

CRITICAL GROUPS HAVING CENTRAL
MONOLITHS OF A NILPOTENT BY ABELIAN
PRODUCT VARIETY OF GROUPS

BY

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ABSTRACT. Let \mathfrak{N} be a variety of groups which has nilpotency class two and finite odd exponent. Let \mathfrak{A} be an abelian variety of groups with finite exponent relatively prime to the exponent of \mathfrak{N} . The existence in the product variety \mathfrak{NA} of nonnilpotent critical groups having central monoliths is established. The structure of these critical groups is studied. This structure is shown to depend on an invariant, k . The join-irreducible subvariety of \mathfrak{NA} generated by the nonnilpotent critical groups of \mathfrak{NA} having central monoliths is determined, in particular, for k odd.

1. Introduction. (Throughout any terminology or notation not explained is described in Hanna Neumann's book [4].) Any nilpotent variety of nilpotency class two and finite exponent is of the form $\mathfrak{B}_n \wedge \mathfrak{A}_n \mathfrak{A}_{n'} \wedge \mathfrak{N}_2$ (where n' divides n); let $\mathfrak{N}_{(n,n')}$ denote this variety. The variety $\mathfrak{N}_{(q,q)} \mathfrak{A}_m$ is a subvariety of the product variety $\mathfrak{N}_{(n,n')} \mathfrak{A}_m$, if q divides n' . Thus we may assume that n equals n' and that n is a power of a prime. In [7], the product variety $\mathfrak{N}_{(q,q)} \mathfrak{A}_m$ was shown to be join-irreducible for q a nontrivial power of an odd prime not dividing m . The critical groups generating these product varieties were also characterized in that paper. For distinct primes p and r , satisfying r dividing $p - 1$, R. G. Burns [2] has obtained a complete description of the lattice of subvarieties of $\mathfrak{N}_{(p,p)} \mathfrak{A}_r$.

Let k be the smallest positive integer such that m divides $p^k - 1$ where p is an odd prime not dividing the positive integer m , and let q equal p^e . The join-irreducible subvariety of $\mathfrak{N}_{(q,q)} \mathfrak{A}_m$ generated by the nonnilpotent critical groups in this variety having central monoliths is determined in the case that m does not divide $p^{k/2} + 1$.

The critical groups of the variety $\mathfrak{N}_{(q,q)} \mathfrak{A}_m$ having central monoliths are contained in the subvariety $\mathfrak{N}_{(q,q)} \mathfrak{A}_m \wedge \mathfrak{C}$, where \mathfrak{C} denotes the variety of groups satisfying the law,

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$$[[x, y], [u, t]]^2 [[u, t], [x, y]] = 1.$$

Thus the third subgroup in the derived series of a group in \mathfrak{C} is central. The following result is obtained in §4:

THEOREM I. *The variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m \wedge \mathfrak{C}$ is join-irreducible, if k is odd or if k is even with m not dividing $p^{k/2} + 1$, where k is the smallest positive integer such that m divides $p^k - 1$.*

The results of §4 are not sufficient to resolve the case for k even and m dividing $p^{k/2} + 1$.

To establish Theorem I it is necessary to study in §2 the structure of the nonnilpotent critical groups in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having central monoliths. A description of this structure is given in Theorems II and III.

THEOREM II. *If C is a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith, then*

- (i) *the p' -complement of the Fitting subgroup, $\text{Fit } C$, is cyclic, and*
- (ii) *the exponent of the commutator subgroup, $(\text{Fit } C)'$, of the Fitting subgroup equals the exponent of the Fitting subgroup.*

The results of R. G. Burns [1] show that if C is a critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ with the exponent of $C/\text{Fit } C$ equaling m , then the Fitting subgroup is generated by k or $2k$ elements, where k is the invariant defined above. The following theorem is also established in §2:

THEOREM III. *Let C be a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith. If $\text{Fit } C$ is generated by k elements and $C/\text{Fit } C$ has exponent m , then k is even and m divides $p^{k/2} + 1$.*

It should be noted that if the word “critical” is replaced by “monolithic”, the theorem remains true.

A nonnilpotent critical group in the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith is constructed in §3. The method of construction depends on the invariant k .

