LEBESGUE SUMMABILITY OF DOUBLE TRIGONOMETRIC SERIES⁽¹⁾

BY

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ABSTRACT. We formulate a definition of symmetric derivatives of *odd* order for functions of two variables. Our definition is based on expanding in a Taylor's series a weighted average of the function about circles.

The definition is applied to derive results on Lebesgue summability for spherically convergent double trigonometric series.

1. Let $f(t)$ be a function defined in a neighborhood of $t_0 \in \mathbf{R}$. Let k be a natural number. We say that f has at t_0 a k th *symmetric derivative* with value a_k if the following holds:

If $k = 2r$ is even,

$$(1.1) \quad \frac{1}{2}\{f(t_0 + t) + f(t_0 - t)\} = a_0 + \frac{a_2}{2!}t^2 + \cdots + \frac{a_{2r}}{(2r)!}t^{2r} + o(t^{2r})$$

as $t \rightarrow 0$.

If $k = 2r + 1$ is odd,

$$(1.2) \quad \frac{1}{2}\{f(t_0 + t) - f(t_0 - t)\} = a_1 t + \frac{a_3}{3!}t^3 + \cdots + \frac{a_{2r+1}}{(2r+1)!}t^{2r+1} + o(t^{2r+1})$$

as $t \rightarrow 0$.

If the limit in (1.1) or (1.2) exists only as $t \rightarrow 0$ through a set having 0 as a point of density, then we say f has a k th *symmetric approximate derivative* at t_0 equal to a_k .

These definitions may be found in [7]. They have the following applications to termwise integrated trigonometric series. Let $T: \sum_{n \in \mathbf{Z}} c_n e^{in\theta}$ be a trigonometric series in one variable.

THEOREM A. *If $c_n \rightarrow 0$ and T converges at θ_0 to s , then the function*

$$F(\theta) = \frac{c_0}{2}\theta^2 - \sum' \frac{c_n}{n^2}e^{in\theta}$$

has at θ_0 a second symmetric derivative with value s .

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THEOREM B. If $c_n \rightarrow 0$ and T converges at θ_0 to a finite sum s then the function

$$L(\theta) = c_0\theta + \sum' \frac{c_n}{in} e^{in\theta}$$

has at θ_0 a first symmetric approximate derivative with value s .

We are concerned in this paper with functions of two variables. We denote points of E_2 by $x = (x_1, x_2) = te^{i\theta}$, and we write integral lattice points $n = (n_1, n_2)$. We set $n \cdot x = n_1 x_1 + n_2 x_2$. We denote the Fourier series of a function F by $S[F]$.

Suppose $F(x)$ is defined in a neighborhood of $x_0 \in E_2$. We say that F has at x_0 an r th generalized Laplacian equal to s if F is integrable over each circle $|x - x_0| = t$, for t small, and if

$$\frac{1}{2\pi} \int_0^{2\pi} F(x_0 + te^{i\theta}) d\theta = a_0 + \frac{a_2}{2!} t^2 + \cdots + \frac{s}{(2^r r!)^2} t^{2r} + o(t^{2r})$$

as $t \rightarrow 0$. This definition is due to V. Shapiro [4] and forms a two dimensional analogue of (1.1) for symmetric derivatives of even order. In [3] and [4], it is used to establish two dimensional analogues of Theorem A.

The purpose of this paper is to give a two dimensional analogue of (1.2) for symmetric derivatives of odd order, and to apply it to Lebesgue summability for double trigonometric series.

2. We make the following definition. Let

$$\Omega(\theta) = \cos \theta + \sin \theta.$$

Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$, and suppose that F is integrable on each circle $|x - x_0| = t$, for t small. Let $k = 2r + 1$ be an odd integer.

DEFINITION. F has at x_0 a generalized symmetric derivative of order $2r + 1$ with value s if

$$(2.1) \quad \begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} F(x_0 + te^{i\theta}) \Omega(\theta) d\theta \\ &= a_1 t + a_3 t^3 + \cdots + \frac{s}{2^{2r+1} r! (r+1)!} t^{2r+1} + o(t^{2r+1}), \end{aligned}$$

as $t \rightarrow 0$.

If the limit in (2.1) exists only as t tends to 0 through a set E having 0 as a point of density, we will say F has at x_0 a generalized symmetric approximate derivative equal to s .

