ON CHAIN VARIETIES OF LINEAR ALGEBRAS

BY

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ABSTRACT. In the present paper we study varieties of linear k -algebras over a commutative associative Noetherian ring k with 1, whose subvarieties form a chain. We describe these varieties in terms of identities in the following cases: residually nilpotent varieties, varieties of alternative, Jordan and $(-1, 1)$ -algebras.

1. Introduction. In [1] we have already described homogeneous chain varieties of linear algebras over a field. Recall that variety is homogeneous iff its free algebras are graded relative to the degree-function. If the ground field is infinite, then any variety of algebras over this field is homogeneous. In the second paper [2] we gave the description of chain varieties of restricted p -algebras Lie over an infinite field.

In the present paper we describe in Theorems 2–15 chain varieties V of linear k -algebras, k -commutative associative Noetherian ring with 1, in the following cases:

- (i) if G is a free algebra in V then $\bigwedge_n G^n = 0$ (this is always valid when V is locally nilpotent), i.e. V is residually nilpotent,
- (ii) V is a variety of alternative algebras,
- (iii) V is a variety of $(-1, 1)$ -algebras,
- (iv) V is a variety of Jordan algebras.

Throughout the paper k is a fixed commutative associative ring with 1, and V is a variety of k -algebras with chain of subvarieties. We also assume that $\text{Ann } V = \{\alpha \in k \mid (\forall A \in V)(\alpha A = 0)\} = 0$.

Finally it is worthy of note that similar problems for associative rings were studied in [3], [4], [5].

2. Residually nilpotent varieties. Let

$$(1) \quad f(x_1, \dots, x_n) = \sum_{i=1}^d f_i = 0$$

be an identity in V and x_1 appears i times in f_i . Then for any $\alpha_1, \dots, \alpha_d \in k$, (1) implies

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$$(1') \quad \alpha_1 \cdots \alpha_d W(\alpha_1, \dots, \alpha_d) f_i = 0$$

where $W(\dots)$ is the Vandermonde determinant. Thus, as usual (see [1], [6]), we have

PROPOSITION 1. *If k has a unique maximal ideal $\mathfrak{m}$, i.e. $k^* = k \setminus \mathfrak{m}$, then if $\bar{k} = k/\mathfrak{m}$ is infinite, (1) is equivalent to the set of identities $f_i = 0$, $i = 1, \dots, d$. If $\bar{k} = F_q$, then $f = 0$ is equivalent to the set of identities*

$$\sum_{i \equiv t \pmod{q-1}} f_i = 0, \quad t = 1, \dots, q-1,$$

and for all $\alpha \in \mathfrak{m}$,

$$\alpha^{d(d-1)/2} f_i = 0.$$

To prove the last affirmation take $\alpha_i = 1 + \alpha^i \in k^*$. Then by (1')

$$\begin{aligned} W(\alpha_1, \dots, \alpha_d) &= \pm \prod_{r < s} (\alpha^r - \alpha^s) = \pm \alpha^{d(d-1)/2} \prod_{r < s} (1 - \alpha^{s-r}) \\ &= \delta \alpha^{d(d-1)/2}, \quad \delta \in k^*. \end{aligned}$$

THEOREM 1. *Let V be a variety of k -algebras with chain of subvarieties k as in Proposition 1. If V is homogeneous then either $xy = 0$ in V , or $\mathfrak{m} = 0$ and V was described in [1]. If $\bar{k} = k/\mathfrak{m}$ is infinite, then V is homogeneous.*

PROOF. By Proposition 1, V is homogeneous if $\bar{k} = k/\mathfrak{m}$ is infinite. Suppose that V is homogeneous and $xy = 0$ in V implies $\mathfrak{m}x = 0$. Then by homogeneity $\mathfrak{m}x = 0$ in V . If $\mathfrak{m}x = 0$ implies $xy = 0$, then V has an identity $xy = \alpha xy + \beta yx$, where $\alpha, \beta \in \mathfrak{m}$. Since $1 - \alpha \in k^*$ we can suppose that $\alpha = 0$. Therefore, $xy = \beta yx$ and $(1 - \beta)x^2 = 0 = x^2$, i.e. $xy + yx = 0$. Eventually $xy = \beta yx = -\beta xy$ and $(1 + \beta)xy = 0 = xy$.

Assume now that V is a residually nilpotent variety, i.e. in a V -free algebra G with free generators x_i , $i = 1, 2, \dots$,

$$(2) \quad \bigwedge_i G^i = 0$$

where $G^i = (\Sigma g_1 \cdots g_i \text{ with different bracketing} | g_j \in G)$. Note that in this case (2) is valid for any V -free algebra.

Let F be a V -free algebra with one free generator x . Let $I_n = \text{Ann } F/F^n \triangleleft k$, $k_n = k/I_n$. Note that $F/F^2 = k_2 x$ with zero multiplication. Thus there is a one-to-one correspondence between ideals of k_2 and subvarieties of $\text{var } F/F^2$ consisting of all k_2 -modules, annihilated by given ideal of k_2 . Thus k_2 has a unique maximal ideal $\mathfrak{m}_2$. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}_2$ in k . Define $\mathfrak{m}_j = \mathfrak{m}/I_j$. If $\beta \in k_n \setminus \mathfrak{m}_n$, then for some $\gamma \in k_n$ we have $\beta\gamma \equiv 1 \pmod{I_2/I_n}$ and since

$I_2^n \subseteq I_n$, $\beta\gamma' = 1$ in k_n for some $\gamma' \in k_n$. Hence k_n is a ring with unique maximal ideal m_n .

Now for each $\beta \in k$ the ideal βF is verbal, so since m is finitely generated, $mF = \alpha F$ for some $\alpha \in m$. In V either $x^2 = 0$ implies $\alpha x = 0$, or $\alpha x = 0$ implies $x^2 = 0$. Thus in F either

$$(3) \quad x^2 = \alpha\beta_1 x + \alpha\beta_2 x^2 + \sum_{i=3}^d \alpha g_i(x), \quad \text{or} \quad \alpha x = \sum_{i=2}^d g_i(x)$$

where $g_i(x)$ is a sum of monomials of degree i . By (3) and Proposition 1 we have in F_n

$$(3') \quad \alpha^t x^2 = 0, \quad t = 1 + \frac{1}{2}d(d-1),$$

where t does not depend on n . Hence $\alpha^t x^2 \in \bigwedge_n F_n = 0$ and, therefore, $m^t F^2 = \alpha^t F^2 = 0$.

