulation of ∂X corresponding to $\rho d(\alpha)$ extends to a simplicial triangulation of X.*

Proof of 2.8. Let W' be the union of $W \times cK$ and $(X|2\overline{\nu}(J)) \times I$ along $2\dot{\nu}(J) \times I \times cK = X|2\dot{\nu}(J) \times I$. Then W' is an s-cobordism between X and a manifold $\overline{X}$. A Whitehead triangulation of $\overline{M}$ and the procedure above provide a simplicial triangulation of $\overline{X}$ which is PL near $\partial\overline{X}$. By the topological s-cobordism theorem, W' is homeomorphic to $X \times I$.

Let $p_1: (W, \partial W) \times K \longrightarrow (W, \partial W)$ be the projection. The obstruction in $H^3(\partial X; Z_2)$ to making $\partial W' - (X \cup \overline{X}) = \partial X \times I$ a PL concordance between ∂X and $\partial\overline{X}$ is then the image of α under the composition

$$H^4(W, \partial W) \xrightarrow[p_1^*]{} H^4(W, \partial W) \times K \xrightarrow[j^*e^{-1}]{} H^4(\partial X \times (I, \partial I))$$

$$\xrightarrow[e(j^*)^{-1}]{} H^3(\partial X \times 0).$$

By naturality this is $p_1^* d(\alpha)$. Clearly $p_1^* d(\alpha)$ and $p^* d(\alpha)$ are cohomologous in $X - \xi'$. Hence $p_1^* d(\alpha) = \rho^* d(\alpha)$.

Remark. If $M - \nu(J)$ can be retriangulated rel $\dot{\nu}(J)$ with obstruction $d(\alpha)$ we get the same conclusion, but M has such a low dimension in the applications that this is not known to be possible.

3. The proof of the main Theorem 0.1.

Case I. $n = 5$. This is treated by Siebenmann [15]. He assumes N is orientable, in which case $H^5(N; Z)$ has no 2-torsion and consequently $\beta(H^4(N; Z_2)) = 0$. He requires orientability to ensure that the Poincaré dual to k_N in $H_1(N; Z_2)$ can be represented by a circle with normal $SO(4)$ bundle. According to 1.2, if $\beta(k_N) = 0$ the Poincaré dual of k_N can be represented by an imbedded circle with orientable normal bundle, and Siebenmann's proof is applicable.

Case II: $n = 6, 7, 8$. By 1.2, the Poincaré dual to k_N may be represented as the inclusion of the fundamental class of some connected smooth submanifold

M with oriented orthogonal normal bundle $\nu(M)$. That is, k_N is the image of the nontrivial element of $H^4(\bar{\nu}(M), \dot{\nu}(M); Z_2)$ under the homomorphism

$$H^4(\bar{\nu}(M), \dot{\nu}(M); Z_2) \xrightarrow{e^{-1}} H^4(N, N-M; Z_2) \xrightarrow{j^*} H^4(N; Z_2).$$

The restriction of k_N to $N-M$ is therefore trivial by exactness in the cohomology sequence of $(N, N-M)$. By naturality of the triangulation obstruction, $N-M$ is PL triangulable. By the product structure theorem [6], the PL structure on $N-M$ is isotopic to one which restricts to a PL structure on $\dot{\nu}(M)$.

By [9], the isotopy classes of PL structures on $\dot{\nu}(M)$ are in 1-1 correspondence with $H^3(\dot{\nu}(M); Z_2)$.

Case IIa: $n = 6$. The following is a portion of the Thom-Gysin sequence for $(\bar{\nu}(M), \dot{\nu}(M))$, coefficients in Z_2.

$$0 = H^3(M) \longrightarrow H^3(\dot{\nu}(M)) \longrightarrow H^0(M) \longrightarrow H^4(M) = 0.$$

Since M is connected, $H^3(\dot{\nu}(M)) \simeq H^0(M) \simeq Z_2$, so there are two possible PL structures on $\dot{\nu}(M)$. One PL structure is that which PL extends to all of $\nu(M)$, induced by the natural Whitehead PL structure on the smooth manifold $\bar{\nu}(M)$.

By 2.1 there is a PL structure $(\dot{\nu}(M))_\Sigma$ which is not isotopic to the Whitehead structure but some PL triangulation of $(\dot{\nu}(M))_\Sigma$ does extend to a simplicial triangulation of $\bar{\nu}(M)$. Hence, in either case, some PL triangulation of $N - \nu(M)$ extends to a simplicial triangulation of $\bar{\nu}(M)$, and N is homeomorphic to a simplicial complex.

