QUASI-SIMILAR MODELS FOR NILPOTENT OPERATORS⁽¹⁾

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ABSTRACT. Every nilpotent operator on a complex Hilbert space is shown to be quasi-similar to a canonical Jordan model. Further, the para-reflexive operators are characterized generalizing a result of Deddens and Fillmore.

A familiar result states that each nilpotent operator on a finite dimensional complex Hilbert space is similar to its adjoint. One proof proceeds by showing that both a nilpotent operator and its adjoint have the same canonical form. In this note we show that although this result does not extend to infinite dimensional spaces, the weaker quasi-similarity version of it, together with the proof indicated above, still holds on any Hilbert space. This yields an affirmative answer to a question raised by P. Rosenthal in connection with the content of [3].

The canonical form exhibited provides positive evidence that the theory of Jordan models might be extended to cover operators of class C_0 of infinite multiplicity and indeed, considerable progress [2] has been made recently in this direction. Although the Jordan model for nilpotent operators on infinite dimensional Hilbert spaces is no longer unique, we single out a "canonical" model. A similar result has been obtained independently by Berkovici [1]. We conclude with an application of our results to extend to infinite dimensional spaces a theorem of Deddens and Fillmore [4] which characterizes reflexive operators on finite dimensional spaces.

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1. In this note, a nilpotent operator T will be called a Jordan operator if $T = \bigoplus_{\alpha} T_{\alpha}$, where each T_{α} operates on some $C^{l_{\alpha}}$ for $0 < l_{\alpha} < \infty$ by the Jordan one-cell matrix

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$$\begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & 0 & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & 1 & 0 \\ 0 & & & & & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Recall that an operator X between Hilbert spaces H and K is said to be a quasi-affinity if $\ker X = (0)$ and $\text{clos}(XH) = K$. An operator A on H is said to be a quasi-affine transform of an operator B on K if there exists a quasi-affinity X such that $XA = BX$. Finally, two operators A and B are quasi-similar if each is a quasi-affine transform of the other. For further information on these concepts see the monograph [8, Chapter II, No. 3.2], or [7].

Our main result is given by the following

THEOREM 1. *Every nilpotent operator T is quasi-similar to a Jordan operator T_0 .*

Since for any Jordan operator T_0 , the operators T_0 and T_0^* are obviously unitarily equivalent, we can infer

THEOREM 2. *If T is a nilpotent operator, then T and T^* are quasi-similar.*

Before starting the proof of Theorem 1, we give an example to show that quasi-similarity cannot be replaced by similarity.

Let X be any compact quasi-affinity on an infinite dimensional Hilbert space H (for example, the Volterra operator on $L^2(0, 1)$) and consider the operator T defined by

$$T = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{on } H \oplus H \oplus H.$$

Clearly $T^3 = 0$ and thus T and T^* are quasi-similar by Theorem 2, but T and T^* are not similar.⁽²⁾ The proof of this is straightforward.

If S were an invertible operator on $H \oplus H \oplus H$ with matrix

(2) The same example was found independently by H. Radjavi (see [3, §6]).

$$S = \begin{pmatrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$$

which satisfied $ST^* = TS$, then a simple computation shows that $B_2 = C_2 = C_1 = 0$, $C_0 = XB_1$ and $A_2 = B_1X^*$. Thus the operator

$$S_0 = \begin{pmatrix} A_0 & B_0 & 0 \\ A_1 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a compact perturbation of S , and hence a Fredholm operator, which is contradicted by the fact that $\ker S_0 = (0) \oplus (0) \oplus H$ is not finite dimensional (cf. [5, Chapter 5]).

2. We start the proof of Theorem 1 with the following

LEMMA 1. *If T_0 and T_1 are two Jordan operators and T_0 is a quasi-affine transform of T_1 , then T_0 and T_1 are quasi-similar.*

PROOF. The fact that T_0 is a quasi-affine transform of T_1 means that there exists a quasi-affinity X such that $XT_0 = T_1X$ and thus $T_0^*X^* = X^*T_1^*$. Since T_i is a Jordan operator, there exists a unitary operator U_j such that $T_j^* = U_j^*T_jU_j$ ($j = 0, 1$). Therefore, $T_0(U_0X^*U_1^*) = (U_0X^*U_1^*)T_1$, where $U_0X^*U_1^*$ is a quasi-affinity, and T_1 is also a quasi-affine transform of T_0 . Consequently, T_0 and T_1 are quasi-similar.