2. Structure of critical groups having central monoliths. Let C be a nonnilpotent critical group in $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$. Since C is a monolithic group, the Fitting subgroup, $\text{Fit } C$, is a p -group where $q = p^e$. Thus C is the semidirect product of $\text{Fit } C$ and the p' -complement D of $\text{Fit } C$ in C ; that is, $C = (\text{Fit } C) \cdot D$. Set $F = \text{Fit } C$ and $\Phi = \Phi(\text{Fit } C)$, the Frattini subgroup of $\text{Fit } C$.

By Maschke’s theorem [3], the Frattini quotient group F/Φ viewed as a $GF(p)D$ -module may be decomposed into the direct product of irreducible $GF(p)D$ -modules, and by Burns’ results there are at most two such irreducible $GF(p)D$ -modules in this decomposition. Thus either F/Φ is a minimal normal subgroup of C/Φ or

$$(2.1) \quad F/\Phi = N_1/\Phi \times N_2/\Phi$$

where N_1/Φ and N_2/Φ are minimal normal subgroups of C/Φ . Since C is a

critical group, D is represented faithfully by conjugation on F/Φ , and thus D is embeddable in the direct product of two m -cycles, $C_m \times C_m$; see Burns [1].

PROOF OF THEOREM II. For this proof set $m' = \exp(D)$, $t = \exp(F)$, and $t' = \exp(F')$. If F/Φ is a minimal normal subgroup in C/Φ , then D is cyclic because it is faithfully and irreducibly represented on F/Φ . On the other hand, if F/Φ has (2.1) as its decomposition into irreducible $GF(p)D$ -modules, then the centralizer of N_1/Φ , $C_D(N_1/\Phi)$, in D is trivial, because the monolith, $[N_1, N_2]^{t'/p}$, of C is centralized by D . Thus again D is faithfully and irreducibly represented on N_1/Φ , and so D is cyclic; hence part (i) has been established.

Part (ii) is proved by contradiction. Assume that $t' < t$. There are elements z_1 and z_2 in F such that F is the normal closure of the subgroup generated by z_1 and z_2 in C (in some cases, one element would suffice). The subgroup F is a regular p -group since p is an odd prime. Thus either the order of z_1 or the order of z_2 is t . However, the powers $z_1^{t'/p}$ and $z_2^{t'/p}$ are central in F as the following calculation shows: $[z_i^{t'/p}, x] = [z_i, x]^{t'/p} = 1$ for any x in F . Since C is monolithic, $z_1^{t'/p}$ and $z_2^{t'/p}$ must be in the monolith. Hence $z_1^{t'/p}$ and $z_2^{t'/p}$ are centralized by D . Thus, $(z_1 z_1^{-\alpha})^{t'/p}$ and $(z_2 z_2^{-\alpha})^{t'/p}$ are trivial for α in D . However, if α is a generator of D , F is the normal closure of the subgroup generated by $(z_1 z_1^{-\alpha})$ and $(z_2 z_2^{-\alpha})$ in C ; that is, F is generated by elements of order t/p or less. This is a contradiction since F is a regular p -group. \square

Theorem III shows that the structure of the nonnilpotent critical groups in $\mathfrak{N}_{(g,q)}\mathfrak{M}_m$ having central monoliths depends on the invariant k . If the exponent of the p' -complement, D , of F in C is m , then the dimension of the irreducible $GF(p)D$ -modules in the decomposition of F/Φ is k ; this follows from M. F. Newman's results in [5].

PROOF OF THEOREM III. Let the critical group $C = F \cdot D$ as above with t the exponent of F . The p' -group D has exponent m by hypothesis and is cyclic by Theorem II, part (i); let α generate D . Also, F/Φ is an irreducible $GF(p)D$ -module by hypothesis.

Suppose there is a y in F but not in $\Phi(F)$ such that some nontrivial p th power, g , of y is in the central monolith. This implies that $(yy^{-\alpha})^g$ is trivial, but the normal closure of the subgroup generated by $yy^{-\alpha}$ in C is F ; contradiction. Therefore F/F^p is a nonabelian monolithic p -group; let F^* denote the quotient group F/F^p . (Note that, F is regular since $p > 2$.)