Consider now k_n -module F/F^n , $n > 1$. Note that m_n is the Jacobson radical of k_n . Thus, by Theorem 4.2 [7, p. 12], $\bigwedge_r m_n^r F/F^n = 0$. Hence, for some $r > 1$ we have $m_n^r F/F^n \subseteq F^2/F^n$ and $m_n^{r+t} F_n \subseteq m_n^t F_n^2 = 0$ by (3'), where $F_n = F/F^n$. Thus, $m^{r+t} F \subseteq F^n$ and $m^{r+t} F \subseteq \bigwedge_n F^n = 0$. So m is nilpotent and $k/m = k_2/m_2 = \bar{k}$ is a field. Finally we have

PROPOSITION 2. *k has a unique maximal nilpotent ideal m , $m^d = 0$.*

Now by Theorem 1 without loss of generality we can assume that $\bar{k} = k/m = F_q$.

Let G be a V -free algebra with free generators x_i , $i > 1$,

$$G^t = \left\{ \sum g_1 \cdots g_t \text{ with different bracketing} \mid g_i \in G \right\}.$$

Put

$$\hat{G} = \bigoplus_{i \geq 1} G^i / G^{i+1}.$$

Since $G^i G^j \subseteq G^{i+j}$ we can define in $\hat{G}$ a multiplication

$$(x + G^{i+1})(y + G^{j+1}) = xy + G^{i+j+1},$$

$x \in G^i$, $y \in G^j$. It is clear that G is generated by $y_i = x_i + G^2$, $i > 1$.

PROPOSITION 3. *$\hat{G}$ is a relatively free k -algebra with free generators y_i , $i \geq 1$. $\text{var } \hat{G}$ has a chain of subvarieties.*

PROOF. Since $\hat{G}$ is homogeneous we need to prove that if $g = g(X_1, \dots, X_n)$ is a homogeneous polynomial of degree d , then $g(y_1, \dots, y_n) = 0$ implies $g(v_1, \dots, v_n) = 0$ for all $v_i \in \hat{G}$. Note that

$$(4) \quad g(y_1, \dots, y_n) = 0 \iff g(x_1, \dots, x_n) \in G^{d+1} \iff g(v_1, \dots, v_n) = 0$$

for all homogeneous $v_i \in \hat{G}$. Thus, if g is multilinear, then $g = 0$ is an identity in $\hat{G}$. Put $D = D(g) = d - n$. So, we have already proved the proposition in the case $D = 0$. Suppose that $D > 1$; then, for example,

$$(4') \quad \begin{aligned} h &= h(x_1, x', x_2, \dots, x_n) \\ &= g(x' + x_1, x_2, \dots, x_n) - g(x_1, \dots, x_n) - g(x', x_2, \dots, x_n) \end{aligned}$$

has $D' = D(h) < D = D(g)$. Note that by (4) $h(y_1, \dots, y_{n+1}) = 0$ and, hence, by induction, $h = 0$ is an identity in $\hat{G}$. Eventually, if $g(v_1, \dots, v_n) = g(v', v_2, \dots, v_n) = 0$, then by (4'), $g(v_1 + v', v_2, \dots, v_n) = 0$. Thus, $\hat{G}$ is relatively free.

Now if f, g are homogeneous in $\hat{G}$ the verbal ideals $\{f\}, \{g\}$ generated by f and g in G are comparable, i.e., for example,

$$g = g(f(\dots, u_j(\dots x_i \dots), \dots), \dots, v_j(\dots x_i \dots), \dots))$$

and, hence, in $\hat{G}$,

$$g = g(f(\dots, u_j(\dots y_i \dots), \dots), \dots, v_j(\dots y_i \dots), \dots),$$

i.e. $f = 0$ implies $g = 0$. Since in [1] we used only this criterion for the description, $\text{var } \hat{G}$ has a chain of subvarieties. This completes the proof.

Now by Theorem 1 since $\hat{G}$ is homogeneous either $\hat{G}^2 = 0$, i.e. $G^2 = G^3$ and $G^2 \in \wedge G^n = 0$, or $\hat{G}$ is an F_q -algebra, $mG = \alpha G \subseteq G^2$ and $\hat{G}$ satisfies identities from [1, Theorems 2–8]. Our aim is to prove that very often $\hat{G} = G$. Note that if G_n is a V -free algebra of rank n , that is, G_n possesses free generators $x_1, \dots, x_n$, then $G_n^r = G_n^{r+1}$ implies $G_n^r = 0$.

THEOREM 2. *If $\hat{G}$ is commutative, associative, $G^2 \neq 0$, and either $x^p = 0$, $p = \text{char } F_q$, or $x_1 \cdots x_{p+1} = 0$, then G has the same property. If $x^p = 0$, then $m = 0$. If $x_1 \cdots x_{p+1} = 0$ and $m \neq 0$ then $x^p = \alpha x$, $q = p$, $m = F_p \alpha$, $\alpha^2 = 0$, $\alpha x^2 = 0$. In all the cases $V = \text{var } G$ has a chain of subvarieties.*

PROOF. If $x^p = 0$ in $\hat{G}$, then $G_1^p = G_1^{p+1}$ and, as we have already noticed, this implies $G_1^p = 0$. Similarly, $G_2^{2p-1} = G_3^{3p-2} = 0$ and by Proposition 1, G is commutative, associative, $x^p = 0$. The case $x_1 \cdots x_{p+1} = 0$ is analogous.

Now if $G^2 \neq 0$ then $\alpha G \leq G^2$ and if $x^p = 0$, by Proposition 1, $\alpha x = 0$, i.e. $m = 0$. If $m \neq 0$ and $x_1 \cdots x_{p+1} = 0$, then by Proposition 1, $\alpha x = \beta x^p$, $\beta \in k$, $p = q$. Multiplying by x we obtain $\alpha x^2 = \beta x^{p+1} = 0$ and, therefore, $\alpha^2 x = \alpha \beta x^p = 0$, i.e. $\alpha^2 = 0$. So, if $\alpha \neq 0$, then $\beta \in k^*$, and without loss of generality we can assume $\beta = 1$. In this case V has a chain of subvarieties

$$0 \subset \{x_1 x_2 = 0\} \subset \cdots \subset \{x_1 \cdots x_p = 0\} \subset \{x^p = \alpha x = 0\} \subset V.$$

Theorem 2 is proved.