3. The numerical value of the derivative is given by the following result.

THEOREM 1. Suppose that $F(x)$ and all partial derivatives of F of order

$\leq 2r + 2$ exist and are continuous in a neighborhood of $x_0 \in E_2$. Then F has at x_0 a $(2r + 1)$ th generalized symmetric derivative with value

$$s = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^r F(x_0).$$

PROOF. We may assume $x_0 = 0$. We apply Taylor's theorem. We write

$$F(r, s) = \frac{\partial^{r+s} F}{\partial x_1^r \partial x_2^s} \Big|_0.$$

$$\begin{aligned} F(te^{i\theta}) &= \sum_{j=0}^{2r+1} \frac{1}{j!} \left(t \cos \theta \frac{\partial}{\partial x_1} + t \sin \theta \frac{\partial}{\partial x_2} \right)^j F(0) \\ &\quad + \frac{1}{(2r+2)!} \left(t \cos \theta \frac{\partial}{\partial x_1} + t \sin \theta \frac{\partial}{\partial x_2} \right)^{2r+2} F(\mu e^{i\theta}), \end{aligned}$$

for some $\mu \in (0, t)$.

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} F(te^{i\theta}) \Omega(\theta) d\theta \\ &= \sum_{j=0}^{2r+1} \frac{t^j}{j!} \frac{1}{2\pi} \int_0^{2\pi} \left(\cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \right)^j F(0) \Omega(\theta) d\theta \\ &\quad + \frac{t^{2r+2}}{(2r+2)!} \frac{1}{2\pi} \int_0^{2\pi} \left(\cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \right)^{2r+2} F(\mu e^{i\theta}) \Omega(\theta) d\theta \\ &= \sum_{j=0}^{2r+1} a_j t^j + R_{2r+2}, \end{aligned} \tag{3.1}$$

where

$$a_j = \frac{1}{j!} \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^j \binom{j}{m} F(m, j-m) \cdot \cos^m \theta \sin^{j-m} \theta \Omega(\theta) d\theta.$$

Clearly $a_j = 0$ when j is even.

When j is odd,

$$\begin{aligned} a_j &= \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j-m) \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos^m \theta \sin^{j-m} \theta \Omega(\theta) d\theta \\ &= \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j-m) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \cos^{m+1} \theta \sin^{j-m} \theta d\theta \right. \\ &\quad \left. + \frac{1}{2\pi} \int_0^{2\pi} \cos^m \theta \sin^{j-m+1} \theta d\theta \right\} \\ &= \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j-m) \{c_{jm} + d_{jm}\}. \end{aligned} \tag{3.2}$$

Using reduction formulae we find,

$$c_{jm} = \begin{cases} \frac{m!(j-m)!}{2^j((j+1)/2)!((m-1)/2)!((j-m)/2)!} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

and

$$d_{jm} = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \frac{m!(j-m+1)!}{2^{j+1}((j+1)/2)!((j-m+1)/2)!(m/2)!} & \text{if } m \text{ is even.} \end{cases}$$

Breaking the sum in (3.2) into two parts,

(3.3)

$$\begin{aligned} a_j &= \sum_{m=0; m \text{ odd}}^j \frac{1}{j!} \binom{j}{m} \frac{m!(j-m)!}{2^j((j+1)/2)!((m-1)/2)!((j-m)/2)!} F(m, j-m) \\ &+ \sum_{m=0; m \text{ even}}^j \frac{1}{j!} \binom{j}{m} \frac{m!(j-m+1)!}{2^{j+1}((j+1)/2)!((j-m+1)/2)!(m/2)!} F(m, j-m) \\ &= \text{I} + \text{II}. \end{aligned}$$

To simplify I, set $s = (m-1)/2$.