Clearly $\Phi(F^*) = (F^*)'$. The center, $Z(F^*)$, of F^* also equals $(F^*)'$ since $Z(F^*)$ is a proper characteristic subgroup of F^* . The order of $(F^*)'$ is p since it is the central monolith of $C/(F)^p$. Thus F^* is an extra-special p -group [3, p. 183].

By [3, Theorem 6.5, p. 213], the order, m , of α divides $(p^{2r} - 1)$ with $0 < 2r \leq k$, which contradicts the choice of k unless $2r = k$. Therefore k is even and m divides $(p^{k/2} + 1)$. \square

In the above proof it is sufficient to assume that C is monolithic; the remaining hypothesis implies that D is cyclic. The author has been able to establish the converse of Theorem III; however, the proof of this is too lengthy to include here.

3. **The existence of critical groups having central monoliths.** Let F_k be a relatively free group of rank k of the variety $\mathfrak{N}_{(q,q)}$. In [7], the existence of an automorphism β and a generating set $\{y_1, \dots, y_k\}$ of F_k such that $y_1\beta = y_2, \dots, y_{k-1}\beta = y_k$ was established. Now let F_{2k} be a relatively free group of $\mathfrak{N}_{(q,q)}$ freely generated by $\{y_{11}, \dots, y_{1k}, y_{21}, \dots, y_{2k}\}$. There are automorphisms β_1 and β_2 of F_{2k} each of order m such that β_i fixes each y_{jn} with $j \neq i$ and acts on $\{y_{i1}, \dots, y_{ik}\}$ as β acts on $\{y_1, \dots, y_k\}$ [7]. Let B be the abelian subgroup of the automorphism group of F_{2k} generated by $\{\beta_1, \beta_2\}$, and let V be the splitting extension of F_{2k} by B . The group V generates the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ [7].

$$T = gp\langle [y_{ij}, y_{in}], ([y_{1j}, y_{2j}][y_{2j}^{\beta_1^{-1}}, y_{1j}^{\beta_1}])^\beta \mid i = 1, 2; 1 \leq j, n \leq k; \beta \in B \rangle$$

defines a normal subgroup of V contained in $(F_{2k})'$. (It turns out that V/T is a critical group generating the variety $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ [7].) Let A denote the subgroup of V/T generated by $\alpha = \beta_1\beta_2^{-1}T$; \mathfrak{Q} denote F_{2k}/T ; and R denote the semidirect product of \mathfrak{Q} by A . The commutator subgroup \mathfrak{Q}' of \mathfrak{Q} is homocyclic and is centralized by A . If $m = 2$, then \mathfrak{Q}' is cyclic and R is monolithic.

Now, assuming $m \neq 2$, let N be the complement of

$$gp\langle [y_{11}, y_{21}^{\beta_1^{-1}}][y_{21}, y_{11}^{\beta_1}]T \rangle$$

in \mathfrak{Q}' which is normal in R given by D. R. Taunt's results [6]. The quotient group $C^* = R/N$ is monolithic with monolith $(\mathfrak{Q}'/N)^{q/p}$, since \mathfrak{Q}'/N is a regular p -group. For $m = 2$, set $C^* = R$. If k is odd, or if k is even with m not dividing $p^{k/2} + 1$, C^* is a nonnilpotent critical group of $\mathfrak{N}_{(q,q)}\mathfrak{A}_m$ having a central monolith. If k is even with m dividing $p^{k/2} + 1$, then the subgroup of C^* generated by α and $y_{11}y_{21}N$ is the desired critical group having a central monolith; denote this group by C^{**} . That is, $C^{**} = gp\langle \alpha, y_{11}y_{21}N \rangle$.

We now define inductively the word ω which is used in showing that the groups C^* and C^{**} are critical groups. First let

$$\mu_2(x_1, x_2) = [x_1, x_2, (x_2^{-1}x_1)],$$

and for n in $\{3, \dots, m\}$, put

$$\mu_n(x_1, \dots, x_n) = [\mu_{n-1}, x_n, (x_n^{-1}x_1), \dots, (x_n^{-1}x_{n-1})].$$

Finally, set

$$\omega = ([\mu_m(x_1, \dots, x_m), \mu_m(z_1, \dots, z_m)])^{q/p}.$$

The following substitution shows that ω is not a law of C^* or C^{**} : $x_j \mapsto \alpha^{(j-1)}, z_j \mapsto \alpha^{(j-1)}, j = 2, \dots, m$, and $x_1 \mapsto y_{11}y_{21}N, z_1 \mapsto (y_{11}y_{21}N)^\alpha$.