Suppose now that $\hat{G}$ is a metabelian Lie algebra. Then $\hat{G}$ satisfies conditions of [1, Theorem 3]. Starting from now we are going to use the following notation

$$x_1 \cdots x_n = ((\cdots ((x_1 x_2) x_3) \cdots) x_{n-1}) x_n, \quad xy^n = xy \cdots y.$$

THEOREM 3. *If $\hat{G}$ is a metabelian Lie algebra from [1, Theorem 3], $G^2 \neq 0$, then $m = 0$ and G is either metabelian Lie algebra, $\hat{G} = G$, with one of the following sets of identities ($p = \text{char } F_q$):*

- (i) $xyz^p = xy^p x^{p-1} = 0$,
- (ii) $x_1 \cdots x_{p+3} = 0$ and if $p = 2$ then either $yx^2y = 0$ or $yxzt + zxyt + txyz = 0$; if $p = q > 3$ then either $xy^{p+1} = 0$ or $xyz^p + yzx^p + zxy^p = 0$, or $p = q \geq 3$, $x_1 \cdots x_{p+3} = 0$ and G satisfies one of the following sets of identities:
 - (i) $p = q \geq 3$, $xy^{p+1} = 0$, $x^2 = 0$, $J(x, y, z) = \beta xyz^p$, $\beta \in F_p^*$, $(xy)(zt) = 0$,
 - (ii) $p = q = 3$, $x^2 = (xy)(zt) = 0$, $J(x, y, z) = \beta(xyz^p + yzx^p + zxy^p)$.

In all the cases V has a chain of subvarieties.

PROOF. Since $x^2 = 0$ in $\hat{G}$, $\hat{G}_1^2 = G_1^2 = 0$, i.e. $x^2 = 0$ in V . As we have already noticed above, $mG_1 \subseteq G_1^2 = 0$, and thus $mx = 0$, $m = 0$.

Suppose now that $xyz^p = xy^p x^{p-1} = 0$ in $\hat{G}$. Then the algebra $\hat{G}_3$ is nilpotent $\hat{G}_3^{3p-2} = 0 = G_3^{3p-2}$. Hence G_3 has these identities. Since $xy^p z = zy^p x + xzy^p$ in $\hat{G}_3$ (see [1, p. 58]), we have in G_3 :

$$(5) \quad J(x, y, z) + \beta xy^p z + \gamma xz^p y + \delta yx^p z = 0, \quad \beta, \gamma, \delta \in F_q.$$

If, for example, $\beta \neq 0$, the linearization of (5) by y gives us $x_1 \cdots x_{p+2} = 0$ (see [1, Lemma 2, p. 58]); thus $J(x, y, z) = 0$, that is, G is a Lie algebra.

Similarly, $(xy)(zt) + \beta xy^p zt + \cdots = 0$ implies $(xy)(zt) = 0$.

Suppose now that $x_1 \cdots x_{p+3} = 0$ in $\hat{G}$ and, hence, in G . Then G has all identities of $\hat{G}$ of degree $p + 2$. By Proposition 1 $(xy)(zt) \in G_4^{p+3} = 0$. Similarly $J(x, y, z) \in G_3^{p+2}$ and therefore

$$J(x, y, z) = (\beta xy^p z + \cdots) + (\gamma xzy^p + \cdots).$$

If $\beta \neq 0$ then, again by [1, Lemma 2, p. 58], we have $xy^p z = 0$. So, we can suppose that

$$(5') \quad J(x, y, z) = \beta xyz^p + \gamma yzx^p + \delta zxy^p, \quad \beta, \gamma, \delta \in F_q.$$

If $q > p$ then, by Proposition 1, $J(x, y, z) = 0$, and in this case the theorem is proved.

Suppose now that $p = 2$, $x_1 \cdots x_5 = 0$ and $yx^2y = 0$. As it has been noticed in [1, p. 59], $yx^2y = 0$ implies $xyz^2 = 0$ and thus $J(x, y, z) = 0$ by (5'). If $yxzt +$

$xyz^2 + txyz = 0$ put $t = z$ and obtain $yxz^2 = 0$ and then $J(x, y, z) = 0$.

If $p \geq 3$ and $xy^{p+1} = 0$, then as it has been noticed in [1, p. 58], we have $xyz^p = zxy^p$ and so $J(x, y, z) = \beta xyz^p$.

If $p \geq 3$ and $xyz^p + yzx^p + zxy^p = 0$, then we can assume that in (5') $\delta = 0$. If $\beta \neq 0$ in (5') put $y = z$. We obtain $xy^{p+1} = 0$ and as above $xyz^p = zxy^p = yzx^p$, which implies $J(x, y, z) = 0$.

Finally suppose that $p = 3$, $xy^4 \neq 0$, put $y = z$ in (5'). We have $\beta xy^4 - \delta xy^4 = 0$ and, therefore, $\beta = \delta$. Similarly $\beta = \gamma = \delta$, i.e.

$$J(x, y, z) = \beta(xyz^p + yzx^p + zxy^p).$$

To finish the proof we need to show that if $p \geq 3$,

$$x^2 = (xy)(zt) = xy^{p+1} = J(x, y, z) - \beta xyz^p = x_1 \cdots x_{p+3} = 0$$

or $p = 3$,

$$x^2 = (xy)(zt) = xyz^3 = J(x, y, z) - \beta(xyz^3 + zxy^3 + yzx^3) = 0,$$

then V has a chain of subvarieties. In the first case if $f = 0$ is any element in G , then modulo verbal ideal generated by xyz^p it is (see [1, p. 61]) equivalent to $x_1 \cdots x_n = 0$, $n < p + 3$, or $\Sigma_i f_{i,p} = 0$, where $f_{i,p} = x_i x_1 \cdots \hat{x}_i \cdots x_p$, and each of them implies $xyz^p = 0$. So subvarieties of V form a chain. The second case is analogous. This completes the proof.

