HYPOELLIPTIC CONVOLUTION EQUATIONS IN K'_p , $p > 1$

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ABSTRACT. We consider convolution equations in the space K'_p , $p > 1$, of distributions which "grow" no faster than $\exp(k|x|^p)$ for some constant k .

Our main result is a complete characterization of hypoelliptic convolution operators in K'_p in terms of their Fourier transforms.

In [7] and [8], the second author studied hypoelliptic convolution equations in the space S' of tempered distributions and in the space K'_1 of distributions of exponential growth. The purpose of the present paper is to extend these investigations to the space K'_p , $p > 1$, of distributions which "grow" no faster than $\exp(k|x|^p)$ for some constant k .

More precisely, we study convolution equations of the form

$$(1) \quad S * U = V,$$

where S is a distribution in $\mathcal{O}'_c(K'_p : K'_p)$, the space of convolution operators in K'_p , and $U, V \in K'_p$. The space EK'_p of C^∞ -functions in K'_p is defined in a natural way, and the equation (1) is said to be hypoelliptic in K'_p if all solutions $U \in K'_p$ are in EK'_p whenever $V \in EK'_p$.

Our main result is the following characterization of hypoelliptic convolution operators in K'_p in terms of their Fourier transforms (which are entire analytic functions).

THEOREM I. *A distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$ is hypoelliptic in K'_p if and only if its Fourier transform $\hat{S}$ satisfies the following conditions:*

(h₁) *There exist positive constants B and M such that*

$$|\hat{S}(\xi)| \geq |\xi|^{-B} \quad \text{if } \xi \in \mathbb{R}^n \text{ and } |\xi| \geq M.$$

(h₂) $|\operatorname{Im} \zeta|^q / \log |\zeta| \rightarrow \infty$ as $|\zeta| \rightarrow \infty$, $\zeta \in \mathbb{C}^n$, $\hat{S}(\zeta) = 0$, where $1/q + 1/p = 1$.

We also prove

THEOREM II. *Conditions (h₁) and (h₂) combined are equivalent to*

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(h₃) Given $\epsilon > 0$ one can find a $B > 0$ such that for every m there exists a constant C_m so that $|1/\hat{S}(\zeta)| \leq |\zeta|^B e^{\epsilon |\operatorname{Im} \zeta|^q}$ if $|\operatorname{Im} \zeta|^q \leq m \log |\zeta|$ and $|\zeta| \geq C_m$.

We note that, if S is in the space E' of distributions with compact support and $q = 1$, then conditions (h₁) and (h₂) are necessary and sufficient for S to be hypoelliptic in $\mathcal{D}'$ (see [1] and [4]).

1. The spaces K_p and K'_p . We denote by K_p , $p \geq 1$, the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$v_k(\varphi) = \sup_{x \in \mathbb{R}^n; |\alpha| \leq k} e^{k|x|^p} |D^\alpha \varphi(x)| < \infty, \quad k = 1, 2, \dots,$$

where $D^\alpha = (i^{-1} \partial / \partial x_1)^{\alpha_1} \cdots (i^{-1} \partial / \partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The topology in K_p is defined by the family of seminorms v_k . Then K_p becomes a Frechet space and the injections $\mathcal{D} \rightarrow K_p \rightarrow E$ are continuous; here E denotes the space of all C^∞ -functions and $\mathcal{D}$ the space of C^∞ -functions with compact support (see [6]).

By K'_p we mean the space of continuous linear functionals on K_p . The restriction $\tilde{T}$ to $\mathcal{D}$ of a functional $T \in K'_p$ is a distribution. Also, since $\mathcal{D}$ is dense in K_p , T is determined by its values on $\mathcal{D}$, i.e. by $\tilde{T}$. Thus we can identify T with $\tilde{T}$ and regard K'_p as a space of distributions. We characterize the distributions in K'_p by their "growth" at infinity.

THEOREM 1. *A distribution $T \in \mathcal{D}'$ is in K'_p if and only if there exist positive integers m, k and a bounded continuous function $f(x)$ on $\mathbb{R}^n$ such that*

$$(2) \quad T = \frac{\partial^{mn}}{\partial x_1^m \cdots \partial x_n^m} [e^{k|x|^p} f(x)].$$

PROOF. In case $p = 1$ the theorem was proved in [3]. For arbitrary $p \geq 1$, one can apply a similar argument which we present here for the sake of completeness.

It is obvious that a distribution of the form (2) defines a continuous linear functional on K_p .

Conversely, suppose that T is in K'_p . We first prove that, for some positive integer k_0 , the set of distributions

$$(3) \quad \{e^{-k_0|y|^p} \tau_y T_x : y \in \mathbb{R}^n\},$$

where $\tau_y T_x$ is the translation of T_x by y , is bounded in $\mathcal{D}'$.

Since T is continuous on K_p and the seminorms v_k are increasing, there exists $\epsilon > 0$ and a positive integer k_1 such that

$$(4) \quad v_k(\varphi) \leq \epsilon \quad \text{implies} \quad |T(\varphi)| \leq 1$$

for all integers $k \geq k_1$ and all $\varphi \in K_p$.

On the other hand, we have $|x - y|^p \leq 2^p(|x|^p + |y|^p)$ and therefore

$$\begin{aligned} v_k(\varphi(x + y)) &= \sup_{x \in \mathbb{R}^n; |\alpha| \leq k} e^{k|x|^p} |D^\alpha \varphi(x + y)| \\ (5) \quad &= \sup_{x \in \mathbb{R}^n; |\alpha| \leq k} e^{k|x-y|^p} |D^\alpha \varphi(x)| \\ &\leq e^{k2^p|y|^p} \sup_{x \in \mathbb{R}^n; |\alpha| \leq k2^p} e^{k2^p|x|^p} |D^\alpha \varphi(x)| = e^{k2^p|y|^p} v_{k2^p}(\varphi) \end{aligned}$$

for all $\varphi \in K_p$. Consequently, if $k_0 \geq k_1 2^p$, we infer from (4) and (5) that

$$|e^{-k_0|y|^p} \tau_y T_x \varphi(x)| = |T_x e^{-k_0|y|^p} \varphi(x + y)| \leq \epsilon^{-1} v_{k_0}(\varphi)$$

for all $\varphi \in \mathcal{D}$, which proves that the set (3) is bounded in $\mathcal{D}'$.

By a theorem of L. Schwartz (see [6, Vol. 2, Theorem XXII]), for every relatively compact open set $\Omega \subset \mathbb{R}^n$ there exists now an integer $N \geq 0$ and a sufficiently small compact neighborhood K of the origin such that, for every $\varphi \in \mathcal{D}_K^N$,

$$\{e^{-k_0|y|^p} \tau_y (T * \varphi) : y \in \mathbb{R}^n\}$$

is a bounded set of continuous functions in Ω . It follows that $e^{-k_0|x|^p} (T * \varphi)(x)$ is a bounded, continuous function in $\mathbb{R}^n$.

Let now E be a fundamental solution for the iterated Laplace operator Δ^m , i.e. $\Delta^m E = \delta$. If m is sufficiently large, E is N times continuously differentiable and $E \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, if $\gamma \in \mathcal{D}_K$ and $\gamma = 1$ in a neighborhood of the origin, we have $\gamma E \in \mathcal{D}_K^N$ and $\delta = \Delta^m(\gamma E) - W$ where $W \in \mathcal{D}_K$. Hence $T = \Delta^m(\gamma E * T) - W * T$ and so

$$(6) \quad T = \sum_{|\alpha| \leq m} D^\alpha [e^{k_0|x|^p} f_\alpha(x)]$$

where f_α are bounded, continuous functions in $\mathbb{R}^n$. Taking primitive functions, if necessary, one can reduce the right-hand side of (6) to one single term of the form (2).

We introduce in K'_p the topology of uniform convergence on all bounded sets in K_p .