LEMMA 2. *Any nilpotent operator T has a quasi-affine transform T_0 which is a Jordan operator.*

PROOF. Suppose that $T^n = 0$, $T^{n-1} \neq 0$ for some $n \geq 1$. If we set

$$X_j = \ker T^j \ominus \ker T^{j-1} \quad \text{for } j = 1, 2, \dots, n,$$

$$Y_n = X_n, \quad Y_{n-1} = X_{n-1} \cap (TY_n)^\perp, \dots,$$

$$Y_1 = X_1 \cap (T^{n-1}Y_n + \dots + TY_2)^\perp \text{ and}$$

$$H_0 = \underbrace{(Y_n \oplus \dots \oplus Y_n)}_{n \text{ times}} \oplus \underbrace{(Y_{n-1} \oplus \dots \oplus Y_{n-1})}_{(n-1) \text{ times}} \oplus \dots \oplus (Y_2 \oplus Y_2) \oplus Y_1$$

we can define the bounded operators T_0 on H_0 and $A: H_0 \rightarrow H$ by the equations

$$\begin{aligned} T_0(y_n^1 \oplus \cdots \oplus y_n^n \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_1^1) \\ = 0 \oplus y_n^1 \oplus \cdots \oplus y_n^{n-1} \oplus \cdots \oplus 0 \oplus y_2^1 \oplus 0, \end{aligned}$$

and

$$\begin{aligned} A(y_n^1 \oplus \cdots \oplus y_n^n \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_1^1) \\ = y_n^1 + \cdots + T^{n-1}y_n^n + \cdots + y_2^1 + Ty_2^2 + y_1^1. \end{aligned}$$

It is easy to see that T_0 is a Jordan operator and that $AT_0 = TA$. Using the fact that

$$\begin{aligned} (y_1 + \cdots + T^{n-1}y_n + y_2 + \cdots + T^{n-2}y_n + \cdots + y_k \\ + \cdots + T^{n-k}y_n)^- = \ker T^k, \end{aligned}$$

which is proved by induction on k , we conclude that $\text{clos}(AH_0) = H$. To complete the proof we must show that A is injective.

If A is not injective, there must exist y_j^k in $\mathcal{V}_j$, $1 \leq j \leq n$, $1 \leq k \leq j$, such that

$$\sum_{j=1}^n \sum_{k=1}^j T^{k-1}y_{n-j+k}^k = 0 \quad \text{but} \quad \sum_{j=1}^n \sum_{k=1}^j \|y_{n-j+k}^k\| \neq 0.$$

Let m be the smallest integer such that $\sum_{k=1}^m \|y_{n-m+k}^k\| \neq 0$ and let p be the smallest integer such that $y_{n-m+p}^p \neq 0$. Because we have

$$\sum_{k=p}^m T^{k-1}y_{n-m+k}^k = - \sum_{j=m+1}^n \sum_{k=1}^j T^{k-1}y_{n-j+k}^k \quad \text{in } \ker T^{n-m},$$

it follows that $y_{n-m+p}^p + \cdots + T^{m-p}y_n^m$ is in $\ker T^{n-m+p-1}$. If we let P denote the orthogonal projection of H onto X_{n-m+p} , then

$$\begin{aligned} y_{n-m+p}^p + P(Ty_{n-m+p+1}^{p+1} + \cdots + T^{m-p}y_n^m) \\ = P(y_{n-m+p}^p + \cdots + T^{m-p}y_n^m) = 0 \end{aligned}$$

since X_{n-m+p} is orthogonal to $\ker T^{n-m+p-1}$ and y_{n-m+p}^p is in X_{n-m+p} . Moreover, since y_{n-m+p}^p is orthogonal to $T^{m-p}y_n + T^{m-p-1}y_{n-1} + \cdots + Ty_{n-p+m+1}$, it follows that

$$y_{n-m+p}^p \perp P(Ty_{n-m+p+1}^{p+1} + \cdots + T^{m-p}y_n^m)$$

and hence that $y_{n-m+p}^p = 0$ which is a contradiction.

