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CLUSTER VALUES OF BOUNDED ANALYTIC FUNCTIONS(¹)

BY

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ABSTRACT. Let D be a bounded domain in the complex plane, and let ζ belong to the topological boundary ∂D of D. We prove two theorems concerning the cluster set $\mathrm{Cl}(f, \zeta)$ of a bounded analytic function f on D. The first theorem asserts that values in $\mathrm{Cl}(f, \zeta) \setminus f(Ш_\zeta)$ are assumed infinitely often in every neighborhood of ζ, with the exception of those lying in a set of zero analytic capacity. The second asserts that all values in $\mathrm{Cl}(f, \zeta) \setminus f(\mathfrak{M}_\zeta \cap \operatorname{supp} \lambda)$ are assumed infinitely often in every neighborhood of ζ, with the exception of those lying in a set of zero logarithmic capacity. Here $\mathfrak{M}_\zeta$ is the fiber of the maximal ideal space $\mathfrak{M}(D)$ of $H^\infty(D)$ lying over ζ, $Ш_\zeta$ is the Shilov boundary of the fiber algebra, and λ is the harmonic measure on $\mathfrak{M}(D)$.

1. Introduction and statement of results. *The cluster set of f at ζ*, denoted by $\mathrm{Cl}(f, \zeta)$, consists of all complex numbers w for which there is a sequence $z_n \in D$ satisfying $z_n \to \zeta$ and $f(z_n) \to w$. *The range of f at ζ*, denoted by $R(f, \zeta)$, consists of all complex numbers w for which there is a sequence $z_n \in D$ satisfying $z_n \to \zeta$ and $f(z_n) = w$. If S is a subset of ∂D, then $\mathrm{Cl}_S(f, \zeta)$ is defined to be the set of all complex numbers w for which there exist $\zeta_n \in S$ and $w_n \in \mathrm{Cl}(f, \zeta_n)$ satisfying $\zeta_n \to \zeta$ and $w_n \to w$. Evidently

$$\mathrm{Cl}_S(f, \zeta) \subset \mathrm{Cl}(f, \zeta).$$

In the case that S coincides with $(\partial D) \setminus \{\zeta\}$, the set $\mathrm{Cl}_S(f, \zeta)$ coincides with the classical boundary cluster set of f at ζ. The Iversen-Gross Theorem [11, p. 14] asserts that the boundary cluster set of f at ζ includes the topological boundary of $\mathrm{Cl}(f, \zeta)$. Furthermore, points of

$$\mathrm{Cl}(f, \zeta) \setminus \mathrm{Cl}_{(\partial D) \setminus \{\zeta\}}(f, \zeta)$$

either belong to $R(f, \zeta)$ or are asymptotic values of f at ζ (or both).

A number of cluster value theorems have appeared since the work of Iversen (1915) and Gross (1918). The main theorem of interest to us was proved by M. Tsuji in 1943 [12, Theorem VIII. 41]. It asserts that if E is a subset of ∂D of zero (outer) logarithmic capacity, and $\zeta \in E$, then the set

$$(*) \qquad \mathrm{Cl}(f, \zeta) \setminus [\mathrm{Cl}_{(\partial D) \setminus E}(f, \zeta) \cup R(f, \zeta)]$$

has zero logarithmic capacity. A related result, due to A. J. Lohwater [9],

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asserts that if D is the open unit disc Δ in the complex plane, if E is a subset of ∂D of zero (outer) length, and if $\zeta \in E$, then the set (*) again has zero logarithmic capacity. The crucial feature of these theorems is that the exceptional set E is required to have zero harmonic measure.

More recently, M. Weiss [13] has studied cluster value theory from the point of view of Banach algebras. He proves that if $\zeta \in \partial\Delta$, $Ш_\zeta$ is the "fiber" over ζ of the Shilov boundary of $H^\infty(\Delta)$ and $f \in H^\infty(\Delta)$, then

$$\mathrm{Cl}(f, \zeta) \setminus [f(Ш_\zeta) \cup R(f, \zeta)]$$

has zero logarithmic capacity. From the discussion of $H^\infty(\Delta)$ as a Banach algebra given in [8], it is clear that $f(Ш_\zeta)$ is included in $\mathrm{Cl}_{(\partial\Delta)\setminus E}(f, \zeta)$ whenever E has zero length, so that the Weiss Theorem includes the Lohwater Theorem.

Our aim is to cast the Tsuji Theorem in a Banach algebra setting, by finding an appropriate extension to arbitrary domains of the Weiss Theorem. One of the extensions (Theorem 1.3), when reinterpreted in the classical setting, will yield a slightly sharpened form (Corollary 1.4) of the Tsuji Theorem, which will be valid for bounded analytic functions.