2. **Convolutions in K'_p .** The convolution of a distribution $T \in K'_p$ and a function $\varphi \in K_p$ is defined as follows

$$(T * \varphi)(x) = (\varphi * T)(x) = \langle T_y, \varphi(x - y) \rangle.$$

Using Theorem 1 one can easily verify that $T * \varphi$ is a C^∞ -function such that, for some integer $k_0 \geq 0$,

$$|D^\alpha(T * \varphi)(x)| \leq C_\alpha e^{k_0|x|^p},$$

where C_α are constants.

More generally, convolution operators in K'_p can be defined and characterized similarly as in K'_1 by applying a method of L. Schwartz (see [5] and [3]). For simplicity we define directly the space $\mathcal{O}'_c(K'_p : K'_p)$ of distributions in K'_p which are convolution operators in K'_p .

Let γ_k , $k = 1, 2, \dots$, be positive functions in $C^\infty(\mathbb{R}^n)$ such that

$$(7) \quad \gamma_k(x) = e^{k|x|^p} \quad \text{for } |x| > 1.$$

THEOREM 2. *For a distribution $S \in K'_p$ the following conditions are equivalent:*

(c₁) *The distributions $S_k = \gamma_k S$, $k = 1, 2, \dots$, are in S' .*

(c₂) *For every integer $k \geq 0$ there exists an integer $m \geq 0$ such that $S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$ where f_α , $|\alpha| \leq m$, are continuous functions in $\mathbb{R}^n$ whose products with $e^{k|x|^p}$ are bounded.*

(c₃) *For every $\varphi \in K_p$, the convolution $S * \varphi$ is in K_p .*

PROOF. We prove the implications (c₁) $\iff$ (c₂) and (c₂) $\iff$ (c₃).

Suppose that condition (c₁) is satisfied and let k be an integer ≥ 1 . Since $S_{k+1} \in S'$, we can write $S_{k+1} = D^\alpha f$ where f is a continuous function in $\mathbb{R}^n$ such that

$$(8) \quad f(x) = O(1 + |x|^l) \quad \text{as } |x| \rightarrow \infty$$

for some integer $l \geq 0$. Hence $S = \gamma_{k+1}^{-1} D^\alpha f = \sum_{\beta \leq \alpha} D^{\alpha-\beta} f_\beta$ where

$$f_\beta(x) = (-1)^{|\alpha-\beta|} f(x) D^\beta \gamma_{k+1}^{-1}(x) = O(e^{-k|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

in view of (7) and (8). This proves the representation (c₂).

Conversely, if (c₂) holds for some given k , then $S_k = \gamma_k S = \sum_{|\alpha| \leq m} \gamma_k D^\alpha f_\alpha$, and applying to each term of the sum the Leibnitz formula one can see that S_k is a sum of derivatives of functions which grow like polynomials. This means that S_k is in S' .

By what has just been said, the convolution $S * \varphi$ of $S \in K'_p$ and $\varphi \in K_p$

is a C^∞ -function. If S satisfies condition (c_2) , then

$$S * \varphi = \sum_{|\alpha| \leq m} (D^\alpha f_\alpha) * \varphi = \sum_{|\alpha| \leq m} f_\alpha * D^\alpha \varphi$$

where f_α , $|\alpha| \leq m$, are continuous functions decreasing as fast as $e^{-k|x|^p}$. Therefore

$$(9) \quad \begin{aligned} |(f_\alpha * D^\alpha \varphi)(x)| &= \left| \int_{-\infty}^{+\infty} f_\alpha(y) D^\alpha \varphi(x-y) dy \right| \\ &\leq C_\alpha \int_{-\infty}^{+\infty} e^{-k\{|y|^p + |x-y|^p\}} dy, \end{aligned}$$

where C_α , $|\alpha| \leq m$, are constants.

Given now any integer $l \geq 0$, we choose k so large that $k > 2^p l + 1$. Since

$$(10) \quad |x|^p \leq 2^p \{|y|^p + |x-y|^p\} \quad \text{for all } x, y \in \mathbb{R}^n,$$

we conclude from (9) that

$$|(f_\alpha * D^\alpha \varphi)(x)| \leq C_\alpha^* e^{-l|x|^p},$$

where C_α^* , $|\alpha| \leq m$, are other constants. Since l was arbitrary, we proved that $S * \varphi \in K_p$, i.e. condition (c_3) is satisfied.

Conversely, from (c_3) it follows that, for any given integer $k \geq 0$, the set of distributions

$$(11) \quad \{e^{k|x|^p} \tau_x S_y : x \in \mathbb{R}^n\}$$

is bounded in $\mathcal{D}'$. In fact, for any $\varphi \in \mathcal{D}$, $\langle \tau_x S_y, \varphi(y) \rangle = (S * \check{\varphi})(-x)$ where $\check{\varphi}(x) = \varphi(-x)$. But $S * \check{\varphi}$ is in K_p , which shows the set (11) is bounded in $\mathcal{D}'$. Applying now an argument analogous to that used in the proof of Theorem 1 one obtains for S the representation (c_2) .

We denote by $\mathcal{O}'_c(K'_p : K'_p)$ the space of all distributions S satisfying the equivalent conditions (c_1) – (c_3) ; it is the space of convolution operators in K'_p .

If $S \in \mathcal{O}'_c(K'_p : K'_p)$ and $T \in K'_p$, we define the convolution $S * T$ by

$$\langle S * T, \varphi \rangle = \langle T * S, \varphi \rangle = \langle T, \check{S} * \varphi \rangle,$$

where $\varphi \in K_p$ and $\langle \check{S}, \varphi \rangle = \langle S, \check{\varphi} \rangle$. The definition is consistent, since $\check{S} * \varphi = (S * \check{\varphi})$ is in K_p and from the proof of the implication $(c_2) \Rightarrow (c_3)$ one can see that the mapping $\varphi \rightarrow S * \varphi$ of K_p into K_p is continuous.

If both S and T are in $\mathcal{O}'_c(K'_p : K'_p)$ then $S * T$ is also in $\mathcal{O}'_c(K'_p : K'_p)$. This follows from condition (c_2) and the associativity of the convolution, when all factors are in $\mathcal{O}'_c(K'_p : K'_p)$.

It is also easy to prove that the convolution commutes with differentiation, i.e. $D^\alpha(S * T) = D^\alpha S * T = S * D^\alpha T$.

3. Fourier transforms. For a function $\varphi \in K_p$, the Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} e^{-i\langle x, \xi \rangle} \varphi(x) dx$$

where $\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_n \xi_n$. Also, the inversion formula holds, i.e.

$$\varphi(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} e^{i\langle x, \xi \rangle} \hat{\varphi}(\xi) d\xi.$$

A distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$ is in S' ; its Fourier transform $\hat{S}$ is defined by the Parseval equality

$$\langle \hat{S}, \psi \rangle = \langle S, \hat{\psi} \rangle \quad \text{for every } \psi \in S$$

(see [6, Vol. 2]).

We establish a Paley-Wiener type theorem for the spaces K_p and $\mathcal{O}'_c(K'_p : K'_p)$. It is based on the following theorem due to G. I. Eskin [2].

THEOREM (ESKIN). *An entire analytic function $F(\xi)$ satisfying the estimate*

$$(12) \quad |F(\xi + i\eta)| \leq C(1 + |\xi|)^N e^{A|\eta|^q}$$

for some constants $A, C, N > 0$ and $q > 1$, is the Fourier transform of a distribution $S \in S'$ of the form

$$(13) \quad S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

where $m = N + n + 2$ and $f_\alpha, |\alpha| \leq m$, are continuous functions which, for every $\epsilon > 0$, fulfill the growth condition

$$(14) \quad f_\alpha(x) = O(e^{-(B-\epsilon)|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

with $B = p^{-1}(qA)^{-p/q}$ and $p = q/(q-1)$.