Suppose now that $p = 2$. The algebra $\hat{G}$ satisfies one of the sets of identities from [1, Theorem 8]. The cases of commutative and associative algebras, metabelian Lie algebras have been already considered. So we can suppose that $\hat{G}$ satisfies identities (3)–(7) from Theorem 8. In the case (3) we have $\hat{G}^4 = G^4 = 0$ and since $\hat{G}$ was defined by identities of degree three, G has the same identities. If $G^2 \neq 0$, then $\alpha G_1 = mG_1 \subseteq G_1^2$, i.e. $\alpha x = \beta x^2 + \gamma x^2 x$, $\alpha \in m$, $\beta, \gamma \in k$. Multiplying twice by x we have $\alpha x^2 = \beta x^2 x$, $\alpha x^2 x = 0$. So $kx^2 x = F_q x^2 x$. If $\gamma \in F_q^*$ then by [1, p. 76], the linearization of $x^2 x$ is equivalent to $x^2 y = 0$. Thus, without loss of generality we can suppose that $\gamma = 0$. So $\alpha x = \beta x^2$. If $\beta \in k^* = k \setminus m$, then

$$0 = (x + y)^2 - \alpha \beta^{-1} (x + y) - x^2 + \alpha \beta^{-1} x - y^2 + \alpha \bar{\beta}^{-1} y = xy + yx$$

and, hence, $xyz = 0$ in G , i.e. G is commutative and associative. If $\beta = \alpha \gamma$, $\gamma \in k$, then $\alpha x^2 = \alpha \gamma x^2 x = 0$ and therefore $\alpha x = \alpha \gamma x^2 = 0$.

Similarly, if $\hat{G}$ satisfies identities of type (6)–(7) from Theorem 8, then $\hat{G} = G$.

Suppose now that $\hat{G}$ is of type (4), that is $J(x, y, z) = 0$, $x(yz) = \gamma xyz + \delta yxz$, $yzx = \beta^2 xyz + (\beta^2 + \beta + 1)yxz$, $\hat{G}^5 = G^5 = 0$. As was mentioned in

[1, p. 77], in this case every $f = f(x, y, z) \in G^4$ is a linear combination of x^2yz , y^2zx , z^2yx and a linearization of x^2yz is equivalent to $xyzt = 0$. Thus, if $J(x, y, z) + \beta x^2yz + \gamma y^2zx + \delta z^2xy = 0$ in G and if $\beta \neq 0$, then the linearization of this identity gives $xyzt = 0$ and, hence, $J(x, y, z) = 0$ in G . Similarly we consider the other identities and a case (5).

Finally, since $J(x, x, x) = J_1(x, x, x) = x^2x = xx^2 = 0$, then if $G^2 \neq 0$, $\alpha G_1 \subseteq G_1^2$, that is $\alpha x = \beta' x^2$ and as in the preceding case if $G^3 \neq 0$ then $\alpha = 0$.

Eventually we have proved

THEOREM 4. *Let $p = 2$ and G is not commutative, associative or metabelian Lie algebra. Then $m = 0$, $\hat{G} = G$ satisfies one of the following sets of identities (numeration from [1, Theorem 8]):*

(iii) $xyzt = 0$ and either $x(yz) = (\alpha + \beta)xyz + \beta yxz$, $\alpha \in F_q \setminus 1$, $xyz = yzx$, or $xyz = \alpha z(xy + yx)$, $x(yz) = y(zx)$,

(iv) $yxztu = 0$, $J(x, y, z) = 0$, $yzx = \beta^2 xyz + (\beta^2 + \beta + 1)yxz$, $x(yz) = \delta' yxz + \gamma' xyz$, $\beta, \delta', \gamma' \in F_q$,

(v) $xyztu = 0$, $J_1(x, y, z) = 0$, $x(zy) = \delta^2 z(yx) + (\delta^2 + \delta + 1)z(xy)$, $xyz = \alpha' z(xy) + \beta' z(yx)$,

(vi) $J(x, y, z) = 0$, $x(yz) = \delta yzx$, $\delta \neq 1$, $xyz = yxz$,

(vii) $J(x, y, z) = xyz = 0$, $x(yz) = x(zy)$.

Here $J(x, y, z) = xyz + yzx + zxy$, $J_1(x, y, z) = x(yz) + y(zx) + z(xy)$.

Assume now that $p > 2$.

THEOREM 5. *If $\hat{G}$, $G^2 \neq 0$, satisfies identities $xy - yx = x^3 = (xy)(zt) = 0$, then $m = 0$, $\hat{G} = G$ and V has a chain of subvarieties.*

PROOF. As in [1] we have $xyzt = -xytz$, $xyz^2 = 0$ and hence $G_3^5 = 0$. Thus, by Proposition 1, $x^3 = (xy)(zt) = 0$. Moreover, if x appears at least three times in a monomial f , then $f = 0$. Thus, G is commutative. Finally, $mG \subseteq G^2$ implies, by Proposition 1, $m = 0$.

THEOREM 6. *If $\hat{G}$ is a 3-Engel Lie algebra, $p > 2$, $G^2 \neq 0$, then $\hat{G} = G$, $m = 0$ and V has a chain of subvarieties.*

PROOF. Since $x^2 = 0$ in $\hat{G}$, $\hat{G}_1^2 = G_1^2 = 0$ and $mG_1 \subseteq G_1^2 = 0$, i.e. $m = 0$. Note that by Theorem 7 [1] if $p = 3$ then $\hat{G}^5 = G^5 = 0$, if $p > 3$ then $\hat{G}_3^6 = G_3^6 = 0$. Thus by Proposition 1 we have $J(x, y, z)$, $xy^3 \in G_3^7 = 0$.

THEOREM 7. *If $p = 3$ and $\hat{G}$ satisfies one of the following sets of identities (numeration from [1, Theorem 4]), $G^2 \neq 0$,*

(v) $x^2 = xyztu = 0$, $xyzt = (\alpha + \beta)(tz)(yx) - \alpha(tv)(zx) + \alpha(tx)(zy)$, where either $\alpha \neq 0$ or $\beta \neq 0$,

- (vi) $x^2 = xyztu = 0$ and either $xytz = (\gamma - 1)xyzt - \gamma(\gamma + 1)\mathcal{V}(x, y, z)t$, $txyz = \gamma xyzt + yzxt - \gamma^2 J(x, y, z)t$, $\gamma \in \mathbb{F}_q$, or $xyzt = xyzt + yztx + zxtx$,
 (vii) $x^2 = xyztu = 0$, $xytz + xyzt = txyz + yzxt = 0$, then $m = 0$, $\hat{G} = G$ and V has a chain of subvarieties.

Proof is obvious, since $x^2 = 0$ in $\hat{G}$ implies $m = x^2 = 0$ in G . Also $\hat{G}^5 = G^5 = 0$ and $\hat{G} = G$ since G is defined by identities of degree four.

THEOREM 8. If $p > 3$ and $\hat{G}$, $G^2 \neq 0$ has either identities $x^2 = xyz + x(yz) = xyzt = 0$ or $xy - yx = x^3 = xyzt - xyzt = xyztu = 0$, then $m = 0$, $\hat{G} = G$ and in these cases V has a chain of subvarieties.