In order to state the results, we introduce some notation. For a more detailed exposition of this circle of ideas, and for precise references, see [2] and [5].

The domain D can be regarded as an open subset of the maximal ideal space $\mathfrak{M}(D)$ of $H^\infty(D)$. We will regard the functions in $H^\infty(D)$ as being continuous functions on $\mathfrak{M}(D)$. [For our purposes, we could take $\mathfrak{M}(D)$ to be any compactification of D of which $H^\infty(D)$ separates points]. The coordinate function z extends to a map Z: $\mathfrak{M}(D) \to \bar{D}$, and Z serves to identify D with an open subset of $\mathfrak{M}(D)$. The fiber $Z^{-1}(\{\zeta\})$ over $\zeta \in \partial D$ is denoted by $\mathfrak{M}_\zeta(D) = \mathfrak{M}_\zeta$. The Cluster Value Theorem of [3] asserts that $\mathrm{Cl}(f, \zeta) = f(\mathfrak{M}_\zeta)$ for all $f \in H^\infty(D)$ and $\zeta \in \partial D$. The fiber algebra $H^\infty(D)|_{\mathfrak{M}_\zeta}$ is a closed subalgebra of $C(\mathfrak{M}_\zeta)$ whose maximal ideal space is $\mathfrak{M}_\zeta$, and whose Shilov boundary will be denoted by $Ш_\zeta$. If ζ is an essential boundary point of D, then $Ш_\zeta$ coincides with the intersection of $\mathfrak{M}_\zeta$ and the Shilov Boundary $Ш(D)$ of $H^\infty(D)$. A well-known principle of Banach algebra theory asserts that $f(Ш_\zeta)$ includes the topological boundary of $f(\mathfrak{M}_\zeta)$, so that $f(\mathfrak{M}_\zeta) \setminus f(Ш_\zeta)$ is an open subset of **C**. Our first result is the following.

1.1 THEOREM. *If $f \in H^\infty(D)$ and $\zeta \in \partial D$, then $f(\mathfrak{M}_\zeta) \setminus [f(Ш_\zeta) \cup R(f, \zeta)]$ has zero analytic capacity.*

Theorem 1.1 is a straightforward consequence of the fact that the local behavior of $\mathfrak{M}(D)$ depends only on the local configuration of D. The proof is given in §2.

Recall that the Ahlfors function G of D, depending on the point $z_0 \in D$, is the extremal function for the problem of maximizing $|f'(z_0)|$ among all $f \in H^\infty(D)$ satisfying $|f| \leqslant 1$; G is normalized so that $G'(z_0) > 0$, and then G is unique. If ζ is an essential boundary point of D, then $|G| = 1$ on $Ш_\zeta$. Furthermore, either

$$\lim_{D \ni z \to \zeta} |G(z)| = 1 \tag{1.1}$$

or

$$\mathrm{Cl}(G, \zeta) = \bar{\Delta} \quad (= \text{closed unit disc}). \tag{1.2}$$

S. Ya. Havinson [7, Theorem 28] has proved that G assumes all values in Δ, with the possible exception of a subset of Δ of zero analytic capacity. From Theorem 1.1 we conclude the following sharper version of Havinson's Theorem.

1.2 Corollary. *Let G be the Ahlfors function of D, and let ζ be an essential boundary point of D for which* (1.2) *is valid. Then values in Δ are assumed infinitely often by G in every neighborhood of ζ, with the exception of those lying in a set of zero analytic capacity.*

Corollary 1.2, and also Theorem 1.1, are sharp. To see this, let W be a domain of "type L," obtained from the open unit disc by excising the origin together with a sequence of disjoint closed subdiscs with centers on the positive real axis converging to 0. Let F be the Ahlfors function of W corresponding to a point on the negative real axis, so that F has the symmetry property

$$F(\bar{z}) = \overline{F(z)}, \qquad z \in W. \tag{1.3}$$

A straightforward application of Lindelöf's Theorem shows that F can have at most one asymptotic value at 0, and (1.3) shows that this value must be real: it is $\lim_{x \to 0-} F(x)$. By the Iverson-Gross Theorem cited earlier the range of F at 0 includes all $w \in \Delta$ with a nonzero imaginary part. On the other hand, it is known that F increases from -1 to $+1$ along the real interval connecting any two adjacent holes of W. We conclude that

$$R(F, 0) = \Delta.$$

Now let S be any relatively closed subset of Δ of zero analytic capacity, and set $D = W \setminus F^{-1}(S)$. Since $F^{-1}(S)$ has zero analytic capacity, it is totally disconnected, and D is a domain. The natural restriction $H^\infty(W) \to H^\infty(D)$ is an isometric isomorphism which induces a natural homeomorphism of $\mathfrak{M}(W)$ and $\mathfrak{M}(D)$. The Ahlfors function G of D is the restriction of F to D. It satisfies

$$G(\mathfrak{M}_0) \setminus [G(Ш_0) \cup R(G, 0)] = S.$$

In other words, any relatively closed subset of Δ of zero analytic capacity can occur as the exceptional set of Corollary 1.2.