Conversely, if $S \in S'$ is of the form (13) with the functions f_α satisfying the growth condition

$$(15) \quad f_\alpha(x) = O(e^{-B|x|^p}) \quad \text{as } |x| \rightarrow \infty,$$

then the Fourier transform of S is an entire analytic function $F(\xi)$ such that

$$(16) \quad |F(\xi + i\eta)| \leq C_\epsilon (1 + |\xi|)^m e^{(A+\epsilon)|\eta|^q}$$

where $\epsilon > 0$ is arbitrary and C_ϵ is a constant (depending on ϵ).

Observe now that by increasing the constant B in (14) and (15) we can make A in (12) and (16) arbitrarily small. Also, for a distribution $S \in S'$ we have the formula $\xi^\alpha \hat{S} = (D^\alpha S)^\wedge$.

The above observations combined with the definition of K_p , condition (c_2) in Theorem 2 and the theorem of Eskin lead immediately to

THEOREM 3. (a) *An entire analytic function $F(\zeta)$ is a Fourier transform of a function $\varphi \in K_p$ if and only if for every N and $\epsilon > 0$ there exists a constant C such that*

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^{-N} e^{\epsilon |\eta|^q}.$$

(b) *An entire analytic function $F(\zeta)$ is a Fourier transform of a distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$ if and only if for every $\epsilon > 0$ there exist constants N and C such that*

$$|F(\xi + i\eta)| \leq C(1 + |\xi|)^N e^{\epsilon |\eta|^q}.$$

In both (a) and (b), $q = p/(p-1)$, $\xi = \operatorname{Re} \zeta$ and $\eta = \operatorname{Im} \zeta$.

Let K_p be the space of Fourier transforms of functions in K_p . We define in K_p a locally convex topology by means of the seminorms

$$w_k(\psi) = \sup_{\xi + i\eta \in \mathbb{C}^n} (1 + |\xi|)^k e^{-|\eta|^q/k} |\psi(\xi + i\eta)|, \quad k = 1, 2, \dots$$

THEOREM 4. *The Fourier transformation is a topological isomorphism of K_p onto K_p .*

PROOF. By Theorem 3 and because the Fourier inversion formula is valid for functions in K_p , the Fourier transformation is an isomorphism of K_p onto K_p . In view of the open mapping theorem it therefore suffices to show that the mapping $\varphi \rightarrow \hat{\varphi}$ of K_p into K_p is continuous. For that purpose we observe that if k is any given integer ≥ 0 and we choose $k' \geq k^{p-1} + 1$ then, for every multi-index α with $|\alpha| \leq k$,

$$\begin{aligned} |\xi^\alpha \hat{\varphi}(\xi + i\eta)| &= \left| \int_{-\infty}^{\infty} e^{-i(x, \xi + i\eta)} D^\alpha \varphi(x) dx \right| \\ &\leq \int_{-\infty}^{\infty} e^{(x, \eta) - k'|x|^p} dx v_{k'}(\varphi) \leq \int_{-\infty}^{\infty} e^{-|x|^p} dx e^{a|\eta|^q} v_{k'}(\varphi) \end{aligned}$$

where $a = q^{-1} \{(k' - 1)p\}^{-1/(p-1)} \leq 1/k$. Hence we conclude that $w_k(\hat{\varphi}) \leq C v_{k'}(\varphi)$ for some constant C (independent of φ), which proves the desired continuity.

Let K'_p be the space of continuous linear functionals on K_p . We equip it with the topology of uniform convergence on all bounded sets in K_p . Each

distribution $T \in K'_p$ has a Fourier transform $\hat{T}$ in K'_p defined by the Parseval formula $\langle \hat{T}, \hat{\varphi} \rangle = (2\pi)^n \langle T, \varphi \rangle$, $\varphi \in K_p$. Moreover, from Theorem 4 we obtain

COROLLARY. *The Fourier transformation is a topological isomorphism of K'_p onto K'_p .*

If $S \in \mathcal{O}'_c(K'_p : K'_p)$ then, by Theorem 3, $\psi \rightarrow \hat{S}\psi$ is a continuous linear mapping of K_p into K_p . Therefore, if $\hat{T}$ is the Fourier transform of a distribution $T \in K'_p$, one can define the product $\hat{S}\hat{T}$ by

$$\langle \hat{S}\hat{T}, \psi \rangle = \langle \hat{T}, \hat{S}\psi \rangle, \quad \psi \in K_p.$$

Moreover, one can easily prove that $(S * T)\hat{=} \hat{S}\hat{T}$.

4. Hypoelliptic convolution equations. We denote by EK'_p the space of all C^∞ -functions f such that

$$(17) \quad D^\alpha f(x) = O(e^{a|x|^p}) \quad \text{as } |x| \rightarrow \infty,$$

for some constant a (depending on f) and all multi-indices α . Obviously EK'_p is a linear subspace of K'_p .

THEOREM 5. *If $S \in \mathcal{O}'_c(K'_p : K'_p)$ and $f \in EK'_p$, then $S * f \in EK'_p$.*

PROOF. Suppose that f satisfies condition (17) for some a . Since S is in $\mathcal{O}'_c(K'_p : K'_p)$, it admits the representation (c_2) in Theorem 2, i.e. for every integer $k \geq 0$ we can write $S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$ where f_α , $|\alpha| \leq m$, are continuous functions such that

$$f_\alpha(x) = O(e^{-k|x|^p}) \quad \text{as } |x| \rightarrow \infty.$$

Choosing $k \geq 2^p a + 1$ and applying the inequality (10) one can see that the functions $f_\alpha(y) D^\beta f(x-y) e^{|\gamma|^p - a|2x|^p}$ are bounded for every multi-index β . Therefore the convolutions

$$h_\alpha(x) = (f_\alpha * f)(x) = \int_{-\infty}^{\infty} f_\alpha(y) f(x-y) dy$$

are C^∞ -functions and fulfill the growth conditions

$$D^\beta h_\alpha(x) = O(e^{a|2x|^p}) \quad \text{as } |x| \rightarrow \infty.$$

It follows that the functions h_α are in EK'_p and consequently $S * f = \sum_{|\alpha| \leq m} D^\alpha h_\alpha$ is in EK'_p .

We now consider the convolution equation (1), i.e. $S * U = V$ where $S \in \mathcal{O}'_c(K'_p : K'_p)$ and $U, V \in K'_p$. If there exists a solution U in EK'_p then, by Theorem 5, V must be in EK'_p .

Conversely, if all solutions $U \in K'_p$ are in EK'_p whenever V is in EK'_p , the

equation (and the distribution S) is said to be hypoelliptic in K'_p . In the next two sections we prove that conditions (h_1) and (h_2) in Theorem I are necessary and sufficient for S to be hypoelliptic in K'_p .

5. **Necessity of conditions (h_1) and (h_2) .** In order to prove the necessity of condition (h_1) we first study series of the form

$$(18) \quad \sum_{j=1}^{\infty} a_j \delta_{(j\xi)}$$

where $\delta_{(j\xi)}$ is the δ -Dirac measure with singularity at $j\xi \in \mathbb{R}^n$ and the coefficients a_j are complex numbers. We assume that

$$(19) \quad |j\xi| > 2|_{j-1}\xi| > 2^j, \quad j = 1, 2, \dots,$$

and

$$(20) \quad a_j = O(|j\xi|^\mu) \quad \text{as } j \rightarrow \infty,$$

for some integer $\mu > 0$. Then the series (18) converges in S' .

The following lemma is a slightly strengthened version of Lemma 1 in [7].

LEMMA 1. *Suppose that T is a distribution in S' whose Fourier transform $\hat{T}$ is of the form (18), i.e.*

$$(21) \quad \hat{T} = \sum_{j=1}^{\infty} a_j \delta_{(j\xi)}$$

where $j\xi$ and a_j , $j = 1, 2, \dots$, satisfy conditions (19) and (20).