Case IIb: $n = 7$. The relevant portion of the Thom-Gysin sequence is

$$0 \longrightarrow H^3(M) \xrightarrow{p^*} H^3(\dot{\nu}(M)) \longrightarrow H^0(M) \longrightarrow H^4(M) = 0.$$

Since $H^3(M) \simeq H^0(M) \simeq Z_2$, $H^3(\dot{\nu}(M)) \simeq Z_2 \oplus Z_2$, and there are now four PL structures possible on $\dot{\nu}(M)$. Since image$(p^*) \simeq Z_2$, two of these PL structures extend to PL structures on $\bar{\nu}(M)$. By 2.1 *one* of the other PL structures has a PL triangulation which extends to a simplicial triangulation of $\bar{\nu}(M)$. We now show that the image of the restriction $H^3(N-M) \xrightarrow{i^*} H^3(\dot{\nu}(M))$ is Z_2. It follows that with a correct choice of a PL structure on $N-M$ the restriction of the structure to $\dot{\nu}(M)$ is *not* the one PL structure which may not extend either simplicially or piecewise-linearly to $\bar{\nu}(M)$.

If i_*: $H_3(M) \longrightarrow H_3(N)$ (all coefficients are Z_2) fails to be injective, $k_N = 0$ by 1.1 so N is PL triangulable and we are done. If the map is injective, i'^*: $H^3(N) \longrightarrow H^3(\bar{\nu}(M))$ is surjective.

$$\begin{array}{ccc} H^3(N) & \longrightarrow & H^3(N - \nu(M)) \\ \downarrow & & \downarrow \\ H^3(\bar{\nu}(M)) & \longrightarrow & H^3(\dot{\nu}(M)) \end{array}$$

is a commutative diagram. Since the bottom map is injective $H^3(N - \nu(M)) \twoheadrightarrow$ $H^3(\dot{\nu}(M))$ has image at least Z_2. As this suffices, we leave to the reader the task of showing that the image is exactly Z_2 as claimed. This completes the proof for $n = 7$.

Case IIc. $n = 8$. Let M be any smooth 4-manifold, possibly with boundary. Following [2], say that two s-cobordisms W and W' from M to a smooth manifold are *equivalent* if there are smooth s-cobordisms V and V' with $\partial_2 W = \partial_1 V$, $\partial_2 W' = \partial_1 V'$ and a homeomorphism of $W \cup V$ onto $W' \cup V'$ which is the identity on $M = \partial_1 W$ and a diffeomorphism from $\partial_2 V$ to $\partial_2 V'$.

Let M_k denote the connected sum of M and k copies of $S^2 \times S^2$.

In [14] we prove the following theorem.

3.1. THEOREM. *There is an integer k such that for any compact* 4*-manifold M there is a* 1-1 *correspondence between* $H^3(M, \partial M; Z_2)$ *and equivalence classes of s-cobordisms of* M_k *to a smooth manifold.*

REMARKS. The equivalence is generated by selecting a representative W of the s-cobordism class and mapping the obstruction to extending the smoothing of ∂W to all of W by the composition

$$H^4(W, \partial W) \xrightarrow[(j^*)^{-1}]{} H^3(\partial W, \partial_2 W) \xrightarrow[e^*]{} H^3(M_k, \partial M_k) \xrightarrow[(q^*)^{-1}]{} H^3(M, \partial M).$$

Here q is the natural projection $M_k \longrightarrow M$.

It is also shown that for M orientable, $k = 1$.

Return to the case $M \subset N$ representing the Poincaré dual of k_N. Let D^4 be a smooth open 4-disk in $M - \bar{\nu}(J)$. Then $\nu(M)$ is trivial over D^4. Perform surgery in the ambient manifold $\nu(M)|D^4 \simeq R^8$ on k trivial smooth circles in D^4 (k as defined in 3.1). The result of the surgery is to change M to M_k and $\nu(M)$ to $q^*(\nu(M)) \simeq \nu(M_k)$, the normal bundle of M_k in N.

Note that $[M_k]$, the fundamental Z_2 homology class of M_k, is homologous in N to $[M]$, via the cobordism given by the surgery. Hence $N - M_k$ is PL triangulable and, as above, we may assume that $\dot{\nu}(M_k)$ has a PL structure for which it is a PL submanifold of $N - M_k$ and that this structure does not extend to $\bar{\nu}(M_k)$. It remains to show that this PL triangulation extends to a simplicial triangulation across $\bar{\nu}(M_k)$.

The remarks preceding 2.1 show that the difference between $\dot{\nu}(M_k)$ and the structure which has been shown to simplicially extend is represented in $H^3(\dot{\nu}(M_k))$ by $p^*(\delta)$ for some δ in $H^3(M_k)$. By 2.8 it suffices to produce a topological cobordism W from $M_k - \nu(J)$ to a smooth manifold such that the obstruction to extending the smooth structure on ∂W to all of W is mapped by d to δ. The remarks following 3.1 show that, since $H^3(M, \nu(J)) \xrightarrow[j^*]{} H^3(M)$ is onto, the required cobordism exists. This completes the proof.