The author does not know whether Theorem 1 or its corollary can be improved upon in the case that every boundary point of D is essential.

The statement of the second main result requires more definitions and notation.

The measure $d\theta$ on $\partial\Delta$ has a natural lift to a measure on $\mathfrak{M}(\Delta)$, which will be denoted by $d\Theta$. The Shilov boundary of $H^\infty(\Delta)$ coincides with the closed support of $d\Theta$.

Let $\pi: \Delta \to D$ denote the universal covering map. Then π extends to a continuous map from $\mathfrak{M}(\Delta)$ to $\mathfrak{M}(D)$, and this extension is also denoted by π. The measure $\lambda = \pi^*(d\Theta/2\pi)$ is called the *harmonic measure* on $\mathfrak{M}(D)$ for the point $z_0 = \pi(0)$. The class of mutual absolute continuity of λ does not depend on the specific choice of π or z_0, nor does the closed support $\operatorname{supp}\lambda$ of λ. Furthermore, $\operatorname{supp}\lambda$ includes $\text{Ш}(D)$, so that

$$f(\mathfrak{M}_\zeta)\setminus[f(\mathfrak{M}_\zeta \cap \operatorname{supp}\lambda) \cup R(f,\zeta)] \subset f(\mathfrak{M}_\zeta)\setminus[f(\text{Ш}_\zeta) \cup R(f,\zeta)].$$

Our second theorem is the following.

1.3 THEOREM. *If $f \in H^\infty(D)$ and $\zeta \in \partial D$, then the set*

$$f(\mathfrak{M}_\zeta)\setminus[f(\mathfrak{M}_\zeta \cap \operatorname{supp}\lambda) \cup R(f,\zeta)]$$

has zero logarithmic capacity.

Theorem 1.3 will be proved in §4.

In the case of the unit disc, λ coincides with $d\Theta/2\pi$, so that $\mathfrak{M}_\zeta \cap \operatorname{supp}\lambda$ coincides with Ш_ζ. Theorem 1.3 is then a direct generalization of the Weiss Theorem.

Theorem 1.3 can be reinterpreted in terms of various concrete cluster sets. As noted earlier, $f(\mathfrak{M}_\zeta)$ coincides with $\operatorname{Cl}(f,\zeta)$. To reinterpret $f(\mathfrak{M}_\zeta \cap \operatorname{supp}\lambda)$, we give some definitions which are based on [4, p. 394].

For $0 \leqslant \theta \leqslant 2\pi$, the image under the universal covering map π of the interval $\{re^{i\theta}: 0 \leqslant r < 1\}$ is called a *conformal ray* and denoted by γ_θ. Let $f \in H^\infty(D)$. If Q is a subset of ∂D, then the *essential cluster set of f along conformal rays terminating in Q*, denoted by $\operatorname{Cl}_\Gamma(f, Q)$, consists of those complex numbers w with the following property: For each $\varepsilon > 0$, there is a set of conformal rays of positive measure (with respect to the parameter θ), each of which terminates at a point of Q, and along each of which f has a limit within ε of w. Let Δ_δ denote the open disc of radius δ centered at ζ. The set

$$\bigcap_{\delta>0} \operatorname{Cl}_\Gamma(f, \Delta_\delta \cap \partial D) \tag{1.4}$$

is then a closed subset of the boundary cluster set of f at ζ. Theorem 2.3 of [4] shows that the set (1.4) coincides with $f(\mathfrak{M}_\zeta \cap \operatorname{supp} \lambda)$, that is, (1.4) is the desired "classical" reinterpretation of $f(\mathfrak{M}_\zeta \cap \operatorname{supp} \lambda)$.

Now the projection $Z^*(\lambda)$ of the measure λ onto $\bar{D}$ coincides with the harmonic measure μ on ∂D for $z_0 \in D$ (cf. [4, Lemma 2.1]). Consequently a Borel subset E of ∂D which has zero harmonic measure corresponds to a subset $Z^{-1}(E) \cap \operatorname{supp} \lambda$ which has no relative interior in $\operatorname{supp} \lambda$. This observation leads immediately to the following version of Tsuji's Theorem, which includes also the Lohwater Theorem.