If, for every integer $\nu > 0$,

$$(22) \quad a_j = O(|j\xi|^{-\nu}) \quad \text{as } j \rightarrow \infty,$$

then T is a C^∞ -function bounded in $\mathbb{R}^n$ together with all its derivatives.

Conversely, if condition (22) is not fulfilled, then T is not in $C^\infty(\mathbb{R}^n)$.

PROOF. By virtue of (20) and (21), $T = (2\pi)^{-n} \sum_{j=1}^{\infty} a_j e^{i(x, j\xi)}$ where the series converges in S' . If the coefficients a_j satisfy condition (22), the last series converges uniformly in $\mathbb{R}^n$ together with all its term-by-term derivatives, which proves the first part of the lemma.

Conversely, if $T \in C^\infty(\mathbb{R}^n)$ then, for every $\nu \geq 0$ and $\varphi \in \mathcal{D}$,

$$\langle e^{-i(h, x)} \Delta^\nu T_x, \varphi(-x) \rangle \rightarrow 0 \quad \text{as } |h| \rightarrow \infty,$$

$h \in \mathbb{R}^n$. Hence

$$(23) \quad \langle \tau_{-h}(|\xi|^{2\nu} \hat{T}_\xi), \hat{\varphi}(\xi) \rangle = \sum_{j=1}^{\infty} a_j |j\xi|^{2\nu} \hat{\varphi}(j\xi - h) \rightarrow 0,$$

by application of the Parseval identity.

We now choose φ so that

$$(24) \quad |\hat{\varphi}(0)| \geq 1.$$

If condition (22) is not satisfied, then there exists $\rho > 0$ and an integer $\nu_0 > 0$ such that

$$(25) \quad |_j \xi|^{2\nu_0} |a_j| \geq \rho$$

for a subsequence of $\{a_j\}$, but without loss of generality we take the whole sequence. Also, since $\hat{\varphi} \in S$, we have

$$(26) \quad \hat{\varphi}(\xi) = O(|\xi|^{-\mu-2\nu_0-1}) \quad \text{as } |\xi| \rightarrow \infty, \xi \in \mathbb{R}^n.$$

Setting $_j h = _j \xi$ and making use of (19), (20) and (26) we obtain

$$\sum_{j=1, j \neq k}^{\infty} a_j |_j \xi|^{2\nu_0} \hat{\varphi}(_j \xi - _k h) = O(2^{-k}) \quad \text{as } k \rightarrow \infty.$$

On the other hand, conditions (24) and (25) imply that $|a_k| |_k \xi|^{2\nu_0} |\hat{\varphi}(0)| \geq \rho$. This contradicts the convergence (23). Our assertion is thus proved.

THEOREM 6. *If a distribution $S \in O'_c(K'_p : K'_p)$ is hypoelliptic in K'_p then its Fourier transform $\hat{S}$ satisfies condition (h_1) .*

PROOF. If condition (h_1) is not satisfied then there exists a sequence $\{_j \xi\} \subset \mathbb{R}^n$ defined as in Lemma 1, such that

$$(27) \quad |\hat{S}(_j \xi)| < |_j \xi|^{-j}.$$

The distribution $\hat{U} = \sum_{j=1}^{\infty} \delta_{(_j \xi)}$ is in S' and, by Lemma 1, U is not in $C^\infty(\mathbb{R}^n)$. But

$$(S * U)^\wedge = \hat{S} \hat{U} = \sum_{j=1}^{\infty} \hat{S}(_j \xi) \delta_{(_j \xi)}$$

whence, applying (27) and again Lemma 1, we conclude that $S * U$ is in EK'_p . Thus S is not hypoelliptic in K'_p .

REMARK. One can modify Lemma 1 and Theorem 6 so as to obtain a proof of the necessity of condition (h_3) .

In the proof of condition (h_2) we may restrict ourselves to continuous solutions of the homogeneous equation.

THEOREM 7. *If every continuous solution of the homogeneous equation*

$$(28) \quad S * U = 0$$

which satisfies the estimate

$$(29) \quad U(x) = O(e^{k|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

for some $k > 0$, has continuous first order derivatives in a neighborhood of the origin, then condition (h_2) holds.

PROOF. Without restriction of generality we may assume that the neighborhood of the origin is the ball $B = \{x: |x| \leq 1\}$.

Fix now $k > 0$ and denote by $H_{p,k}$ the space of continuous solutions of equation (28) satisfying the estimate (29). In $H_{p,k}$ we define the norm

$$\|U\| = \sup_{x \in \mathbb{R}^n} |U(x)|e^{-k|x|^p}.$$

We also denote by $H_{p,k}^*$ the subspace of $H_{p,k}$ consisting of functions with continuous first order derivatives in B ; a norm in $H_{p,k}^*$ is defined by

$$\|U\|^* = \|U\| + \sup_{|x| < 1} \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(x) \right|.$$

It is easy to see that $H_{p,k}$ and $H_{p,k}^*$ are Banach spaces. By assumption $H_{p,k}$ is mapped by the identity mapping onto $H_{p,k}^*$. The mapping is closed and therefore continuous. Thus there exists a constant $C > 0$ such that

$$(30) \quad \|U\|^* \leq C\|U\| \quad \text{for all } U \in H_{p,k}.$$

If $\hat{S}(\zeta) = 0$ for some $\zeta \in \mathbb{C}^n$, then $U_0(x) = e^{i\langle \zeta, x \rangle}$ is a solution of equation (28) and $U_0 \in H_{p,k}$. But

$$\|U_0\| = \sup_{x \in \mathbb{R}^n} e^{-(\eta, x) - k|x|^p} = e^{B|\eta|^q}$$

where $B = q^{-1}(kp)^{-q/p}$, and $\|U_0\|^* = \|U_0\| + \sum_{j=1}^n |\zeta_j|e^{|\eta|} \geq \|U_0\| + |\zeta|$. Thus, from (30) we obtain

$$\lim_{|\zeta| \rightarrow \infty} \frac{|\eta|^q}{\log |\zeta|} \geq q(kp)^{q/p},$$

and since k is arbitrary we conclude that condition (h_2) is valid.

COROLLARY. If $S \in \mathcal{O}'_c(K'_p : K'_p)$ is hypoelliptic in K'_p , then condition (h_2) holds.

6. The implication $(h_1), (h_2) \Rightarrow (h_3)$. For $q = 1$ this implication was proved by L. Hörmander [4]. In this section, we modify suitably Hörmander's argument to our case. First we establish the following lemma on harmonic functions in $\mathbb{R}^2$.

LEMMA 2. Given $A, B, b > 0$ and $q > 1$ one can find a constant $N > 0$

such that if u is a harmonic function for $x^2 + y^2 < \rho^2$ and satisfies the inequalities

$$(31) \quad u(x, 0) \leq 0, \quad u(x, y) \geq -a|y|^q - Br^q, \quad x^2 + y^2 < \rho^2,$$

then it follows that

$$(32) \quad u(x, y) \leq a|y|^q + (B + b)r^q, \quad x^2 + y^2 < r^2$$

provided that $0 < a < A$ and $0 < r \leq \rho/N$.

PROOF. Assuming that the lemma is false one can find a harmonic function v in $\mathbb{R}^2$ and real numbers a_0, x_0, y_0 such that

$$(33) \quad v(x, 0) \leq 0, \quad v(x, y) \geq -a_0|y|^q - B, \quad v(x_0, y_0) \geq a_0|y_0|^q + (B + b).$$

This can be accomplished in the same way as in the proof of Lemma 1 in [4].