(3.4)

$$\begin{aligned} \text{I} &= \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{s!((j-1)/2-s)!} F(2s+1, j-2s-1) \\ &= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \sum_{s=0}^{(j-1)/2} \binom{(j-1)/2}{s} F(2s+1, j-2s-1) \\ &= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \frac{\partial}{\partial x_1} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^{(j-1)/2} F(0). \end{aligned}$$

To simplify II, set $s = m/2$.

$$\begin{aligned} \text{II} &= \sum_{m=0; m \text{ even}}^j \frac{j-m+1}{2 \cdot 2^j((j+1)/2)!((j-m+1)/2)!(m/2)!} F(m, j-m) \\ &= \sum_{m=0; m \text{ even}}^j \frac{1}{2^j((j+1)/2)!((j-m-1)/2)!(m/2)!} F(m, j-m) \\ (3.5) \quad &= \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{((j-1)/2-s)!s!} F(2s, j-2s) \\ &= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \frac{\partial}{\partial x_2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^{(j-1)/2} F(0). \end{aligned}$$

Combining (3.4) and (3.5), we get

(3.6)

$$a_j = \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^{(j-1)/2} F(0).$$

For the remainder term,

$$(3.7) \quad R_{2r+2} = t^{2r+2} O(1) = o(t^{2r+1}).$$

Substituting (3.6) and (3.7) into (3.1), the proof of Theorem 1 is complete.

4. We now apply the definition in (2.1) to deduce two dimensional versions of Lebesgue summability for spherically convergent double trigonometric series. The role of (2.1) in the extension of Theorem B to two dimensions is parallel to the role played by generalized Laplacians in the extension of Theorem A to two dimensions. Our proof is similar to the methods used in [5], where a different multi-dimensional analogue of Theorem B is given.

THEOREM 2. Let $T: \sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$ be a double trigonometric series which converges spherically at x_0 to s , $s < \infty$. Suppose the coefficients of T satisfy

$$(4.1) \quad \sum_{n_1+n_2=0} |n|^\alpha |c_n|^2 + \sum_{n_1+n_2 \neq 0} |n|^\alpha (n_1 + n_2)^{-2} |c_n|^2 < \infty,$$

for some number $\alpha > 1$. Then the series

$$(4.2) \quad \sum_{n_1+n_2=0} \frac{1}{2} (x_1 + x_2) c_n e^{in \cdot x} + \sum_{n_1+n_2 \neq 0} \frac{-ic_n}{n_1 + n_2} e^{in \cdot x}$$

converges spherically a.e. on T_2 to a function $L(x)$ which has at x_0 a first generalized symmetric approximate derivative equal to s .

THEOREM 3. Suppose $\sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$ converges spherically at x_0 to s , $s < \infty$. Suppose there are functions $L_1(x)$ and $L_2(x)$ such that

$$\sum_{n_1+n_2=0} c_n e^{in \cdot x} = S[L_1]$$

and

$$\sum_{n_1+n_2 \neq 0} \frac{-ic_n}{n_1 + n_2} e^{in \cdot x} = S[L_2].$$

Let $L(x) = \frac{1}{2} (x_1 + x_2) L_1(x) + L_2(x)$. Then $L(x)$ has at x_0 a first generalized symmetric approximate derivative with value s .

5. Before starting the proofs of Theorems 2 and 3 we establish the following result. Here $J_\nu(z)$ represents the Bessel function of the first kind of order ν .

$$J_\nu(z) = \frac{1}{\pi i^\nu} \int_0^\pi e^{iz \cos \varphi} \cos(\nu \varphi) d\varphi.$$

LEMMA. Let $x = te^{i\theta} \in E_2$ and let $n = (n_1, n_2) \in \mathbb{Z}_2$, with $|n| \neq 0$. Define

$$g_n(x) = \begin{cases} \frac{-ie^{in \cdot x}}{n_1 + n_2} & \text{if } n_1 + n_2 \neq 0, \\ \frac{1}{2}(x_1 + x_2)e^{in \cdot x} & \text{if } n_1 + n_2 = 0. \end{cases}$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta = \frac{J_1(|n|t)}{|n|}.$$

PROOF. Let $n_1/|n| = \cos \varphi$, $n_2/|n| = \sin \varphi$.

We first consider $g_n(x)$ for $n_1 + n_2 \neq 0$.

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta \\ &= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} \exp \left\{ it|n| \left(\frac{n_1}{|n|} \cos \theta + \frac{n_2}{|n|} \sin \theta \right) \right\} \\ & \quad \cdot (\cos \theta + \sin \theta) \left(\frac{n_1}{|n|} + \frac{n_2}{|n|} \right) d\theta \\ &= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos(\theta - \varphi)} (\cos(\theta - \varphi) + \sin(\theta + \varphi)) d\theta \\ &= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos(\theta - \varphi)} \cos(\theta - \varphi) d\theta \\ & \quad + \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos(\theta - \varphi)} \sin(\theta + \varphi) d\theta \\ &= A_1 + B_1. \\ A_1 &= \frac{|n|}{(n_1 + n_2)^2} J_1(|n|t). \end{aligned}$$