1.4 COROLLARY. *Let $f \in H^\infty(D)$, let $\zeta \in \partial D$, and let E be a Borel subset of ∂D of zero harmonic measure. Then*

$$\operatorname{Cl}(f, \zeta) \setminus [\operatorname{Cl}_{(\partial D) \setminus E}(f, \zeta) \cup R(f, \zeta)]$$

has zero logarithmic capacity.

The example constructed earlier can be used to show that Theorem 1.3 is also sharp. Indeed, if the set S of the example is taken to have zero logarithmic capacity, then the harmonic measure λ on $\mathfrak{M}(D)$ coincides with the harmonic measure on $\mathfrak{M}(W)$ via the natural identification $\mathfrak{M}(D) \cong \mathfrak{M}(W)$. Furthermore, the Ahlfors function G of D is unimodular on $\operatorname{supp} \lambda$, so that

$$G(\mathfrak{M}_0) \setminus [G(\mathfrak{M}_0 \cap \operatorname{supp} \lambda) \cup R(G, 0)] = S.$$

2. **Proof of Theorem 1.1.** Since Theorem 1.1 is trivially valid when ζ is an inessential boundary point of D, we assume that ζ is an essential boundary point of D.

The inessential boundary points of D form a set of zero analytic capacity, across which all functions in $H^\infty(D)$ extend analytically. By adjoining this set to D, we increase $R(f, \zeta)$ by at most a set of zero analytic capacity. Consequently we can assume that every boundary point of D is essential.

For $\delta > 0$, let Δ_δ denote the open disc centered at ζ with radius δ. Then

$$R(f, \zeta) = \bigcap_{\delta > 0} f(D \cap \Delta_\delta). \tag{2.1}$$

Now suppose that $f(\mathfrak{M}_\zeta) \setminus [f(Ш_\zeta) \cup R(f, \zeta)]$ has positive analytic capacity. From (2.1) it follows that $f(\mathfrak{M}_\zeta) \setminus [f(Ш_\zeta) \cup f(D \cap \Delta_\delta)]$ has positive analytic capacity for some $\delta > 0$. There is then a compact subset E of $f(\mathfrak{M}_\zeta)$ such that

$$E \text{ has positive analytic capacity,} \tag{2.2}$$

$$E \text{ is at a positive distance from } f(Ш_\zeta), \text{ and} \tag{2.3}$$

$$E \text{ does not meet } f(D \cap \Delta_\delta). \tag{2.4}$$

The closure of $f(D \cap \Delta_\delta)$ includes $\operatorname{Cl}(f, \zeta) = f(\mathfrak{M}_\zeta)$. Hence (2.4) shows that E is nowhere dense in $f(\mathfrak{M}_\zeta)$. Since $f(Ш_\zeta)$ includes the topological boundary of $f(\mathfrak{M}_\zeta)$, the set $f(\mathfrak{M}_\zeta) \setminus f(Ш_\zeta)$ is an open subset of the complex plane **C**, and hence

$$E \text{ is nowhere dense in } \mathbf{C}. \tag{2.5}$$

On account of (2.2) and (2.3) there is a bounded analytic function g on $\mathbf{C} \setminus E$ which satisfies

$$|g(z)| < 1, \qquad z \in \mathbf{C} \setminus E, \tag{2.6}$$

$$\limsup_{z \to E} |g(z)| = 1, \tag{2.7}$$

$$|g(z)| < 1/4, \qquad z \in f(Ш_\zeta). \tag{2.8}$$

On account of (2.4), the function $g \circ f$ is defined and analytic on $D \cap \Delta_\delta$, and satisfies $|g \circ f| < 1$ there.

Now choose a sequence $z_n \in \mathbf{C} \setminus E$ such that $|g(z_n)| \to 1$. Then $\{z_n\}$ accumulates on E, so that eventually $z_n \in f(\mathfrak{M}_\zeta) = \mathrm{Cl}(f, \zeta)$. Consequently there are $\zeta_{nm} \in D \cap \Delta_\delta$ such that $\zeta_{nm} \to \zeta$ as $m \to \infty$, while $f(\zeta_{nm}) \to z_n$. Hence $(g \circ f)(\zeta_{nm}) \to f(z_n)$ as $m \to \infty$. Letting $n \to \infty$, we conclude that

$$\limsup_{D \cap \Delta_\delta \ni z \to \zeta} |(g \circ f)(z)| = 1. \tag{2.9}$$