From the first two inequalities in (33) it follows that v is a linear function of y , since v must be a harmonic polynomial which is bounded when y is bounded. Suppose that $v(x, y) = cy + d$. Then $d \leq 0$, by the first inequality in (33), and from the remaining two inequalities in (29) it follows that

$$(34) \quad a_0|y|^q + cy + B + d \geq 0$$

and

$$(35) \quad a_0|y_0|^q - cy_0 + (B + b) - d \leq 0.$$

In particular, setting $y = -y_0$ in (34) we obtain $a_0|y_0|^q - cy_0 + B + d \geq 0$ which contradicts the inequality (35), because $d \leq 0$.

Let now S be a distribution in $\mathcal{O}'_c(K'_p : K'_p)$ whose Fourier transform satisfies conditions (h_1) and (h_2) . Then there exist constants $B_1, M_1 > 0$ such that

$$(36) \quad |\hat{S}(\xi)| \geq |\xi|^{-B_1} \quad \text{if } \xi \in \mathbb{R}^n, |\xi| \geq M_1,$$

and

$$(37) \quad |\eta|^q / \log |\zeta| \rightarrow \infty \quad \text{if } \zeta = \xi + i\eta \in \mathbb{C}^n, |\zeta| \rightarrow \infty, \hat{S}(\zeta) = 0.$$

Furthermore, by Theorem 3, to every $\epsilon > 0$ there exist constants $B_2, M_2 > 0$ so that

$$(38) \quad |\hat{S}(\zeta)| \leq |\xi|^{B_2} e^{\epsilon|\eta|^q} \quad \text{if } \zeta \in \mathbb{C}^n, |\zeta| \geq M_2.$$

THEOREM 8. If the Fourier transform $\hat{S}$ of a distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$ satisfies conditions (36), (37) and (38), then to every $m > 0$ one can find a

constant C_m such that

$$(39) \quad |1/\hat{S}(\zeta)| \leq |\xi|^{2(B_1+B_2+1)} e^{\epsilon|\eta|^q} \quad \text{if } |\eta|^q \leq m \log|\zeta|$$

and $|\zeta| \geq C_m$.

PROOF. Let M be a positive constant which we fix later. Given $\zeta \in \mathbb{C}^n$ such that $0 < |\eta|^q < m \log|\zeta|$ we consider the analytic function of one complex variable z defined by

$$F_\zeta(z) = \hat{S}(\xi + z\eta/|\eta|) \quad \text{for } |z|^q \leq M \log|\xi|.$$

If $|\zeta|$ is sufficiently large, conditions (36), (37) and (38) imply that

$$(40) \quad |F_\zeta(x)| \geq (2|\xi|)^{-B_1} \quad \text{if } x \in \mathbb{R}, |x|^q < M \log|\xi|,$$

and

$$(41) \quad |F_\zeta(z)| \leq (2|\xi|)^{B_2} e^{\epsilon|y|^q} \quad \text{if } |z|^q < M \log|\xi|.$$

The function $u_\zeta(z) = \log \{(2|\xi|)^{-B_1} |F_\zeta(z)|^{-1}\}$ is harmonic for $|z|^q < M \log|\xi|$ and large $|\zeta|$, by virtue of (37). Moreover, from (40) and (41) it follows that

$$(42) \quad u_\zeta(x) \leq 0 \quad \text{if } x \in \mathbb{R} \text{ and } |x|^q < M \log|\xi|$$

and

$$(43) \quad u_\zeta(z) \geq -\epsilon|y|^q - (B_1 + B_2)\log(2|\xi|), \quad |z|^q < M \log|\xi|.$$

We now apply Lemma 2 with the constants $A = 1 + \epsilon$, $B = (B_1 + B_2 + 1)/(m + 1)$, $b = 1/(m + 1)$ and $r^q = (m + 1)\log|\xi|$. If N is the constant in the lemma, we set $M = N^q(m + 1)$ and we observe that

$$(B_1 + B_2)\log(2|\xi|) \leq (B_1 + B_2 + 1)\log|\xi| = Br^q,$$

if $|\zeta|$ is sufficiently large. Thus, by Lemma 2, we have

$$(44) \quad u_\zeta(z) \leq \epsilon|y|^q + (B_1 + B_2 + 2)\log|\xi| \quad \text{if } |z|^q \leq r^q = (m + 1)\log|\xi|.$$

Since

$$|\eta|^q \leq m \log|\zeta| \leq (m + 1)\log|\xi| = r^q$$

if $|\zeta|$ is sufficiently large, we may substitute $z = i|\eta|$ in (44). Then we obtain

$$\log \{(2|\xi|)^{-B_1} |\hat{S}(\zeta)|^{-1}\} \leq \epsilon|\eta|^q + (B_1 + B_2 + 2)\log|\xi|$$

whence we conclude that

$$|1/\hat{S}(\xi)| \leq |\xi|^{2B_1+B_2+2} e^{\epsilon|\eta|^q} \leq |\xi|^{2(B_1+B_2+1)} e^{\epsilon|\eta|^q}$$

where $|\xi|$ is sufficiently large. This proves Theorem 8.

COROLLARY. For a distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$, conditions (h_1) and (h_2) combined imply condition (h_3) .

7. Parametrices. We define suitable parametrices for a distribution $S \in \mathcal{O}'_c(K'_p : K'_p)$ and prove that these parametrices exist if S fulfills condition (h_3) .

In what follows b is a positive number and k a positive integer.

DEFINITION. A distribution $P \in K'_p$ is said to be a (b, k) -parametrix for S if it has the following properties:

(p₁) There exists an integer $m > 0$ such that $P = \sum_{|\alpha| \leq m} D^\alpha F_\alpha$ where F_α , $|\alpha| \leq m$, are continuous functions in $\mathbb{R}^n$ such that $F_\alpha(x) = O(e^{-b|x|^p})$ as $|x| \rightarrow \infty$,

(p₂) $S * P = \delta - W$, where δ is the Dirac measure and W is a function in $C^k(\mathbb{R}^n)$ satisfying the growth condition $D^\alpha W(x) = O(e^{-b|x|^p})$ as $|x| \rightarrow \infty$ when $|\alpha| \leq k$.

THEOREM 9. Let S be a distribution in $\mathcal{O}'_c(K'_p : K'_p)$ which satisfies condition (h_3) . Then for each pair (b, k) there exists a (b, k) -parametrix for S .

PROOF. In order to simplify the notation we present the proof of Theorem 9 for $n = 2$. The general case can be handled in the same way.

We apply condition (h_3) with ϵ and m to be fixed later. Suppose that (h_3) holds for some given $\epsilon, m, B > 0$ and $C_m > 1$. Then the function

$$F(x, \xi) = \{(2\pi)^2 \hat{S}(\xi) \langle \xi, \xi \rangle^\mu\}^{-1} e^{i\langle x, \xi \rangle}$$

is analytic in ξ , when $|\eta|^q \leq m \log|\xi|$ and $|\xi| \geq C_m$, provided that C_m is sufficiently large. If $\mu > B/2 + 1$, then $F(x, \xi)$ is integrable over $R^2 \setminus Q$, where $Q = \{\xi \in R^2 : |\xi_j| \leq C_m, j = 1, 2\}$. Moreover, if μ is even and

$$(45) \quad h(x) = \iint_{R^2 \setminus Q} F(x, \xi) d\xi_2 d\xi_1,$$

then it is easy to verify that the distribution

$$(46) \quad H = \Delta^\mu h$$

satisfies the equation

$$(47) \quad S * H = \delta - \frac{1}{(2\pi)^2} \iint_Q e^{i\langle x, \xi \rangle} d\xi_2 d\xi_1. \quad (1)$$

The integral in (45) can be represented as

(1) In what follows we assume that μ is even; otherwise we should need $(-\Delta)^\mu$ in (46).

$$(48) \quad \iint_{R^2 \setminus Q} = \int_{|\xi_1| > C_m} \int_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \int_{|\xi_2| > C_m} - \int_{|\xi_1| > C_m} \int_{|\xi_2| > C_m}.$$

We now replace the iterated integrals in (48) by integrals over contours in the respective complex planes.