Let $\mu = \theta - \varphi$.

$$\begin{aligned} B_1 &= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos \mu} \sin(\mu + 2\varphi) d\mu \\ &= \frac{|n|}{(n_1 + n_2)^2} \cos 2\varphi \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos \mu} \sin \mu d\mu \\ & \quad + \frac{|n|}{(n_1 + n_2)^2} \sin 2\varphi \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|t \cos \mu} \cos \mu d\mu \\ &= 0 + \frac{|n|}{(n_1 + n_2)^2} \sin(2\varphi) J_1(|n|t) = \frac{|n|}{(n_1 + n_2)^2} \frac{2n_1 n_2}{|n|^2} J_1(|n|t). \end{aligned}$$

Combining,

$$\begin{aligned} \frac{1}{2\pi i} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta \\ = A_1 + B_1 = \left(1 + \frac{2n_1 n_2}{|n|^2}\right) \frac{|n|}{(n_1 + n_2)^2} J_1(|n|t) = \frac{J_1(|n|t)}{|n|}. \end{aligned}$$

In the case $n_1 + n_2 = 0$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta \\ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}(t \cos \theta + t \sin \theta) e^{in \cdot te^{i\theta}} (\cos \theta + \sin \theta) d\theta \\ = \frac{t}{4\pi} \int_0^{2\pi} (\cos \theta + \sin \theta)^2 e^{i|n|t \cos(\theta-\varphi)} d\theta \\ = \frac{t}{4\pi} \int_0^{2\pi} e^{i|n|t \cos(\theta-\varphi)} d\theta + \frac{t}{4\pi} \int_0^{2\pi} 2 \cos \theta \sin \theta e^{i|n|t \cos(\theta-\varphi)} d\theta \\ = A_2 + B_2. \\ A_2 = \frac{1}{2} t J_0(|n|t). \\ B_2 = \frac{t}{4\pi} \int_0^{2\pi} \sin 2(\mu + \varphi) e^{i|n|t \cos \mu} d\mu \\ = \cos(2\varphi) \frac{t}{4\pi} \int_0^{2\pi} \sin(2\mu) e^{i|n|t \cos \mu} d\mu \\ + \sin(2\varphi) \frac{t}{4\pi} \int_0^{2\pi} \cos(2\mu) e^{i|n|t \cos \mu} d\mu \\ = 0 - \sin(-\pi/2) \frac{1}{2} t J_2(|n|t) = \frac{1}{2} t J_2(|n|t). \end{aligned}$$

Combining A_2 and B_2 ,

$$\frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta = \frac{1}{2} t (J_0(|n|t) + J_2(|n|t)) = \frac{J_1(|n|t)}{|n|}$$

by a formula from [1, p. 12]. Thus the proof of the Lemma is complete.

6. Proof of Theorem 3. We will assume, as we may, that $x_0 = 0$ and $s = 0$. We must show

$$\lim_{t \rightarrow 0} \text{app} \frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) d\theta = 0.$$

Set

$$L_1(x, r) = \sum_{n_1+n_2=0} c_n e^{in \cdot x} e^{-|n|r},$$

$$L_2(x, r) = \sum_{n_1+n_2 \neq 0} \frac{-ic_n}{n_1 + n_2} e^{in \cdot x} e^{-|n|r}$$

and let $L(x, r) = \frac{1}{2}(x_1 + x_2)L_1(x, r) + L_2(x, r)$. Using results found in [6], for example, we obtain

$$\begin{aligned} & \lim_{r \rightarrow 0} \int_{T_2} |L(x) - L(x, r)| dx \\ & \leq \lim_{r \rightarrow 0} \int_{T_2} |L_1(x) - L_1(x, r)| dx + \lim_{r \rightarrow 0} \int_{T_2} |L_2(x) - L_2(x, r)| dx \\ & = 0. \end{aligned}$$

Choose a sequence μ_k decreasing to 0 such that

$$\int_{T_2} |L(x) - L(x, \mu_k)| dx \leq 2^{-3k-1}.$$

Let

$$C_k = \left\{ t \in (0, 1) \mid \int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta}, \mu_k)| d\theta > 2^{-k} \right\}.$$