THEOREM 3.1. *The groups C^* and C^{**} are critical.*

PROOF. Since the groups C^* and C^{**} are monolithic and since ω is not a law for C^* or C^{**} , to establish that C^* and C^{**} are critical groups, we need only show by a result of Kovács and Newman (see 53.41 in [4]) that ω is a law in their proper subgroups. Only the proof for C^* is given; the proof for C^{**} is similar. Let \mathcal{Q}^* denote the Fitting subgroup of C^* ; that is, $\mathcal{Q}^* = \mathcal{Q}/N$.

Suppose H is a subgroup of C^* which does not have ω as a law. Since ω is not a law in H , there is a substitution for the $x_1, \dots, x_m, z_1, \dots, z_m$ in ω with values in H such that ω is not the identity. Under this substitution $\mu_m(x_1, \dots, x_m)$ and $\mu_m(z_1, \dots, z_m)$ have values in \mathcal{Q}^* outside of the Frattini subgroup, $\Phi(\mathcal{Q}^*)$. It is also clear that each of the values substituted for x_1, \dots, x_m (or for z_1, \dots, z_m) lie in distinct cosets of \mathcal{Q}^* in C^* . Hence the order of H is divisible by m . Set $\mathcal{Q}_H = \text{Fit } H$. The p -group $P = \mathcal{Q}_H(\mathcal{Q}^*)^p/(\mathcal{Q}^*)^p$ is an extra-special p -group. Since k is odd or k is even with m not dividing $p^{k/2} + 1$, P/P' cannot be an irreducible $GF(p)A$ -module [3, p. 213]. Thus $\mathcal{Q}_H\Phi(\mathcal{Q}^*)/\Phi(\mathcal{Q}^*)$ is not an irreducible $GF(p)A$ -module; hence $\mathcal{Q}_H = \mathcal{Q}^*$. Therefore H is not a proper subgroup of C^* . \square

4. **Proof of Theorem I.** For notational convenience set $\mathfrak{U} = \mathfrak{N}_{(q,q)}\mathfrak{U}_m \wedge \mathbb{C}$ and from §3, $R = gp\langle \beta_1, \beta_2^{-1}T, F_{2k} \rangle/T$. The proof of Theorem I is accomplished by establishing the following three propositions:

- (i) The group R generates the variety \mathfrak{U} .
- (ii) There is a proper subvariety of \mathfrak{U} which contains any critical group C of \mathfrak{U} in which either (a) the exponent of the commutator subgroup of $\text{Fit } C$ is less than q , or (b) the order of $C/\text{Fit } C$ is less than m .
- (iii) For k odd or k even with m not dividing $p^{k/2} + 1$, a nonnilpotent critical group C having a central monolith generates the variety \mathfrak{U} if the exponent of $\text{Fit } C$ is q and $C/\text{Fit } C$ has order m .

Proposition (i) is established by showing that any critical group C of \mathfrak{U} is a factor of a direct power of R . If C is nilpotent this result is obvious. For any nontrivial divisor m^* of m and corresponding smallest positive integer k^* such that m^* divides $p^{k^*} - 1$, there is, by Clifford's theorem [3], a subgroup of R which has a corresponding structure like R for m^* and k^* . Hence we may assume that C/F has order m , where as before $F = \text{Fit } C$. It was shown in [7] that any critical group C in $\mathfrak{N}_{(q,q)}\mathfrak{U}_m$ is a factor of V , the splitting extension of F_{2k} by B . There are three cases: $\dim(F/\Phi) = k$, $\dim(F/\Phi) = 2k$ with m not dividing $p^{k/2} + 1$, and $\dim(F/\Phi) = 2k$ with m dividing $p^{k/2} + 1$, where $\Phi = \Phi(F)$.