This completes the proof of the lemma.

3. Proof of Theorem 1. By applying Lemma 2 to T and T^* we obtain quasi-affinities X and X_* together with Jordan operators T_0 and T_1 such that

$TX = XT_0$, and $T^*X_* = X_*T_1$. Hence, $X_*^*T = T_1^*X_*^*$ and T is a quasi-affine transform of the Jordan operator T_1^* . Thus T_0 is a quasi-affine transform of T_1^* and hence T_0 and T_1^* are quasi-similar by Lemma 1. Consequently, T is a quasi-affine transform of T_0 and we have established that T and T_0 are quasi-similar.

4. We make several remarks before continuing.

Since there exist quasi-nilpotent operators T such that $\ker T = (0) \neq \ker T^*$ (for example, take T to be the weighted shift with weights $1, 1/2, 1/3, \dots$), Theorem 2 is not valid for quasi-nilpotent operators.

As a consequence of Theorem 2, observe that Lemma 1 holds for all nilpotent operators, that is, if one nilpotent operator is a quasi-affine transform of another, then the two operators are actually quasi-similar.

Lastly, by using the Dunford-Riesz spectral decomposition Theorem 2 can be shown to hold for algebraic operators with *real* spectrum.

5. Theorem 1 provides a Jordan model for every nilpotent operator on Hilbert space. However, in contrast with the finite dimensional case, distinct Jordan models may be quasi-similar. Fortunately, the situation is not as complicated as it might first appear. We obtain a canonical choice and hence a complete set of quasi-similarity invariants for nilpotent operators after introducing some terminology.

For each integer m ($1 \leq m < \infty$) and each infinite cardinal $\aleph$, let $J_m^\aleph$ denote the Jordan operator defined by the $m \times m$ operator matrix

$$\begin{pmatrix} 0 & I_H & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I_H & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & & & & \cdot & & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

$$\text{on } mH = \underbrace{H \oplus \cdots \oplus H}_{m \text{ times}},$$

where H is a Hilbert space of dimension $\aleph$.

THEOREM 3. *Every nilpotent operator is quasi-similar to a unique Jordan*

model of the form $\oplus J_{m_i}^{\aleph_i} \oplus N$, where $1 \leq m_1 < m_2 < \dots < m_k < \infty$, $\aleph_1 > \aleph_2 > \dots > \aleph_k$, and N is a finite rank Jordan model $\oplus_{j=1}^n T_j$ on $\oplus_{j=1}^n C^j$ with $m_k < l_j$ for $j = 1, 2, \dots, n$

PROOF. By Theorem 1 we need only consider Jordan models and easy arguments reduce the result to proving that $J_k^{\aleph} \oplus J_{k-1}^{\aleph}$ and $J_k^{\aleph}$ are quasi-similar for each $1 \leq k < \infty$ and infinite cardinal $\aleph$. Moreover, since $J_k^{\aleph*}$ and $J_{k-1}^{\aleph*}$ are unitarily equivalent to $J_k^{\aleph}$ and $J_{k-1}^{\aleph}$, respectively, it is sufficient to show that $J_k^{\aleph}$ is a quasi-affine transform of $J_k^{\aleph} \oplus J_{k-1}^{\aleph}$. Let H be a Hilbert space of dimension $\aleph$ and suppose A and B are operators on H which satisfy

- (1) $\ker A = (0)$,
- (2) $\text{clos}(AH) = H$, and
- (3) $\text{clos}\{Ax \oplus Bx : x \in H\} = H \oplus H$.