Then

$$\begin{aligned} 2^{-3k-1} & \geq \int_0^1 t dt \int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta}, \mu_k)| d\theta \\ & \geq \int_{C_k} t 2^{-k} dt \geq \int_0^{|C_k|} t 2^{-k} dt \\ & = 2^{-k-1} |C_k|^2. \end{aligned}$$

Hence, $|C_k| \leq 2^{-k}$. Thus if we let

$$T = (0, 1) - \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} C_k \right),$$

then $|T| = 0$ and, outside of T ,

$$\lim_{k \rightarrow \infty} \int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta}, \mu_k)| d\theta = 0,$$

so $\lim_{k \rightarrow \infty} \int_0^{2\pi} |L(te^{i\theta})\Omega(\theta) - L(te^{i\theta}, \mu_k)\Omega(\theta)| d\theta = 0$. Thus, for almost all $t \in (0, 1)$,

$$(6.1) \quad \lim_{k \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta}, \mu_k) \Omega(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) d\theta.$$

For $t \in (0, 1)$, define

$$(6.2) \quad \varphi(t) = \lim_{k \rightarrow \infty} \frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}, \mu_k) \Omega(\theta) d\theta.$$

Then, applying the Lemma,

$$(6.3) \quad \begin{aligned} \varphi(t) &= \lim_{k \rightarrow \infty} \frac{1}{2\pi t} \int_0^{2\pi} \lim_{R \rightarrow \infty} \left(\sum_{|n| < R} c_n g_n(te^{i\theta}) e^{-|n|\mu_k} \right) \cdot \Omega(\theta) d\theta \\ &= \lim_{k \rightarrow \infty} \lim_{R \rightarrow \infty} \sum_{|n| < R} t^{-1} c_n \cdot \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta e^{-|n|\mu_k} \\ &= \lim_{k \rightarrow \infty} \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k}. \end{aligned}$$

Let $S_u = \sum_{|n| < u} c_n$. Then, summing by parts,

$$(6.4) \quad \begin{aligned} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k} \\ = - \int_0^R S_u \frac{d}{du} \left(\frac{J_1(ut)}{ut} e^{-u\mu_k} \right) du + S_R \frac{J_1(Rt)}{Rt} e^{-R\mu_k}. \end{aligned}$$

Since $S_R = o(1)$ as $R \rightarrow \infty$, and using the identity $d(t^{-\nu} J_\nu(t))/dt = -t^{-\nu} J_{\nu+1}(t)$, we get

$$S_R \frac{J_1(Rt)}{Rt} e^{-R\mu_k} \rightarrow 0$$

as $R \rightarrow \infty$. Hence the last term on the right side of (6.4) drops out, and

$$\begin{aligned} \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k} \\ = - \int_0^\infty S_u \frac{d}{du} \left(\frac{J_1(ut)}{ut} e^{-u\mu_k} \right) du \\ = - \int_0^\infty S_u \left\{ \frac{J_2(ut)}{u} e^{-u\mu_k} - \mu_k \frac{J_1(ut)}{ut} e^{-u\mu_k} \right\} du. \end{aligned}$$

Returning to (6.3),

$$\begin{aligned} \varphi(t) &= - \lim_{k \rightarrow \infty} \int_0^\infty S_u \frac{J_2(ut)}{u} e^{-u\mu_k} du + \lim_{k \rightarrow \infty} \mu_k \int_0^\infty S_u \frac{J_1(ut)}{ut} e^{-u\mu_k} du \\ &= - \lim_{k \rightarrow \infty} \int_0^\infty S_u \frac{J_2(ut)}{u} e^{-u\mu_k} du. \end{aligned}$$

We claim

$$(6.5) \quad \int_\rho^{2\rho} |\varphi(t)| dt = o(\rho) \quad \text{as } \rho \rightarrow 0.$$

For,

$$\begin{aligned} \int_{\rho}^{2\rho} |\varphi(t)| dt &= \int_{\rho}^{2\rho} \left| \lim_{k \rightarrow \infty} \int_0^{\infty} S_u \frac{J_2(ut)}{u} e^{-u\mu_k} du \right| dt \\ &\leq \int_{\rho}^{2\rho} \int_0^{\infty} \left| S_u \frac{J_2(ut)}{u} \right| du dt = \int_0^{\infty} \int_{\rho}^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt du \\ &= \int_0^{1/\rho} \int_{\rho}^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt du + \int_{1/\rho}^{\infty} \int_{\rho}^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt du \\ &= P + Q. \end{aligned}$$

We use the relations $|J_{\nu}(t)| \leq ct^{\nu}$ for $0 < t \leq 2$, and $|J_{\nu}(t)| \leq c't^{-1/2}$ for $t > 1$.