PROOF. In the first case $x^2 = 0$ implies as usual $m = 0$ and since $xyzt = 0$ by Proposition 1, $\hat{G} = G$. In the second case $mG_1 \subseteq G_1^2$ and $x^3 = 0$, $p > 3$, imply by Proposition 1, $m = 0$. Now $xy - yx \in \hat{G}_2^6 = G_2^6 = 0$. Thus, $\hat{G} = G$.

3. Power-associative algebras. Let k be a Noetherian commutative, associative ring, V variety of power-associative algebras, $F = G_1$ a V -free algebra with one generator x . Then F is a Noetherian commutative associative ring. Let J be the Jacobson radical of F . If $J = F$ then by results from [7, p. 12], $\bigwedge_n F^n = 0$, so we can apply to $\text{var } F$ Theorem 2 and obtain

THEOREM 9. If $J = F \neq 0$, then $x^n = 0$, $2 \leq n \leq p + 1$, k has a unique maximal ideal m and if $m \neq 0$, then either $k/m = \mathbb{F}_p$, $m = \mathbb{F}_p \alpha$, $\alpha x = x^p$, $x^{p+1} = \alpha^2 = \alpha x^2 = 0$ or $x^2 = 0$.

Suppose now that $F \neq J$. Then F/J is a subdirect product of fields K_j , $F/J \subseteq \prod K_j$. Each K_j from this decomposition is a k -algebra, that is we have ring homomorphisms $f_j: k \rightarrow K_j$, $\text{Ker } f_j = \mathfrak{p}_j$. The ideal $\mathfrak{p}_j$ is prime and $k_j = k/\mathfrak{p}_j \in \text{var } K_j \subseteq V$. If $L \triangleleft k_j$, then $L = \text{Ann var } k_j/L$ and, therefore, ideals of k_j form a chain, that is k_j is a valuation ring (see [7]) with unique maximal principal ideal (δ_j) and every ideal of k_j is equal to some (δ_j^t) , $t \geq 0$.

Now K_j is generated by k_j and the image of x , which is different from zero, since $F/J \neq 0$. By (3), (3') we have in K_j : $\delta_j^{(d(d-1)+2)/2} x^2 = 0$, and since K_j is a field, $\delta_j = 0$. Thus, k_j is a field as well. For distinct i, j we have, for example,

$$\mathfrak{p}_i = \text{Ann var } k_i \subseteq \text{Ann var } k_j = \mathfrak{p}_j$$

and since k_i, k_j are fields, $\mathfrak{p}_i = \mathfrak{p}_j = m$. Note that $\text{var } J/J^n \not\subseteq K_j$, hence, $\text{var } K_j \supseteq J/J^n$ and, therefore,

$$m = \text{Ann var } K_j = \text{Ann var } F/J \subseteq \text{Ann var } J/J^n$$

for all $n > 1$. But m is a maximal ideal. Thus, either $m = \text{Ann var } J/J^n$ and

$mJ \subseteq \bigwedge_n J^n = 0$ (see [7, p. 12]) or $J = J^n = \bigwedge_n J^n = 0$. In both cases $m^2 F \subseteq mJ = 0$ and therefore m is a unique maximal ideal.

Suppose now, that $k/m = F_q$. Then each K_j is a F_q -algebra, $\text{var } F_q \subseteq V$ and hence $\text{var } F_q \supseteq \text{var } J/J^2$. But in $\text{var } F_q$ each finite F_q -algebra is a direct product of F_q (see, for example, [8]). So, $J/J^2 \in \text{var } F_q$ implies $J = J^2$ and therefore $J = 0$, $mF \subseteq J = 0$. Eventually, we have

THEOREM 10. *If $J \neq F$, then k has a unique maximal ideal m , $m^2 = 0$. If $k/m = F_q$, then $J = 0$, $m = 0$, $k = F_q$.*

Since the case $|k/m| = \infty$ has been reduced by Theorem 1 to the results of [1] we assume throughout this paper, by Theorems 9 and 10, $k/m = F_q$, and if $J = F$, then $m = F_q \alpha$, $\alpha^2 = 0$, if $J \neq F$, then $J = m = 0$.

4. Alternative algebras. We need

PROPOSITION 5. *Let A be a right alternative k -algebra (see [9]). If A is anticommutative, then $yxz + y(xz) = 0$ and if $\text{char } F_q \neq 2$, then $xyzt = 0$. If A is commutative (and, hence, alternative), $p = \text{char } F_q \neq 3$, then A is associative. If A is commutative, $p = 3$, then A is alternative and $J(x, y, z) = 0$. Conversely, if A is commutative, $p = 3$ and $J(x, y, z) = 0$, then A is alternative.*

PROOF. If A is anticommutative and $(x, y, z) + (x, z, y) = 0$, then

$$\begin{aligned} xyz &= (x, y, z) + x(yz) = -(x, z, y) - yzx = -xzy + x(zy) - yzx \\ &= zxy - zyx - yzx = zxy = yzx = -x(yz). \end{aligned}$$

If $p > 2$, then

$$xyzt = -(x(yz))t = x((yz)t) = -x(y(zt)) = (xy)(zt) = -xyzt,$$

i.e. $xyzt = 0$.

If A is commutative, then

$$xyz = (x, y, z) + x(yz) = -(y, x, z) + yzx = -xyz + xzy + yzx,$$

and, hence,

$$(6) \quad 2xyz = xzy + yzx, \quad 2yzx = yxz + zxy.$$

From (6) $3xyz = 3xzy$ and since $p \neq 3$, $xyz = xzy = yxz = yzx = x(yz)$. If $p = 3$, then in (6) $J(x, y, z) = 0$.

Conversely, if $J(x, y, z) = 0$, $p = 3$, then

$$x^2y - x(xy) = x^2y - xyx = x^2y + 2xyx = J(x, y, x) = 0.$$

Similarly, $yx^2 = yxx$.