Let σ be a C^∞ -function of one real variable r defined for $r > 0$ in such a way that $\sigma(r) = C_m$ for $0 < r \leq C_m$, σ is increasing for $r \geq C_m$, and $\sigma(r) = C_m e^{ar^p}$ for $r \geq 2C_m$, where a is a positive constant. We extend σ to negative values of r by setting $\sigma(-r) = -\sigma(r)$. We also denote by σ^* an increasing, odd C^∞ -function on R such that $d^k \sigma / dx^k|_{x=0} = 0$, $k = 1, 2, \dots$, and $\sigma^*(r) = \sigma(r)$ for $r \geq C_m$.

Further, let τ be an even, C^∞ -function on R such that $\tau(r) = 0$ for $|r| \leq C_m$, τ is increasing for $r \geq C_m$, and $\tau(r) = c|r|^{p/q}$ for $|r| \geq 2C_m$, where c is a positive constant.

We assume that

$$(49) \quad (\sqrt{2} \tau(r))^q \leq m \log |\sigma(r)| \quad \text{for } |r| \geq C_m$$

which implies that

$$(50) \quad (\sqrt{2} c)^q \leq ma.$$

Given any $x = (x_1, x_2)$ in R^2 we denote by Λ_j (or Λ_j^* , respectively) the contour in the complex ζ_j plane defined by $\zeta_j(r) = \sigma(r) + i \operatorname{sgn} x_j \tau(r)$, where r runs from $-\infty$ to $-C_m$ and from C_m to ∞ (or by $\zeta_j(r) = \sigma^*(r) + i \operatorname{sgn} x_j \tau(r)$, where r runs from $-\infty$ to ∞ , respectively). By the inequality (49), the contours $\Lambda_1 \times \Lambda_2^*$, $\Lambda_1^* \times \Lambda_2$ and $\Lambda_1 \times \Lambda_2$ lie in the domain $|\eta|^q < m \log |\zeta|$. If, in addition,

$$(51) \quad \mu > B + \epsilon m + 1$$

then we can write

$$(52) \quad h(x) = \left(\int_{\Lambda_1} \int_{\Lambda_2^*} + \int_{\Lambda_1^*} \int_{\Lambda_2} - \int_{\Lambda_1} \int_{\Lambda_2} \right) F(x, \zeta) d\zeta_2 d\zeta_1.$$

We denote by λ_j and λ_j^* the parts of the contours Λ_j and Λ_j^* , respectively, obtained by restricting the values of the parameter r to the interval $(-|x|, |x|)$.

The remaining portions of Λ_j and Λ_j^* will be denoted by Γ_j and Γ_j^* , respectively.

If

$$h_1(x) = \left(\int_{\Gamma_1} \int_{\Gamma_2^*} + \int_{\Gamma_1^*} \int_{\Gamma_2} - \int_{\Gamma_1} \int_{\Gamma_2} \right) F(x, \zeta) d\zeta_2 d\zeta_1,$$

we now set $P = \Delta^\mu h_1$. Then, by virtue of (46), (47) and (52), we have

$$(53) \quad S * P = \delta - W$$

where

$$W = S * \Delta^\mu h_2 + \frac{1}{(2\pi)^2} \iint_Q e^{i(x, \xi)} d\xi_2 d\xi_1$$

and

$$h_2(x) = \left(\int_{\lambda_1} \int_{\lambda_2} + \int_{\lambda_1} \int_{\lambda_2} - \int_{\lambda_1} \int_{\lambda_2} \right) F(x, \zeta) d\zeta_2 d\zeta_1.$$

The proof of the theorem follows immediately from the next two technical lemmas.

LEMMA 3. *The function h_1 satisfies the growth condition*

$$(54) \quad h_1(x) = O(e^{-c|x|^p}) \quad \text{as } |x| \rightarrow \infty.$$

PROOF. Consider first the integral

$$\int_{\Gamma_1} \int_{\Gamma_2} F(x, \zeta) d\zeta_2 d\zeta_1 = \int_{|t_1| \geq |x|} \int_{|t_2| \geq |x|} F(x, \zeta(t)) \zeta'_1(t_1) \zeta'_2(t_2) dt_2 dt_1$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$ and $\zeta_j(t_j) = \sigma(t_j) + i \operatorname{sgn} x_j \tau(t_j)$, $j = 1, 2$.

For $|t_1|$ and $|t_2|$ sufficiently large, we have

$$|\zeta(t), \zeta(t)|^\mu \geq \exp \{ \mu a(|t_1|^p + |t_2|^p) \}$$

and

$$|\zeta'_1(t_1) \zeta'_2(t_2)| \leq \frac{1}{2} |t|^p \exp \{ a(|t_1|^p + |t_2|^p) \}.$$

Also, from (h₃) it follows that

$$\begin{aligned} \left| \frac{1}{\hat{S}(\zeta(t))} \right| &\leq |\zeta(t)|^B \exp \{ \epsilon |\eta(t)|^q \} \\ &\leq 2 C_m^B \exp \{ (Ba + \epsilon(c\sqrt{2})^q)(|t_1|^p + |t_2|^p) \}, \end{aligned}$$

provided that $|t_1|$ and $|t_2|$ are sufficiently large.

Further, if $|t_1|, |t_2| \geq |x|$, we have

$$\begin{aligned} |\exp \{ i(x, \zeta(t)) \}| &= \exp \{ -c(|x_1| |t_1|^{p/q} + |x_2| |t_2|^{p/q}) \} \\ &\leq \exp \{ -c(|x_1| + |x_2|) |x|^{p/q} \} \leq \exp \{ -c|x|^p \}. \end{aligned}$$

Consequently, for $|t_1|$ and $|t_2|$ sufficiently large and $\geq |x|$,

$$\begin{aligned}
 (55) \quad & |F(x, \zeta(t))\zeta'_1(t_1)\zeta'_2(t_2)| \\
 & \leq (2\pi)^{-2} C_m^B \exp\{-c|x|^p\} |t|^p \\
 & \quad \times \exp\{[Ba + a - \mu a + \epsilon(c\sqrt{2})^q] (|t_1|^p + |t_2|^p)\}.
 \end{aligned}$$

But

$$Ba + a - \mu a + \epsilon(c\sqrt{2})^q \leq a(B + 1 - \mu + \epsilon m) < 0,$$

because of (50) and (51). Therefore the right-hand side of (55) is integrable with respect to t_1 and t_2 over $|t_1|, |t_2| \geq C_m$, which proves the growth condition (54) for the integral under consideration.

We now observe that

$$(56) \quad \int_{\Gamma_1} \int_{\Gamma_2} F(x, \zeta) d\zeta_2 d\zeta_1 = \int_{l_1} \int_{\Gamma_2} F(x, \zeta) d\zeta_2 d\zeta_1$$

where l_1 is the line $\zeta_1(t_1) = t_1 + i \operatorname{sgn} x_1 \tau(|x|)$ directed from $-\infty$ to ∞ . The last integral is equal to

$$\int_{-\infty}^{\infty} \int_{|t_2| \geq |x|} F(x, \zeta(t))\zeta'_2(t_2) dt_2 dt_1,$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$, $\zeta_1(t_1) = t_1 + i \operatorname{sgn} x_1 \tau(|x|)$ and $\zeta_2(t_2) = \sigma(t_2) + i \operatorname{sgn} x_2 \tau(t_2)$.