By [3, Lemma 1.1], there exist $F \in H^\infty(D)$ and $h \in H^\infty(D \cap \Delta_\delta)$ such that h is analytic at ζ, $h(\zeta) = 0$, and $g \circ h = F + h$. From (2.9) we obtain

$$\limsup_{D \ni z \to \zeta} |F(z)| = 1. \tag{2.10}$$

Let $\varphi \in Ш_\zeta$. Then there is a net $\{z_\alpha\}$ in D which converges to φ. In the topology of **C**, z_α converges to ζ, so that $F(z_\alpha) - g(f(z_\alpha)) \to 0$, and $F(\varphi) = g(f(\varphi))$. From (2.8) we obtain $|F(\varphi)| \leq \frac{1}{4}$, this for all $\varphi \in Ш_\zeta$. Since $Ш_\zeta$ is the Shilov boundary of the fiber algebra, $|F| \leq \frac{1}{4}$ on $\mathfrak{M}_\zeta$. This contradicts (2.10). The theorem is established.

3. The space of bounded harmonic functions on D. For the purposes of proving Theorem 1.3, it will be convenient to replace $\mathfrak{M}(D)$ by an appropriate compactification $\mathfrak{Q}(D)$ of D, and to redefine λ as a measure on $\mathfrak{Q}(D)$.

The space of complex-valued bounded harmonic functions on D will be denoted by BH(D). The smallest compactification of D to which all the functions in BH(D) extend continuously will be denoted by $\mathfrak{Q}(D)$. Then $\mathfrak{Q}(D)$ can be obtained from the Stone-Čech compactification of D by identifying pairs of points identified by BH(D).

In this section we will establish a "localization" result, Theorem 3.6, for $\mathfrak{Q}(D)$. Most of the material preliminary to this result is well known. For a detailed treatment of various compactifications of Riemann surfaces, see [1].

The closure of D in $\mathfrak{M}(D)$ is obtained from $\mathfrak{Q}(D)$ by identifying pairs of points which are identified by $H^\infty(D)$. In the case of the open unit disc Δ,

$\mathfrak{Q}(\Delta)$ coincides with $\mathfrak{M}(\Delta)$. Indeed Carleson's Corona Theorem asserts that Δ is dense in $\mathfrak{M}(\Delta)$. Since every real-valued function $u \in \mathrm{BH}(\Delta)$ is of the form $u = \log|f|$ for some $f \in H^\infty(D)$, the functions in $H^\infty(\Delta)$ already separate the points of $\mathfrak{Q}(\Delta)$, and hence $\mathfrak{Q}(\Delta) = \mathfrak{M}(\Delta)$.

If h is an analytic map from a domain D' to D, then h extends to a continuous map from $\mathfrak{Q}(D')$ to $\mathfrak{Q}(D)$. In particular, the universal covering map $\pi: \Delta \to D$ extends to a continuous map,

$$\pi: \mathfrak{M}(\Delta) \to \mathfrak{Q}(D).$$

For $w \in \Delta$, let m_w be the lift to $\mathfrak{M}(\Delta)$ of the usual Poisson representing measure for w. If $z \in D$ satisfies $\pi(z) = w$, then $\lambda_z = \pi^*(m_w)$ is the *harmonic measure on* $\mathfrak{Q}(D)$ *for* z. It is easy to check that the measure λ_z does not depend on the choice of $z \in \pi^{-1}(w)$. Furthermore,

$$u(z) = \int u\,d\lambda_z, \qquad z \in D,\ u \in \mathrm{BH}(D).$$

Since the m_w are all mutually absolutely continuous with respect to $d\Theta$, the λ_z are all mutually absolutely continuous. When we are concerned only with the class of mutual absolute continuity of λ_z, we will abbreviate λ_z to λ.

3.1 Lemma. *The correspondence*

$$u \to \tilde{u}, \qquad \tilde{u}(z) = \int u\,d\lambda_z,$$

determines an isometric isomorphism of $L^\infty(\lambda)$ *and* $\mathrm{BH}(D)$. *Consequently*

$$L^\infty(\lambda) \cong C(\operatorname{supp}\lambda) \cong \mathrm{BH}(D).$$

Furthermore, the closed support $\operatorname{supp}\lambda$ *of* λ *is homeomorphic to the maximal ideal space* $\Sigma(\lambda)$ *of* $L^\infty(\lambda)$.