Let V be a variety of alternative k -algebras. Then $F = G_1$ satisfies conditions of Theorems 9 and 10. If $J = F$ and, therefore, $x^{p+1} = 0$, then by a theorem of Shirshov-McCrimmon (see [10]), V is locally nilpotent and V -free algebras have property (2). Applying Theorems 2–8 we have

THEOREM 11. *Let V be a variety of alternative k -algebras with $x^{p+1} = 0$. Then V is one of varieties:*

- (i) *any variety of associative commutative algebras from Theorem 2;*
- (ii) $p \neq 2$, $xy + yx = xyz + x(yz) = xyz = 0$, $m = 0$;
- (iii) $p = 3$, $xy = yx$, $x^3 = (xy)(zt) = 0$, $m = 0$;
- (iv) $p = 2$, $xyz = 0$ and either $x(yz) = \alpha xyz$, $\alpha \neq 1$, $xyz = yzx$, or $x(yz) = y(zx) = y(xz)$, $xyz = 0$; in both cases $m = 0$;
- (v) $p = 2$, $(x, y, z) = J(x, y, z) = xyz = 0$, $m = 0$;
- (vi) $p = 2$, $(x, y, z) = J(x, y, z) = yzx + \beta^2 xyz + (\beta^2 + \beta + 1)yxz = xyz = 0$, $\beta \neq 1$, $m = 0$.

Note. This theorem is valid when k is an infinite field, $\text{char } k > 0$. If k is a field of characteristic zero, then V is either of type (ii) or any variety of commutative and associative algebras.

PROOF. By Proposition 5 we can suppose that V is not anticommutative and if G is commutative, but not associative, then $p = 3$, $J(x, y, z) = 0$ and in this case, by Theorems 5–6, V has identities $x^3 = (xy)(zt)$. Note that linearization of $x^3 = 0$ gives us $J(x, y, z)$, so we have (iii).

Suppose now that $p = 2$ and V of types (iii)–(vii) from Theorem 4. In case (iii), $x^3 = x^2x = (\alpha + \beta)x^2x + \beta x^2x = \alpha x^2x$, where $\alpha \neq 1$. So, $x^3 = 0$. But, by results from [1, p. 79], the linearization of $x^3 = 0$ implies $x^2y = 0$ and, hence, $x(yz) = \alpha xyz$, $\alpha \neq 1$. The dual case in (iii) is similar.

Now if G is of types (iv)–(vii) from Theorem 4, then $J(x, y, z) = J_1(x, y, z) = 0$ and by alternativity,

$$0 = xyz + x(yz) + yzx + y(zx) = zxy + z(yx),$$

i.e. V is associative. In cases (iv) and (v) we have

$$J(x, y, z) = yzx + \beta^2 xyz + (\beta^2 + \beta + 1) = 0$$

or

$$xzx = \beta^2 x^2 z + (\beta^2 + \beta + 1)x^2 z = (\beta + 1)x^2 z,$$

and by $J(x, x, z) = 0$ we have $zx^2 = \beta x^2 z$. As we have noticed in [1, p. 77], in this variety $xyz = yxtz = ztxy$. Thus, $x^2 zt = ztx^2 = \beta x^2 zt$. So, if $\beta \neq 1$, then $x^2 zt = 0$ and $xyz = yxzt$, i.e. since $F_q S_4 xyz$ is the irreducible two-dimensional $F_q S_4$ -module, $xyz = 0$.

Now consider case (vi) from Theorem 4. V is associative and $xyz = yxz$, $xyz = \delta yzx$, $\delta \neq 1$. Then $xyz = \delta yzx = \delta zy x = \delta^2 yxz = \delta^2 xyz$. Since $\delta \neq 1$ this implies $xyz = 0$. Similarly, in (vii) we have $xyz = 0$. So, in both cases V is of type (v). By Proposition 5 this completes the proof.

THEOREM 12. *If in $F = G_1$ we have $J(F) = 0$, $k = F_q$, then $V = \text{var } F_{q^r}$, where $r = 1$, or r is a prime-power. In this case subvarieties of V form a chain and V is defined by identity $X = X^{q^r}$.*

PROOF. If $x \in F$ is not algebraic, then $F = (X) \triangleleft F_q[X]$ and, thus, F contains two verbal ideals (X^2) , $(X - X^q)$, which do not contain one another. Thus, x is algebraic, $\dim_k F < \infty$ and since $J(F) = 0$, $F = \bigoplus_{i=1}^n F_q d_i$ and therefore, $\text{var } F = \text{var } F_{q^r}$ for some F_{q^r} from this decomposition. By Jacobson's theorem (see [6], [11]) two-generated algebras in V are commutative and hence, V is associative (see [12, Lemma 3]), since V has an identity $X = X^{q^r}$.

LEMMA 1. *If $z \in \mathbb{Z}$, $z > 1$, then $(z^s - 1) \mid (z^t - 1)$ iff $s \mid t$.*

PROOF. Consider the ring $R = \mathbb{Z}/(z^s - 1)$. The element $z \in R^*$ has order s . So $z^t = 1$ iff $s \mid t$.

LEMMA 2. *The field F_{q^s} satisfies the identity $X = X^{q^t}$ iff $s \mid t$.*

PROOF. Take β to be a generator of $F_{q^s}^*$. Then β has order $q^s - 1$. Thus, $\beta^{q^t - 1} = 1$ iff $(q^s - 1) \mid (q^t - 1)$ and iff $s \mid t$ by Lemma 1.

Now V has the identity $X = X^{q^r}$ and by Lemma 2 and Jacobson's theorem each algebra $A \in V$ is a subdirect product of subfields of F_{q^r} . So $V = \text{var } F_{q^r}$ and V has a chain of subvarieties iff $r = 1$ or r is a prime power.

5. (γ, δ) -algebras. In this part we make some remarks concerning (γ, δ) -algebras. Recall that algebra A over a field k is a (γ, δ) -algebra iff

$$(x, y, z) + \gamma(y, x, z) + \delta(z, x, y) = 0,$$

$$(x, y, z) - \gamma(x, z, y) + (1 - \delta)(y, z, x) = 0$$

where $\gamma, \delta \in k$ and $\gamma^2 - \delta^2 + \delta - 1 = 0$.

As it has been shown in [13, p. 518], if $\text{char } k \neq 2, 3, 5$, then A is power-associative. Moreover, if $\text{char } k \neq 2, 3$ and A has no nilpotent elements, then A is associative [13, Theorem 1, p. 516]. In the special case $\gamma = -1$, $\delta = 1$, if $\text{char } k \neq 2$, then A is power-associative, and if $\text{char } k \neq 2, 3$ and $x^n = 0$ for all $x \in A$, then A is locally nilpotent (see [14]), and, hence, $\text{var } A$ has property (2). Thus, we have

THEOREM 13. *Let V be a variety of (γ, δ) -algebras over a field F_q , where $p = \text{char } F_q > 5$ for arbitrary (γ, δ) and $p > 3$ for $\gamma = -1$, $\delta = 1$. If V is not a*

variety of nilalgebras, then by Theorem 12, $V = \text{var } F_{q^r}$, where $r = 1$ or r is a prime-power. If $\gamma = -1$, $\delta = 1$, $p > 3$ and V is a variety of nilalgebras, then V is one of the following varieties:

- (i) variety of commutative, associative algebras from Theorem 2,
- (ii) $p > 2$, $xy + yx = xyz + x(yz) = xyz = 0$,
- (iii) $p = 3$, $xy = yx$, $x^3 = (xy)(zt) = 0$.