Let F/Φ have dimension k . In this case, it can be shown that C is a factor of V^* , the splitting extension of F_k by the m -cycle β of §3; in the same way that C was shown to be a factor of V in [7]. The group V^* is isomorphic to the subgroup \mathcal{V}^* of V generated by $y_{11}y_{21}$ and $\beta_1\beta_2^{-1}$. Hence C is a homomorphic image of \mathcal{V}^* , and clearly the intersection of \mathcal{V}^* and T is contained in the kernel of this homomorphism of \mathcal{V}^* onto C . Thus C is a factor of R .

Now suppose $\dim(F/\Phi)$ is $2k$ and m does not divide $p^{k/2} + 1$. Recall the minimal normal subgroups N_1/Φ and N_2/Φ of C/Φ in the decomposition (2.1)

of F/Φ and the p' -subgroup D of C in §2. The quotient group

$$(gp\langle N_i, D \rangle)F^p/F^p$$

is monolithic, and so by Theorem III (see the note at the end of the proof of Theorem III), it must be metabelian. Thus N_1 and N_2 can be chosen so that $(N_1)'$ and $(N_2)'$ are trivial. The critical group C is a homomorphic image of the subgroup \mathcal{V} of V generated by F_{2k} and $\beta_1\beta_2^{-1}$. The kernel of this homomorphism contains T , since $(N_1)'$ and $(N_2)'$ are trivial and since the monolith of C is central. Therefore C is a factor of R .

In the third case where $\dim(F/\Phi)$ is $2k$ but k is even with m dividing $p^{k/2} + 1$, N_1 and N_2 need not be abelian. The critical group C is again a homomorphic image of \mathcal{V} . The group \mathcal{V} can be embedded in the direct product $V^* \times \mathcal{V}/T_1 \times V^*$, where T_1 is the normal closure of the subgroup of V generated by commutators of the form $[y_{1j}, y_{1j'}]$ and $[y_{2j}, y_{2j'}]$ for $1 \leq j, j' \leq k$ in V . It is easy to verify that V^* is isomorphic to the subgroup of \mathcal{V}/T_1 generated by $\beta_1\beta_2^{-1}$ and $y_{11}y_{21}T_1$; hence C is a factor of the direct power $\mathcal{V}/T_1 \times \mathcal{V}/T_1 \times \mathcal{V}/T_1$. Since the monolith of C is central, C is a factor of the direct power $R \times R \times R$.

The proper subvariety of proposition (ii) is the subvariety of \mathfrak{U} determined by ω . To establish (ii) all that needs to be shown is that if C is a critical group of \mathfrak{U} which does not have ω as a law, then the exponent of $(\text{Fit } C)'$ is q and $C/\text{Fit } C$ has order m . That the exponent of the commutator subgroup is q is obvious. The argument that $C/\text{Fit } C$ has order m is essentially the argument in the proof of Theorem 3.1, that the order of H is divisible by m . Therefore proposition (ii) is established.

Let C be a critical group of \mathfrak{U} satisfying the hypothesis of proposition (iii). The critical group C is a homomorphic image of R by the proof of (i). If \mathcal{K} is the kernel of this homomorphism, then \mathcal{K} is contained in the commutator subgroup of F_{2k}/T since $C/\text{Fit } C$ has order m . The kernel \mathcal{K} cannot contain the socle, S , of R because the exponent of $(\text{Fit } C)'$ is q by Theorem II(ii). Thus, R must lie in the variety generated by C , if the socle S of R is minimal characteristic. By proposition (i), this implies that C generates the variety \mathfrak{U} . Therefore the final step in the proof is to show that the socle, S , of R is minimal characteristic in R .

To establish that S is minimal characteristic, it is necessary to study the action of conjugation by β_2 on the group R . First, the subgroup \mathcal{V} of V generated by $\beta_1\beta_2^{-1}$ and F_{2k} is clearly normalized by β_2 , and the subgroup T of F_{2k} is also normalized by β_2 . Thus conjugation by β_2 induces an automorphism on R . Now consider the action of conjugation by β_2 on the element $[y_{11}, y_{21}]^{q/p}T$ in the socle of R . Clearly the elements $([y_{11}, y_{21}]^{q/p}T)^\beta$ for $\beta \in gp\langle \beta_1\beta_2^{-1}, \beta_2 \rangle$ generate the socle, S , of R . Therefore S is minimal characteristic in R . \square

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