Then the identity

$$\left(\begin{array}{ccc|ccc} A & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & 0 & & A & \\ \hline 0 & B & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & 0 & & \\ 0 & & & & & B \end{array} \right) (J_k^{\aleph}) = \left(\begin{array}{c|c} J_k^{\aleph} & 0 \\ \hline 0 & J_{k-1}^{\aleph} \end{array} \right) \left(\begin{array}{ccc|ccc} A & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & 0 & & A & \\ \hline 0 & B & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & 0 & & \\ 0 & & & & & B \end{array} \right)$$

would complete the proof since (1), (2) and (3) imply that the matrix

$$\left(\begin{array}{ccc|ccc} A & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & 0 & & A & \\ \hline 0 & B & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & 0 & & \\ 0 & & & & & B \end{array} \right)$$

defines a quasi-affinity from kH to $(2k-1)H$.

There are various ways of exhibiting operators satisfying (1), (2) and (3). For example, let M_1 and M_2 denote multiplication by the characteristic functions of the first and second quarters Q_1 and Q_2 of the unit circle respectively, defined from the Hardy space H^2 to the L^2 spaces $L^2(Q_1)$ and $L^2(Q_2)$ respectively. If V_1 , V_2 , and V_3 are unitary maps from H onto $H^2 \otimes H$, $L^2(Q_1) \otimes H$, and $L^2(Q_2) \otimes H$, then $A = V_2^*(M_1 \otimes I_H)V_1$ and $B = V_3^*(M_2 \otimes I_H)V_1$ have the desired properties.

This theorem is probably indicative of the kind of uniqueness one can expect for Jordan models for C_0 -operators of infinite multiplicity.

We conclude this section with a corollary which completes the classification of nilpotents up to quasi-similarity.

COROLLARY. *If T_1 and T_2 are nilpotent operators on the Hilbert spaces H_1 and H_2 , respectively, then T_1 and T_2 are quasi-similar if and only if $\dim \text{clos } [T_1^l H_1] = \dim \text{clos } [T_2^l H_2]$ for $l = 0, 1, \dots$.*

PROOF. If X is a quasi-affinity from H_1 to H_2 such that $T_2 X = X T_1$, then

$$\text{clos } [X T_1^l H_1] = \text{clos } [T_2^l X H_1] = \text{clos } [T_2^l H_2]$$

which implies that $\dim \text{clos } [T_1^l H_1] = \dim \text{clos } [T_2^l H_2]$ for $l = 0, 1, 2, \dots$. Conversely, an easy argument shows that the Jordan model given in the theorem is uniquely determined by these dimensions.

6. The results of this note enable us to extend a characterization of reflexive operator of Deddens and Fillmore [4] to infinite dimensional spaces. Recall that a linear subspace M of the Hilbert space H is said to be para-closed for the operator T on H if M is the range of some bounded operator on H . Let us call an operator T on H *para-reflexive* if any operator U on H leaving invariant the para-closed-invariant spaces of T is an entire function of T . The definition is one of the possible natural extensions to infinite dimensional spaces of the concept of reflexive operators on a finite dimensional space.

We begin this section with a result which may have some independent interest.

PROPOSITION 1. *Para-reflexivity is preserved under quasi-similarity.*

PROOF. If T and S are quasi-similar and S is para-reflexive we must show that T is also para-reflexive. If T is not algebraic, then by virtue of Theorem 2 [6], T is para-reflexive. Thus we can assume that T (and consequently S also) is algebraic. Suppose $TA = AS$, $BT = SB$ where A, B are quasi-affinities, and let Z be an operator leaving invariant every finite dimensional subspace invariant for T , that is, for every h in H there exists some polynomial p_h such that $Zh = p_h(T)h$. If we set $Z_0 = BZA$, then

$$Z_0 h_0 = BZA h_0 = B p_{A h_0}(T) A h_0 = B A p_{A h_0}(S) h_0 \quad \text{is in } BAH$$

for every h_0 in H . Thus $X = (BA)^{-1} Z_0$ is, by the closed graph theorem, an operator on H such that

$$X h_0 \text{ is in the finite dimensional space } \bigvee_{j \geq 0} S^j h_0$$

for every h_0 in H . It follows that X leaves invariant every finite dimensional

subspace of H invariant under S . Thus, since S is para-reflexive, we infer from Corollary 2 [6] that $X = q(S)$, where q is a suitable polynomial. Consequently, $BZA = Z_0 = BAq(S) = Bq(T)A$ and hence $Z = q(T)$. Using Corollary 2 [6] once again, we conclude that T is para-reflexive.