Proof. Every function in $\mathrm{BH}(D)$ is the Poisson integral of a continuous function on $\operatorname{supp}\lambda$ with the same norm. On the other hand, if $u \in L^\infty(\lambda)$ is arbitrary, then $\tilde{u} \in \mathrm{BH}(D)$, so that u and (the extension of) $\tilde{u}$ have the same Poisson integrals. It suffices then to show that if $u \in L^\infty(\lambda)$ satisfies $\tilde{u} = 0$, then $u = 0$ a.e. $(d\lambda)$.

Suppose $u \in L^\infty(\lambda)$ satisfies $\tilde{u} = 0$. Then $u \circ \pi \in L^\infty(d\Theta)$ satisfies

$$\int (u \circ \pi)\,dm_w = \int u\,d\lambda_{\pi(w)} = 0, \qquad w \in \Delta.$$

By a classical result [8], $u \circ \pi = 0$ a.e. $(d\Theta)$. Hence $u = 0$ a.e. $(d\lambda)$. Q.E.D.

From Lemma 3.1 it follows that $\operatorname{supp}\lambda$ is the Choquet boundary of $\mathrm{BH}(D)$. Furthermore λ is a normal measure on $\operatorname{supp}\lambda$. In fact, λ is characterized, up to mutual absolute continuity, as the normal measure on the Choquet boundary of $\mathrm{BH}(D)$ whose closed support coincides with the Choquet boundary.

Roughly the same state of affairs holds if D is any bounded open set. If $D_1, D_2, \ldots$ are the constituent components of D, then $\mathfrak{Q}(D_j)$ can be regarded as a clopen subset of $\mathfrak{Q}(D)$. If λ_j is the harmonic measure on $\mathfrak{Q}(D_j)$, then the measure $\lambda = \Sigma\lambda_j/2^j$ can be referred to as the harmonic measure on $\mathfrak{Q}(D)$. Again there are isometric isomorphisms

$$C(\operatorname{supp}\lambda) \cong L^\infty(\lambda) \cong \mathrm{BH}(D),$$

and a homeomorphism $\operatorname{supp}\lambda \cong \Sigma(d\lambda)$.

Redefine Z to be the extension of the coordinate function z to $\mathfrak{Q}(D)$, so that Z maps $\mathfrak{Q}(D)$ onto $\bar{D}$. As noted earlier, $Z^*(\lambda_z)$ coincides with the harmonic measure μ_z on ∂D for $z \in D$. Since the set R of regular boundary points of D has full harmonic measure, the set $Z^{-1}(R) \subset \mathfrak{Q}(D)$ has full λ-measure. In particular, we obtain the following.

3.2 LEMMA. *Let R be the set of regular boundary points of D. Then* $Z^{-1}(R) \cap \operatorname{supp}\lambda$ *is dense in* $\operatorname{supp}\lambda$.

Let $\zeta \in \partial D$. The fiber $\mathfrak{Q}_\zeta(D)$, or $\mathfrak{Q}_\zeta$, is defined to be the set of all $\varphi \in \mathfrak{Q}(D)$ such that $Z(\varphi) = \zeta$:

$$\mathfrak{Q}_\zeta = Z^{-1}(\{\zeta\}) \subset \mathfrak{Q}(D).$$

3.3 LEMMA. *Let ζ be a regular boundary point of D. Let $u \in$ BH(D), and let p be a strictly positive continuous function on $\mathfrak{Q}(D)$ such that $|u| \leq p$ on $\mathfrak{Q}_\zeta$. Then there is $v \in$ BH(D) such that $|v| \leq p$ on $\mathfrak{Q}(D)$, while $v = u$ on $\mathfrak{Q}_\zeta$.*

PROOF. By Lemma 6.1 of [6] (which stems from a classical construction of Keldysh and Bishop), it suffices to show that there is a sequence $\{u_n\}$ in BH(D) and $M > 0$ such that $|u_n| \leq M$ for all n, $u_n = u$ on $\mathfrak{Q}_\zeta$, and $\{u_n\}$ converges uniformly to zero on each subset of D at a positive distance from ζ.

Define $u_n \in$ BH(D) by

$$u_n(z) = \int_{Z^{-1}(\Delta_\delta)} u\,d\lambda_z, \qquad z \in D,$$

where $\delta = 1/n$, and Δ_δ is the open disc of radius δ centered at ζ. The estimate $|u_n| \leq \|u\|$ is immediate.

Since ζ is regular, the harmonic measures μ_z on ∂D for z cluster at the point mass at ζ as $z \in D$ tends to ζ. Since $Z^*(\lambda_z) = \mu_z$, the measures λ_z cluster towards measures on the fiber $\mathfrak{Q}_\zeta$ as $z \in D$ tends to ζ. Consequently

$$u_n(z) - u(z) = \int_{Z^{-1}(\bar{D}\setminus\Delta_\delta)} u\,d\lambda_z$$

tends to zero as $z \in D$ approaches ζ. Hence $u_n = u$ on $\mathfrak{Q}_\zeta$.