The proof by Proposition 5 follows from Theorems 5–8 since $(-1, 1)$ -algebras are right alternative.

Note that if k is an infinite field, $\text{char } k \neq 2, 3$ then (i)–(ii) are the only varieties of $(-1, 1)$ -algebras with a chain of subvarieties, since in this case they are homogeneous.

6. Jordan algebras. Throughout this part, V is a variety of Jordan algebras, k from Theorems 9 and 10, and as usual in this case, $p = \text{char } F_q > 2$.

Assume first that $F = G_1$ satisfies conditions of Theorem 10. Consider the Jordan F_q -algebra A_δ which is a subalgebra of the matrix algebra $\text{Mat}(2, F_{q^s})^+$, consisting of all matrices

$$(7) \quad T = \begin{pmatrix} \alpha & \beta\delta \\ \beta & \gamma \end{pmatrix}$$

where $\alpha, \beta, \delta, \gamma \in F_{q^s}$, δ is fixed, $-\delta \notin F_{q^s}^2$. So, $\dim_{F_{q^s}} A_\delta = 3$.

PROPOSITION 6. Algebras A_δ for distinct δ are isomorphic.

PROOF. Define the map $f: A_\delta \rightarrow A_{\delta\gamma^2}$, $\delta, \gamma \in F_{q^s}^*$, by

$$\begin{pmatrix} \alpha_1 & \beta\delta \\ \beta & \alpha_2 \end{pmatrix}^f = \begin{pmatrix} \alpha_1 & \beta\delta\gamma \\ \beta\gamma^{-1} & \alpha_2 \end{pmatrix}.$$

An easy calculation shows that f is a homomorphism, and, therefore, an isomorphism.

PROPOSITION 7. If $\beta \neq 0$ in (7), then eigenvalues of T are distinct.

PROOF. The characteristic polynomial of T is $X^2 - X(\alpha + \gamma) + (\alpha\gamma - \beta^2\delta)$. If its roots coincide, then

$$0 = (\alpha + \gamma)^2 - 4\alpha\gamma + 4\beta^2\delta = (\alpha - \gamma)^2 + (2\beta)^2\delta$$

or $-\delta = [(2\beta)^{-1}(\alpha - \gamma)]^2$, which is impossible.

Note that eigenvalues of T belong to $F_{q^{2s}}$.

PROPOSITION 8. *There exists $T \in A_\delta$ such that the eigenvalues of T belong to $F_{q^{2s}} \setminus F_{q^s}$.*

PROOF. Suppose that for all $T \in A_\delta$ the eigenvalues belong to F_{q^s} . Then for all $\alpha, \beta, \gamma \in F_{q^s}$,

$$(\alpha + \gamma)^2 - 4\alpha\gamma + 4\delta\beta^2 = (\alpha - \gamma)^2 + (2\beta)^2\delta \in F_{q^s}^2,$$

that is for all $x, y \in F_{q^s}$ there exists $z \in F_{q^s}$ such that $x^2 + y^2\delta = z^2$. Hence, $0 + 1 \cdot \delta = u^2$ and $x^2 + y^2\delta = x^2 + (uy)^2 = z^2$. Since $\delta \neq 0, u \neq 0$ and for all $x, y \in F_{q^s}$ there exists $z \in F_{q^s}$ such that $x^2 + y^2 = z^2$. Thus, $F_{q^s}^2$ is a subgroup of F_{q^s} . Since $\delta \in F_{q^s}^2, -\delta \in F_{q^s}^2$. However, this contradicts our assumption.

PROPOSITION 9. *The algebra A_δ satisfies the identity $X = X^{q^{2s}}$, not the identity $X = X^{q^s}$.*

PROOF. By Proposition 7 the matrix T from equation (7) is conjugate in $\text{Mat}(2, F_{q^{2s}})$ to the matrix

$$T_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad \alpha_1, \alpha_2 \in F_{q^{2s}},$$

and, therefore, $T_1^{q^{2s}} = T_1$. The affirmation follows now from Propositions 7 and 8.

THEOREM 14. *Let V be a variety of Jordan F_q -algebras, $p = \text{char } F_q > 2$, and $X = X^{q^r}$ in V , where $r = 1$ or r is a prime-power. Then either $V = \text{var } F_{q^r}$ or $r = 2$, $V = \text{var } A_\delta$, where A_δ is the F_q -algebra, defined above (with $s = 1$).*

PROOF. Let $B \in V$. By Theorems 15.11, 15.4 from [6] and Proposition 6, B is a subdirect product of fields F_{q^t} , $t|r$, and algebras A_δ over some fields F_{q^s} . By Proposition 6 we assume $\delta \in F_q$. By Proposition 9 since $F_{q^s} \in V$ and $s|r$ we have $F_q[\alpha] \in V$ for some $\alpha \in F_{q^{2s}} \setminus F_{q^s}$, i.e. by Lemma 2 from Theorem 12 we have $2|r$. Thus, $r = 2^n$ and $F_{q^{2s}} = F_q[\alpha] \in V$.

LEMMA 1. *If $A_\delta \in V$, then $s = 1$.*

PROOF. Let A'_δ be a F_q -subalgebra of A_δ with $\alpha, \beta, \gamma \in F_q$ in (7). Note that by our assumption $\delta \in F_q$. If $\text{var } F_{q^{2s}} \supseteq A'_\delta$, then as we have already noticed at the end of §4 of this paper, A'_δ is a subdirect product of fields F_{q^t} . But A'_δ is a simple algebra (see [6, Proposition 15.4]), so $A'_\delta \cong F_{q^t}$, which is impossible since A'_δ has nontrivial idempotents. Thus, $F_{q^{2s}} \in \text{var } A'_\delta$ and by

Proposition 9, Lemma 2 from Theorem 12, we have $2s|2$, i.e. $s = 1$, $A'_\delta = A_\delta$.