A nilpotent operator on H is said to satisfy the Deddens-Fillmore condition [4], if either $\dim H \leq 1$ or its Jordan model $\bigoplus_{\alpha \in A} J_\alpha$ has the following property: If n_α denotes the order of the matrix of J_α ($\alpha \in A$) and α_0 is chosen in A such that

$$(0) \quad n_{\alpha_0} = \max \{n_\alpha | \alpha \in A\},$$

then

$$(1) \quad \max \{n_\alpha | \alpha \in A \setminus \{\alpha_0\}\} \geq n_{\alpha_0} - 1.$$

PROPOSITION 2. *A nilpotent operator T on H is para-reflexive if and only if it satisfies the Deddens-Fillmore condition.*

PROOF. By virtue of Proposition 1 and Theorem 1, it is sufficient to prove the statement in case $T = \bigoplus_{\alpha \in A} J_\alpha$. Exactly as in [4] we can prove that if this T does not fulfill the Deddens-Fillmore condition then T does not have property (A) or (B) of Corollary 2 [6]. Thus, by this corollary, T is not para-reflexive.

Let us now show the sufficiency of the Deddens-Fillmore condition. It is clear that we can assume that

$$T = J_0 \oplus J_1 \oplus \left(\bigoplus_{\alpha \in B} J_\alpha \right),$$

where the order n_i of J_i is the maximum occurring in formula (i) above ($i = 0, 1$); thus the order of any J_α ($\alpha \in B$) is not greater than n_1 . Now let Z be an operator leaving invariant all para-closed subspaces invariant for T . Then obviously

$$Z = Z_0 \oplus Z_1 \oplus \left(\bigoplus_{\alpha \in B} Z_\alpha \right)$$

and for any $h = h_0 \oplus h_1 \oplus \left(\bigoplus_{\alpha \in B} h_\alpha \right)$ there exists a polynomial p_h such that

$$Zh = p_h(T)h, \quad \text{that is,}$$

$$(Z_0 \oplus Z_1 \oplus Z_\alpha)(h_0 \oplus h_1 \oplus h_\alpha) = p_h(J_0 \oplus J_1 \oplus J_\alpha)(h_0 \oplus h_1 \oplus h_\alpha)$$

for every α in B . The above relation shows in particular that $Z_0 \oplus Z_1 \oplus Z_\alpha$ leaves invariant every invariant subspace of $J_0 \oplus J_1 \oplus J_\alpha$. By virtue of the Deddens-Fillmore theorem there exists a unique polynomial q_α of degree $\leq n_0$ such that

such that

$$(2) \quad \begin{aligned} Z_0 \oplus Z_1 \oplus Z_\alpha &= q_\alpha(J_0 \oplus J_1 \oplus J_\alpha) \\ &= q_\alpha(J_0) \oplus q_\alpha(J_1) \oplus q_\alpha(J_\alpha), \end{aligned}$$

for every α in B . Thus for α, β in B we have

$$(3) \quad q_\alpha(J_0) = Z_0 = q_\beta(J_0).$$

Since J_0 is of order n_0 and q_α, q_β are of degree $\leq n_0$, (3) implies $q_\alpha = q_\beta$. Consequently, there exists a polynomial q of degree $\leq n_0$ such that $q_\alpha \equiv q$ for every α in B . From (2) we infer

$$Z = Z_0 \oplus Z_1 \oplus \left(\bigoplus_{\alpha \in B} Z_\alpha \right) = q(J_0) \oplus q(J_1) \oplus \left(\bigoplus_{\alpha \in B} q(J_\alpha) \right) = q(T)$$

which finishes our proof.

THEOREM 4. *An operator T is para-reflexive if and only if either it is nonalgebraic or it is algebraic and the nilpotents corresponding to the points of the spectrum of T satisfy the Deddens-Fillmore condition.*

PROOF. This follows at once from Proposition 2 above, the Dunford-Riesz spectral decomposition of an algebraic operator, and Corollary 1.

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