In the interval of integration involving P , $|ut| \leq 2$, so $|u^{-1}J_2(ut)| \leq cut^2$.

$$P = \int_0^{1/\rho} \int_{\rho}^{2\rho} o(1)O(ut^2) dt du = o(\rho).$$

In the interval of integration for Q , $ut > 1$, so $|J_2(ut)| \leq c(ut)^{-1/2}$.

$$Q = \int_{1/\rho}^{\infty} \int_{\rho}^{2\rho} o(1)u^{-1}O(ut)^{-1/2} dt du = o(\rho).$$

Thus the claim is established.

We complete the proof of Theorem 2 as follows. Let

$$(6.6) \quad \int_{2^{-n-1}}^{2^{-n}} |\varphi(t)| dt = 2^{-n} \varepsilon_n,$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Let $E_n = \{t \in [2^{-n-1}, 2^{-n}]: |\varphi(t)| > \sqrt{\varepsilon_n}\}$. Then

$$\int_{2^{-n-1}}^{2^{-n}} |\varphi(t)| dt \geq |E_n| \sqrt{\varepsilon_n},$$

so using (6.6), $2^{-n} \varepsilon_n \geq \sqrt{\varepsilon_n} |E_n|$, and $|E_n| \leq 2^{-n} \sqrt{\varepsilon_n}$. Now let $E = T - \bigcup_{n=1}^{\infty} E_n$. Then E has 0 as a point of density. In E , $\varphi(t) \rightarrow 0$, and $\varphi(t) = 1/(2\pi t) \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) d\theta$. Thus, the theorem is established.

7. Proof of Theorem 2. Let

$$T_R(x) = \sum_{|n| < R; n_1 + n_2 = 0} \frac{1}{2} (x_1 + x_2) c_n e^{in \cdot x} + \sum_{|n| < R; n_1 + n_2 \neq 0} \frac{-ic_n}{n_1 + n_2} e^{in \cdot x}.$$

The condition (4.1) insures that $L(x) = \lim_{R \rightarrow \infty} T_R(x)$ exists a.e. on each circle $|x| = t$. This is a consequence of Theorem 1 of [2]. Moreover, by Theorem 2 of [2], $\int_0^{2\pi} \sup_R |T_R(te^{i\theta})| d\theta < \infty$, so

$$\begin{aligned}
\frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) d\theta &= \lim_{R \rightarrow \infty} \frac{1}{2\pi t} \int_0^{2\pi} T_R(te^{i\theta}) \Omega(\theta) d\theta \\
&= \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \cdot \frac{1}{2\pi t} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) d\theta \\
&= \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t}.
\end{aligned}$$

We now let

$$\varphi(t) = \frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) d\theta.$$

Summing by parts,

$$\varphi(t) = \int_0^\infty S_u \frac{J_2(ut)}{u} du.$$

The verification of the claim (6.5) and the completion of the proof follow exactly the lines of the completion of the proof of Theorem 3.

REFERENCES

1. A. Erdélyi, W. Magnus, F. Oberheittinger and F. G. Tricomi, *Higher transcendental functions*, Vol. II, McGraw-Hill, New York, 1953. MR 15, 419.
2. M. Kohn, *Spherical convergence and integrability of multiple trigonometric series on hypersurfaces*, *Studia Math.* **44** (1972), 345–354. MR 49 #9535.
3. ———, *Riemann summability of multiple trigonometric series*, *Indiana Univ. Math. J.* **24** (1975), 813–823.
4. V. L. Shapiro, *Circular summability C of double trigonometric series*, *Trans. Amer. Math. Soc.* **76** (1954), 223–233. MR 15, 866.
5. ———, *The approximate divergence operator*, *Proc. Amer. Math. Soc.* **20** (1969), 55–60. MR 38 #4918.
6. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971. MR 46 #4102.
7. A. Zygmund, *Trigonometric series*, Vol. I, Cambridge Univ. Press, London, 1968. MR 38 #4882.

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