An elementary estimate on harmonic measure shows that $\mu_z(\Delta_\delta)$ tends to zero uniformly on each subset of D at a positive distance from ζ. Consequently $\{u_n\}$ tends to zero uniformly on each such set. Q.E.D.

3.4 COROLLARY. *If ζ is a regular boundary point of D, then the restriction* *space* $\mathrm{BH}(D)|_{\mathcal{Q}_\zeta}$ *is a closed subspace of* $C(\mathcal{Q}_\zeta)$ *whose Choquet boundary is* $\mathcal{Q}_\zeta \cap \operatorname{supp} \lambda$.

The next lemma shows that the fiber $\mathcal{Q}_\zeta$ depends only on the local configuration of D near ζ.

3.5 LEMMA. *Let* $\zeta \in \partial D$, *let* U *be an open neighborhood of* ζ, *and let* $u \in \mathrm{BH}(D \cap U)$. *Then there exists* $v \in \mathrm{BH}(D)$ *such that* $v - u$ *extends harmonically to a neighborhood of* ζ.

PROOF. Let g be a smooth function supported on a compact subset of U, such that $g = 1$ near ζ. Declare u to be zero off $D \cap U$, and define

$$v(z) = u(z)g(z) - \frac{1}{2\pi} \iint u(w)(\Delta g)(w) \log \frac{1}{|z - w|}\, ds\, dt$$
$$+ \frac{1}{\pi} \iint u(w) \left[\frac{\partial g}{\partial x}(w) \frac{s - x}{|w - z|^2} + \frac{\partial g}{\partial y}(w) \frac{t - y}{|w - z|^2} \right] ds\, dt,$$

where $w = s + it$. Then v satisfies the differential equation $\Delta v = g \Delta u$ in the sense of distributions. It is easy to check (cf. [10]) that v has the desired properties. Q.E.D.

3.6 THEOREM. *Let* U *be an open subset of* $\mathbf{C}$. *Then the inclusion* $D \cap U \to D$ *induces a homeomorphism*

$$\mathcal{Q}_\zeta(D \cap U) \cong \mathcal{Q}_\zeta(D), \quad \text{all } \zeta \in \partial D \cap U. \tag{3.1}$$

Furthermore, the natural map

$$\mathcal{Q}(D \cap U) \cap Z^{-1}(U \cap \partial D) \to \mathcal{Q}(D) \cap Z^{-1}(U \cap \partial D) \tag{3.2}$$

is a homeomorphism. The restriction to $Z^{-1}(U \cap \partial D)$ *of the harmonic measure on* $\mathcal{Q}(D \cap U)$ *corresponds to a measure which is mutually absolutely continuous with the restriction to* $Z^{-1}(U \cap \partial D)$ *of the harmonic measure on* $\mathcal{Q}(D)$.

PROOF. The inclusion $D \cap U \to D$ induces a continuous map $\mathcal{Q}_\zeta(D \cap U) \to \mathcal{Q}_\zeta(D)$, which identifies points of $\mathcal{Q}_\zeta(D \cap U)$ which are identified by $\mathrm{BH}(D)$. By Lemma 3.5, no such identification occurs, so the fibers are homeomorphic. The map given by (3.2) is then a homeomorphism.

The homeomorphism of fibers induces an isomorphism

$$\mathrm{BH}(D \cap U)|_{\mathcal{Q}_\zeta(D \cap U)} \cong \mathrm{BH}(D)|_{\mathcal{Q}_\zeta(D)}.$$

In particular, the Choquet boundaries of these restriction spaces correspond to each other under the fiber homeomorphism.

It will be convenient henceforth to identify $\mathcal{Q}_\zeta(D \cap U)$ and $\mathcal{Q}_\zeta(D)$ via (3.1), for $\zeta \in U \cap \partial D$.

Since the Wiener criterion is local, the point $\zeta \in U \cap \partial D$ is a regular boundary point of D if and only if it is a regular boundary point of $D \cap U$.

In this case, Lemma 3.4 (which applies, even if $D \cap U$ is not connected) shows that the supports for the harmonic measures on $\mathfrak{Q}(D \cap U)$ and $\mathfrak{Q}(D)$ meet the fiber over ζ in the same set. Lemma 3.2 then shows that the supports of the harmonic measures meet $Z^{-1}(D \cap U)$ in the same set. Since both measures are normal measures on extremely disconnected spaces, their restrictions to $Z^{-1}(D \cap U)$ must be mutually absolutely continuous. Q.E.D.