Finally, every algebra $B \in \text{var } A_\delta$ over F_q satisfies the identity $X = Xq^2$ and by results of [6], B is a subdirect product of some copies of F_q , F_{q^2} and A_δ (see also Proposition 6). Hence there is the only chain

$$0 \subset \text{var } F_q \subset \text{var } F_{q^2} \subset \text{var } A_\delta = V$$

of subvarieties. This completes the proof.

Now suppose that the V -free algebra $G_1 = F$ enjoys conditions of Theorem 9.

THEOREM 15. *If F enjoys conditions of Theorem 9, $G^2 \neq 0$, then V is one of the varieties:*

- (i) *commutative and associative algebras from Theorem 2, $p \neq 2$,*
- (ii) *$p \neq 2$, $xy = yx$, $x^3 = (xy)(zt) = m = 0$,*
- (iii) *$p \neq 2, 3$, $xy = yx$, $x^3 = xyzt - xytz = xyztu = m = 0$.*

PROOF. Since V is a variety of nilalgebras, $x^{p+1} = 0$, by results of Shirshov [15] special algebras in V are locally nilpotent. Thus $U = \text{var } G_2$ has property (2) and therefore U is a commutative variety from Theorems 2, 5 and 8. If U is associative, then V is a variety of alternative algebras and by Theorem 11, V is of types (i) or (ii), since V is commutative.

Suppose now, that in U we have $xy - yx = x^3 = (xy)(zt) = 0$. Then in V we have $xy - yx = x^3 = x^2 \cdot y^2 = 0$. Since V is commutative, linearization of $x^2 \cdot y^2 = 0$ gives us

$$0 = (x + y)^2 \cdot z^2 - x^2 \cdot z^2 - y^2 \cdot z^2 = 2(xy) \cdot z^2$$

and $(xy) \cdot z^2 = 0$. Similarly, $(xy)(zt) = 0$, i.e. V is of type (ii).

Suppose finally that U has identities

$$xy = yx, \quad x^3 = 0, \quad xyzt = xytz, \quad xyztu = 0, \quad p \neq 2, 3.$$

The linearization of $x^3 = 0$ gives us $J(x, y, z) = 0$, i.e. in V we have $xy = yx$, $x^3 = J(x, y, z) = 0$. Moreover, in U , and, hence, in V we have

$$(8) \quad x^2yx = x^3y = 0, \quad yxxx = xyxx = -\frac{1}{2}x^3y = 0,$$

$$J(x, x, y) = x^2y + 2xyx = 0, \quad xyx = yxx.$$

By linearization of $x^2yx = 0$ we obtain

$$(8') \quad x^2yz + 2zxyx = 0.$$

Also, by linearization of $xyxy - xy yx = 0$ by x and y we have

$$(8'') \quad xyxz + xzxy + yzxx = 0, \quad xyzz + xzzy + yzzx = 0.$$

LEMMA. Let G_3 be a V -free algebra with free generators $x = x_1$, $y = x_2$, $z = x_3$, and $M \in G_3$ a monomial

$$M = t_1 \cdots t_n, \quad n > 2, \quad t_i = x_1, x_2, x_3.$$

If x_m appears j_m times in M , then

$$M = \beta x_{i_1}^{j_{i_1}} x_{i_2}^{j_{i_2}} \cdots x_{i_2}^{j_{i_2}} x_{i_3}^{j_{i_3}} \cdots x_{i_3}^{j_{i_3}}, \quad \beta \in k^*,$$

$\underbrace{\hspace{1.5cm}}_{j_{i_2}} \quad \underbrace{\hspace{1.5cm}}_{j_{i_3}}$

(i_1, i_2, i_3) is a permutation of $(1, 2, 3)$.

COROLLARY. If x (or y, z) appears in M at least three times, then by (8), $M = 0$.

PROOF OF LEMMA. The case $n = 3$ is trivial by (8). Suppose that $n > 3$, and for $n - 1$ the Lemma was proved. By induction, if $M \neq 0$ we can assume that

$$t_1 \cdots t_{n-1} = \underbrace{x \cdots x}_{j_1} \underbrace{xy \cdots yz}_{j_2} \underbrace{yz \cdots z}_{j_3}, \quad 0 \leq j_i < 3, j_1 + j_2 + j_3 > 2.$$

The case $t_n = z$ is trivial. So, $t_n = x, y$. Note that by (8) and $xyxy = xyyx$,

$$\underbrace{x \cdots x}_{j_1} \underbrace{xy \cdots y}_{j_2} = \gamma \underbrace{y \cdots y}_{j_2} \underbrace{yx \cdots x}_{j_1}, \quad \gamma \in k^*.$$

So without loss of generality we can suppose that $t_n = y$, $j_2 > 0$. If $j_1 = 2$, then put $u = x^2$ and applying an induction and identity

$$(8''') \quad (yz) \cdot x^2 = x^2(yz) = -x(yz)x - yzxx = -2yzxx,$$

we prove our Lemma.

So $j_1 = 1$ (the case $j_1 = 0$ is trivial as we have already noticed),

$$M = xy \underbrace{\cdots yz}_{j_2} \underbrace{\cdots zy}_{j_3}.$$

If $j_3 = 2$, then by (8''') we have

$$M = -\frac{1}{2}(xy \cdots y) \cdot z^2 \cdot y.$$

So applying induction to $M' = xy \cdots yzy$ and using (8''') we prove the Lemma. Suppose now that $j_3 = 1$. Then by (8') if $j_2 = 1$

$$M = xyzy = -\frac{1}{2}y^2zx.$$

If $j_2 = 2$, then by (8), (8''')

$$M = xyyzy = -\frac{1}{2}y^2xzy = -\frac{1}{2}((xz) \cdot y^2)y = xzyyy = 0.$$

This completes the proof.

Now by the Corollary of the Lemma G_3 is nilpotent, so $\text{var } G_3$ has (2) and $\text{var } G_3$ is one of the varieties (i)–(iii) from the conditions of Theorem 15, $\text{var } G_3 \supseteq \text{var } G_2$, and therefore, in $\text{var } G_3$,

$$xy = yx, \quad x^3 = J(x, y, z) = 0, \quad xyzt = xytz.$$

Hence, in V we have $x^2yz = x^2zy$. Linearizing this identity we obtain by commutativity in V , $xyzt = xytz$. Now, as it was proved in [1], this implies $xyztu = 0$. The theorem is proved.

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