If $|t_1|, |t_2|$ are sufficiently large and $|t_2| \geq |x|$, we now have

$$|\zeta(t), \zeta(t)|^\mu \geq \frac{1}{2}(t_1^2 + e^{2a|t_2|^p})^\mu,$$

$$|\zeta'_2(t_2)| \leq \frac{1}{2}|t_2|^p e^{a|t_2|^p},$$

and, in view of (h₃) and (49),

$$\begin{aligned}
 |1/\hat{S}(\zeta(t))| & \leq (t_1^2 + e^{2a|t_2|^p})^B e^{\epsilon(c\sqrt{2})^q |t_2|^p} \\
 & \leq (t_1^2 + e^{2a|t_2|^p})^{B+m\epsilon/2}.
 \end{aligned}$$

Also, for $|t_2| \geq |x|$,

$$\begin{aligned}
 |\exp\{i(x, \zeta(t))\}| & \leq \exp\{-c(|x_1| |x|^{p/q} + |x_2| |t_2|^{p/q})\} \\
 & \leq \exp\{-c(|x_1| + |x_2|)|x|^{p/q}\} \leq \exp\{-c|x|^p\},
 \end{aligned}$$

so that, finally,

$$\begin{aligned}
 (57) \quad & |F(x, \zeta(t))\zeta'_2(t_2)| \leq (2\pi)^{-2} e^{-c|x|^p} |t_2|^p e^{a|t_2|^p} (t_1^2 + e^{2a|t_2|^p})^{B-\mu+m\epsilon/2} \\
 & \leq (2\pi)^{-2} e^{-c|x|^p} |t_2|^p e^{a|t_2|^p} (t_1^2 + e^{2a|t_2|^p})^{-1-m\epsilon/2}
 \end{aligned}$$

because of (51). Since the last expression in (57) is integrable with respect to t_1 and t_2 for $t_1 \in \mathbb{R}$ and $|t_2| \geq C_m$, condition (54) holds for the integral (56). The integral $\int_{\Gamma_1} \int_{\Gamma_2} F(x, \xi) d\xi_2 d\xi_1$ can be estimated in the same way. Lemma 3 is now established.

LEMMA 4. *Given any pair (b, k) we can choose the constants ϵ , a (sufficiently small) and m, c (sufficiently large) so that*

$$D^\alpha W(x) = O(e^{-b|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

for $|\alpha| \leq k$.

PROOF. Assume that $|x| \rightarrow \infty$ through $x_1, x_2 \geq 0$; otherwise we could modify suitably our argument.

By definition,

$$(58) \quad D^\alpha W = S * D^\alpha \Delta^\mu h_2 + \frac{1}{(2\pi)^2} D^\alpha \iint_Q e^{i\langle x, \xi \rangle} d\xi_2 d\xi_1$$

where

$$h_2(x) = \left(\int_{\lambda_1} \int_{\lambda_2^*} + \int_{\lambda_1^*} \int_{\lambda_2} - \int_{\lambda_1} \int_{\lambda_2} \right) F(x, \xi) d\xi_2 d\xi_1.$$

It is easy to verify that h_2 is a C^∞ -function such that $h_2(x) = 0$ for $|x| \leq C_m$ and

$$D^\alpha h_2(x) = O(e^{a(|\alpha|+1)|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

for all α .

On the other hand, by Theorem 2, for every integer ρ one can find an integer l such that $S = \sum_{|\beta| \leq l} D^\beta f_\beta$, where $f_\beta, |\beta| \leq l$, are continuous functions in $\mathbb{R}^n$ satisfying the condition

$$(59) \quad f_\beta(x) = O(e^{-\rho|x|^p}) \quad \text{as } |x| \rightarrow \infty.$$

Therefore, if a is so small that $(2\mu + k + l + 1)a < \rho$, we can write

$$(60) \quad S * D^\alpha \Delta^\mu h_2 = \sum_{|\beta| \leq l} (-1)^{|\alpha|+|\beta|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu h_2(x-y) dy_2 dy_1$$

where $|\alpha| \leq k$.

We decompose $h_2(x-y)$ as follows: $h_2(x-y) = g_1(x, y) + g_2(x, y)$ where $g_1(x, y)$ is defined similarly as $h_2(x-y)$ except for the contours which are parts of $\Lambda_1, \Lambda_2, \Lambda_1^*, \Lambda_2^*$ obtained by restricting the parameters t_1, t_2 to the interval $[-|x|, |x|]$ (instead of $[-|x-y|, |x-y|]$).

The function $g_1(x, y)$ is a sum of three integrals one of which is

$$(61) \quad \int_{-|x|}^{|x|} \int_{|x| > |t_2| > C_m} F(x-y, \zeta(t)) \zeta'_1(t_1) \zeta'_2(t_2) dt_2 dt_1$$

where $\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2))$, $\zeta_1(t_1) = \sigma^*(t_1) + \operatorname{sgn}(x_1 - y_1)\tau(t_1)$ and $\zeta_2(t_2) = \sigma(t_2) + \operatorname{sgn}(x_2 - y_2)\tau(t_2)$. Its contribution toward the right-hand side of (60) is

$$(62) \quad \begin{aligned} & \frac{1}{(2\pi)^2} D^\alpha \int_{\lambda_1} \int_{\lambda_2} e^{i\langle x, \zeta \rangle} d\zeta_2 d\zeta_1 \\ & + \sum_{|\beta| \leq l} (-1)^{|\alpha+\beta|} \left\{ \int_{-\infty}^{x_1} \int_{x_2}^{\infty} \left[f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu \int_{-|x|}^{|x|} \int_{-\tau(|x|)}^{\tau(|x|)} F_1(x, y, t) dt_2 dt_1 \right] dy_2 dy_1 \right. \\ & + \int_{x_1}^{\infty} \int_{-\infty}^{x_2} \left[f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu \int_{-\tau(|x|)}^{\tau(|x|)} \int_{|x| > |t_2| > C_m} F_2(x, y, t) dt_2 dt_1 \right] dy_2 dy_1 \\ & \left. + \int_{x_1}^{\infty} \int_{x_2}^{\infty} \left[f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu \int_{-\tau(|x|)}^{\tau(|x|)} \int_{-\tau(|x|)}^{\tau(|x|)} F_3(x, y, t) dt_2 dt_1 \right] dy_2 dy_1 \right\} \end{aligned}$$

where the functions F_1, F_2, F_3 are defined by

$$\begin{aligned} F_1(x, y, t) &= \sum_{j=0}^1 (-1)^j F(x-y, {}_j\zeta(t)) {}_j\zeta'_1(t_1) {}_j\zeta'_2(t_2), \\ F_2(x, y, t) &= \sum_{j=2}^3 (-1)^j F(x-y, {}_j\zeta(t)) {}_j\zeta'_1(t_1) {}_j\zeta'_2(t_2), \\ F_3(x, y, t) &= \sum_{r,s=0}^1 (-1)^{r+s} F(x-y, {}_{r,s}\zeta(t)) {}_{r,s}\zeta'_1(t_1) {}_{r,s}\zeta'_2(t_2), \end{aligned}$$

where

$$\begin{aligned} {}_j\zeta(t) &= ({}_j\zeta_1(t_1), {}_j\zeta_2(t_2)), & {}_{r,s}\zeta(t) &= ({}_{r,s}\zeta_1(t_1), {}_{r,s}\zeta_2(t_2)) \\ {}_j\zeta_1(t_1) &= \sigma^*(t_1) + i\tau(|x|), & {}_j\zeta_2(t_2) &= (-1)^j \sigma(|x|) + it_2, \quad j = 0, 1, \\ {}_j\zeta_1(t_1) &= (-1)^j \sigma^*(|x|) + it_1, & {}_j\zeta_2(t_2) &= \sigma(t_2) + i\tau(|x|), \quad j = 2, 3, \\ {}_{r,s}\zeta_1(t_1) &= (-1)^r \sigma^*(|x|) + it_1, & {}_{r,s}\zeta_2(t_2) &= (-1)^s \sigma(|x|) + it_2. \end{aligned}$$

In deriving (62) we made repeated use of Cauchy's integral theorem.