Note again that the hypothesis that D be connected is irrelevant, providing harmonic measure is defined as indicated earlier.

4. **Proof of Theorem 1.3.** Since $\mathfrak{Q}_\zeta$ is obtained from the subset of $\mathfrak{M}_\zeta$ adherent to D by identifying those pairs of points which are identified by $H^\infty(D)$, and since the harmonic measure on $\mathfrak{M}(D)$ is collapsed to the harmonic measure on $\mathfrak{Q}(D)$ under this identification, it suffices to prove that

$$f(\mathfrak{Q}_\zeta) \setminus [f(\mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda) \cup R(f, \zeta)]$$

has zero logarithmic capacity whenever $f \in H^\infty(D)$.

Suppose, on the contrary, that this statement fails for certain f and ζ. Then there exist a disc Δ_δ centered at ζ with radius δ and a compact set

$$E \subset f(\mathfrak{Q}_\zeta) \setminus f(\mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda),$$

such that E has positive logarithmic capacity, while

$$E \cap f(\Delta_\delta \cap D) = \varnothing. \tag{4.1}$$

Let u be a real-valued harmonic function on $\mathbf{C} \setminus E$ such that

$$u < 0 \quad \text{on } \mathbf{C} \setminus E, \qquad \limsup_{z \to E} u(z) = 0.$$

On account of (4.1) the function $v = u \circ f$ is well defined and harmonic on $D \cap \Delta_\delta$.

Choose $w_n \in \mathbf{C} \setminus E$ such that $u(w_n) \to 0$. Now E is a compact subset of the interior $\mathrm{Cl}(f, \zeta)$. Consequently for n large, there is z_n near ζ such that $f(z_n)$ is near w_n. In this manner we obtain a sequence $\{z_n\}$ in D such that $z_n \to \zeta$ and $u(f(z_n)) \to 0$. In other words,

$$\limsup_{D \ni z \to \zeta} v(z) = 0. \tag{4.2}$$

Let $\varphi \in \mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda$. Suppose $\{z_\alpha\}$ is a net in $D \cap \Delta_\delta$ which converges in the topology of $\mathfrak{Q}(D \cap \Delta_\delta)$ to φ. Setting

$$a = \sup\{u(z) \colon z \in f(\mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda)\} < 0,$$

we obtain

$$v(\varphi) = \lim u(f(z_\alpha)) = u(f(\varphi)) \leqslant a.$$

Consequently

$$v \leqslant a < 0 \quad \text{on } \mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda. \tag{4.3}$$

Note that $\mathfrak{Q}_\zeta \cap \operatorname{supp} \lambda$ refers here both to a subset of $\mathfrak{Q}_\zeta(D)$ and a subset of $\mathfrak{Q}_\zeta(D \cap \Delta_\delta)$. This is permitted, on account of the identification furnished by Theorem 3.6.

Suppose ζ is a regular boundary point of D. From (4.3) and Lemma 3.4 we conclude that $v \leqslant a < 0$ on $\mathfrak{Q}_\zeta$. This contradicts (4.2), and the theorem is established for regular boundary points.

Suppose that ζ is an irregular boundary point of D. In this case, $\{\zeta\}$ is a connected component of ∂D; in fact, Beurling's condition for regular boundary points [12] shows that for arbitrarily small values of δ, the boundary $\partial\Delta_\delta$ of Δ_δ is contained in D. By shrinking E and choosing a small, appropriate $\delta > 0$, we can make the following further assumptions:

$$E \cap f(\partial\Delta_\delta) = \varnothing, \tag{4.4}$$

$$E \cap f(\mathfrak{Q}_\xi \cap \operatorname{supp} \lambda) = \varnothing \quad \text{for all } \xi \in \Delta_\delta \cap \partial D. \tag{4.5}$$

Now $v = u \circ f$ is harmonic on $\overline{\Delta}_\delta \cap D$. As before, (4.5) shows that

$$\sup\{v(\varphi) \colon \varphi \in Z^{-1}(\Delta_\delta) \cap \operatorname{supp} \lambda\} < 0,$$

while from (4.4) we obtain

$$\sup\{v(z) \colon z \in \partial\Delta_\delta\} < 0.$$

Consequently there is a constant $b < 0$ such that $v \leqslant b$ on the closed support of the harmonic measure for $\mathfrak{Q}(D \cap \Delta_\delta)$. It follows that $v \leqslant b < 0$ on $D \cap \Delta_\delta$. This contradicts (4.2), so that the theorem is also established for irregular boundary points. Q.E.D.

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