Appendix. The bundle straightening theorem. The following theorem was used in Assertion 4 of the proof of Proposition 2.5.

THEOREM. *Let* $f: M^m \longrightarrow Q^q$ *be a map of* PL *manifolds such that*

(i) f *is a topological fiber bundle.*

(ii) *There is a* PL *submanifold* $N \subset M$ *such that* f *is* PL *near* N *and* $f|N$: $N \longrightarrow f(N)$ *is a* PL *fiber bundle.*

(iii) $m - q \geqslant 5$.

Then there is an isotopy $h_t: M \longrightarrow M$ rel N *such that* h_0 *is the identity and* fh_1 *is a* PL *fiber bundle.*

PROOF OF THEOREM. Choose a PL triangulation of Q such that $f(N)$ is a full subpolyhedron.

Suppose inductively that for some $0 \leqslant i \leqslant q$ an isotopy has already been defined rel N altering f to a PL bundle over a neighborhood of a subcomplex $Q^{(i-1)}$ of Q containing the $(i-1)$ skeleton and properly contained in the i-skeleton. Let Δ be an i-simplex of Q not in $Q^{(i-1)}$.

Since f is a PL map near N and near $f^{-1}(Q^{(i-1)})$, it follows from the PL product structure theorem that M may be isotoped rel $N \cup f^{-1}(Q^{(i-1)})$ so that $f^{-1}(\Delta)$ is a PL submanifold of M [6].

Let F denote the TOP fiber of f. Since Δ is contractible there is a homeomorphism $g: f^{-1}(\Delta) \longrightarrow F \times \Delta$ such that $p_2 g = f: f^{-1}(\Delta) \longrightarrow \Delta$.

Let K be the full subcomplex $f(N) \cap \Delta$ of Δ and let F' be the fiber of the bundle $N \longrightarrow f(N)$. Since K is full in Δ, K is contractible and there is a homeomorphism $g': N \cap f^{-1}(\Delta) \longrightarrow F' \times K$ such that $p_2 g' = f: N \cap f^{-1}(\Delta) \longrightarrow K$. Then $g(g')^{-1}: F' \times K \longrightarrow F \times \Delta$ is an imbedding which commutes with projection to Δ. For a fixed vertex v in K, $g(g')^{-1}|F' \times \{v\}$ determines an imbedding $F' \xrightarrow{i} F$. By the TOP isotopy extension theorem the trivialization g may be altered so that $g(g')^{-1} = i \times (\text{identity})_K: F' \times K \longrightarrow F \times \Delta$.

Since $f|N$ is a PL fiber bundle, $g'^{-1}(F' \times K)$ is a PL submanifold of $f^{-1}(\Delta)$ on which f is a PL map. Since f is a PL fiber bundle over a neighborhood of $Q^{(i-1)}$, $f^{-1}(\partial\Delta)$ is a PL submanifold of $f^{-1}(\Delta)$ with $f|f^{-1}(\partial\Delta)$ a PL map. The homeomorphisms g and g' therefore assign PL structures to $F \times \Delta$ and $i(F') \times K$ such that $i(F') \times K$ is a PL submanifold of $F \times \Delta$, and $F \times \Delta$ is *sliced* near $F \times \partial\Delta$ and $i(F') \times K$ [7].

By the sliced concordance implies isotopy theorem [7], there is a PL structure $(F \times \Delta)_\Sigma$ on $F \times \Delta$ and an isotopy $h_t: F \times \Delta \longrightarrow (F \times \Delta)_\Sigma$ from the identity to a PL homeomorphism such that $p_2 h_t = p_2$ on a neighborhood of $(F \times \partial\Delta) \cup (F' \times K)$ and the projection $(F \times \Delta)_\Sigma \longrightarrow \Delta$ is a PL bundle.

Damp out the action of $g^{-1}h_t g: f^{-1}(\Delta) \longrightarrow f^{-1}(\Delta)$ through a tubular neighborhood of $f^{-1}(\Delta)$ in M, and denote the resultant isotopy of M by $\bar{h}_t$.

Since $\bar{h}_t$ is an isotopy, the map $f\bar{h}_t$ is always a TOP fiber bundle. Moreover $f\bar{h}_t$ is fixed on a neighborhood of $N \cup Q^{(i-1)}$ and $f\bar{h}_1$ is a PL fiber bundle over Δ. The theorem then follows by induction over simplices of Q.

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