For sufficiently large $|x|$ each of the integrals in curly brackets in (62) is less than

$$(63) \quad \exp \{ \{-c + c^q [e 2^{q/2} + (2/q)(1/p\rho)^{1/(p-1)}] + a(B+k+l+1)\} |x|^p \}.$$

Given $b > 0$, we now fix ϵ so small and ρ (or equivalently l) so large that

$$\epsilon 2^{q/2} + (2/q)(1/p\rho) < 1/q(2bp)^{q-1}.$$

Next we choose $c = 2bp$ and $a = b/(B+k+l+1)$. Then the expression in (63) is less than $e^{-b|x|^p}$. In order to satisfy condition (50) we have, however, to choose $m \geq a^{-1}(c\sqrt{2})^q$.

A representation similar to (62) can be obtained for the contribution toward the right-hand side of (60) of the two remaining integrals which make up $g_1(x, y)$. The sum of the first terms in these representations is

$$\begin{aligned} \frac{1}{(2\pi)^2} D^\alpha \left(\int_{\lambda_1} \int_{\lambda_2} + \int_{\lambda_1} \int_{\lambda_2} - \int_{\lambda_1} \int_{\lambda_2} \right) e^{i\langle x, \zeta \rangle} d\zeta_2 d\zeta_1 \\ = \frac{1}{(2\pi)^2} D^\alpha \int_{\lambda_1} \int_{\lambda_2} e^{i\langle x, \zeta \rangle} d\zeta_2 d\zeta_1 - \frac{1}{(2\pi)^2} D^\alpha \iint_Q e^{i\langle x, \zeta \rangle} d\zeta_2 d\zeta_1 \end{aligned}$$

and, for sufficiently large $|x|$, we have

$$\left| D^\alpha \int_{\lambda_1} \int_{\lambda_2} e^{i\langle x, \zeta \rangle} d\zeta_2 d\zeta_1 \right| \leq e^{[-c+a(k+2)]|x|^p} \leq e^{-b|x|^p},$$

under the above choice of c and a .

In view of (58) the proof of the lemma will be complete if we can choose ϵ , a sufficiently small and c , m sufficiently large to insure that

$$(64) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu g_2(x, y) dy_2 dy_1 = O(e^{-b|x|^p})$$

as $|x| \rightarrow \infty$, for $|\alpha| \leq k$ and $|\beta| \leq l$. We again prove (64) for one of the three integrals of which $g_2(x, y)$ is composed, namely for

$$(65) \quad \int_{-|x-y|}^{|x-y|} \int_{|x-y| \geq |t_2| \geq |x|} F(x-y, \zeta(t)) \zeta'_1(t_1) \zeta'_2(t_2) dt_2 dt_1$$

where $\zeta(t)$ is as in (61). The other integrals can be estimated analogously.

The part of (64) generated by the integral (65) can be written in the form

$$\sum_{r,s=0}^1 \int_{I_r^1 \times I_s^2} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu G_{r,s}(x, y) dy_2 dy_1$$

where $I_0^j = (-\infty, x_j)$, $I_1^j = (x_j, \infty)$, $j = 1, 2$, and

$$G_{r,s}(x, y) = \int_{-|x-y|}^{|x-y|} \int_{|x-y| \geq |t_2| \geq |x|} F(x-y, {}_{r,s}\tilde{\zeta}(t)) {}_r\tilde{\zeta}'_1(t_1) {}_s\tilde{\zeta}'_2(t_2) dt_2 dt_1$$

with

$$\begin{aligned} {}_{r,s}\tilde{\zeta}(t) &= ({}_r\tilde{\zeta}_1(t_1), {}_s\tilde{\zeta}_2(t_2)), \\ {}_r\tilde{\zeta}_1(t_1) &= \sigma^*(t_1) + (-1)^r i r(t_1), \\ {}_s\tilde{\zeta}_2(t_2) &= \sigma(t_2) + (-1)^s i r(t_2). \end{aligned}$$

But, for $|\eta| \leq m \log|\zeta|$ and $|\zeta| \geq C_m$, the function $[\hat{S}(\zeta)\zeta, \zeta^\mu]^{-1}$ is bounded together with all its derivatives, because of (51). Therefore, if $|x-y|$ is sufficiently large and $\geq |x|$, we have

$$|D_y^{\alpha+\beta} \Delta_y^\mu G_{r,s}(x, y)| \leq \exp\{-c|x|^p + a(2\mu + k + l + 3)|x - y|^p\},$$

$$r, s = 0, 1.$$

For a given $b > 0$ we now set $c = 2b$ and choose ρ (or equivalently l) so that $\rho > b$. Next we choose a so small that $a(2\mu + k + l + 3) < b/2^p$. Again m must be sufficiently large, because of (50). It follows that

$$\iint_{I_r^1 \times I_s^2} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu G_{r,s}(x, y) dy_2 dy_1 = O(e^{-b|x|^p})$$

as $|x| \rightarrow \infty$, which proves (63) and thus the lemma.

REMARK. Note that condition (51) is not inconsistent with the choice of the constants in the proof of Lemma 4, since ϵ can be chosen arbitrarily small.

8. Sufficiency of condition (h_3) . In order to establish the sufficiency of (h_3) we need only to prove

THEOREM 10. Let S be a distribution in $\mathcal{O}'_c(K'_p : K'_p)$. If for every pair (b, k) there exists a (b, k) -parametrix for S , then S is hypoelliptic in K'_p .

PROOF. Suppose that U is a solution in K'_p of the equation

$$(66) \quad S * U = V$$

where $V \in EK'_p$. We have to prove that, in fact, $U \in EK'_p$.

By Theorem 1, we can write $U = D^\beta f$ where f is a continuous function such that

$$(67) \quad f(x) = O(e^{b_1|x|^p}) \quad \text{as } |x| \rightarrow \infty,$$

for some constant $b_1 > 0$.

On the other hand, since $V \in EK'_p$, V is a C^∞ -function such that

$$(68) \quad D^\alpha V(x) = O(e^{b_2|x|^p}) \quad \text{as } |x| \rightarrow \infty,$$

for some $b_2 > 0$ and all α .

Suppose now that l is any given positive integer. By assumption there exists a (b, k) -parametrix P for S with $b = b_1 + b_2 + 1$ and $k = l + |\beta|$. Thus we have

$$(69) \quad S * P = \delta - W$$

where P and W satisfy the growth conditions in (p_1) and (p_2) .

From (66) and (69) it now follows that

$$\begin{aligned} U &= U * \delta = U * (S * P) + U * W \\ &= (U * S) * P + U * W = V * P + U * W \end{aligned}$$

where the convolutions are well defined and the associativity is legitimate because of the rate of decrease of P and W .

But $V * P$ is in EK'_P , since by (p_1) ,

$$D^\alpha(V * P) = \sum_{|\alpha'| \leq m} (D^{\alpha+\alpha'} V) * f_{\alpha'},$$

where

$$f_{\alpha'}(x) = O(e^{-b|x|^p}) \quad \text{as } |x| \rightarrow \infty,$$

so that $V * P$ is a C^∞ -function and, by (68),

$$D^\alpha(V * P)(x) = O(e^{b_2|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

for all α .

Also $U * W = f * D^\beta W$, which shows that $U * W$ is a function in C^l and, by (67),

$$D^\alpha(U * W)(x) = O(e^{b_1|x|^p}) \quad \text{as } |x| \rightarrow \infty$$

for $|\alpha| \leq l$.

Consequently U is a function in C^l and

$$D^\alpha U(x) = e^{(b_1+b_2)|x|^p} \quad \text{as } |x| \rightarrow \infty$$

for $|\alpha| \leq l$. But l was arbitrary and therefore U must be in EK'_P .

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