COMPLETELY UNSTABLE FLOWS ON 2-MANIFOLDS

BY

DEAN A. NEUMANN

ABSTRACT. Completely unstable flows on 2-manifolds are classified under both topological and C^r -equivalence ($1 < r < \infty$), in terms of the corresponding orbit spaces.

1. Statement of results. Let $\phi: M \times \mathbf{R}^1 \rightarrow M$ be a continuous flow on the differentiable manifold M . A point $x \in M$ is *nonwandering* if $x \in J^+(x)$; here $J^+(x)$ denotes the set of limits of sequences $\{\phi(x_n, t_n)\}$, where $\{x_n\}$ converges to x and $\{t_n\}$ tends to infinity. We say that ϕ is *completely unstable* if there are *no* nonwandering points of ϕ .

If $\psi: N \times \mathbf{R}^1 \rightarrow N$ is another continuous flow, we say that ϕ and ψ are (*topologically*) *equivalent* if there is a homeomorphism h of M onto N that takes orbits of ϕ onto orbits of ψ , preserving sense. If ϕ and ψ are C^r ($1 < r < \infty$), they are *C^r -equivalent* if there is such an h that is a C^r -diffeomorphism.

We are concerned with flows (M, ϕ) in which M is an arbitrary 2-manifold (separable metric and without boundary), with given C^∞ structure, and ϕ is completely unstable. Our results constitute a classification of such flows, under both topological and C^r -equivalence, in terms of the associated orbit spaces. Let M/ϕ denote the space of orbits of ϕ with the quotient topology (the finest topology in which the projection $\pi: M \rightarrow M/\phi$ is continuous). It is known that M/ϕ has a countable basis of open sets homeomorphic with $\mathbf{R}^1$ (see [5]); but M/ϕ need not be Hausdorff. We refer to a space with these properties of M/ϕ as a *nonseparated 1-manifold*. In general, there are many topological types (M, ϕ) with a given orbit space; to obtain a classification we must impose additional structure. In §3 we define an “order” relation on certain points and subsets of M/ϕ —essentially: $p < q$ in M/ϕ iff $\pi^{-1}(q) \subseteq J^+(\pi^{-1}(p))$ in M . The resulting ordered orbit space, still denoted M/ϕ , completely determines both M and ϕ :

THEOREM 1. *If ϕ, ϕ' are completely unstable continuous flows on 2-manifolds M, M' respectively, then ϕ and ϕ' are topologically equivalent if and only if M/ϕ and M'/ϕ' are order isomorphic.*

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Essentially this result, for the case $M, M' = \mathbb{R}^2$, is stated in [2]. Note that any noncritical flow on the plane is completely unstable. An earlier classification of noncritical flows in the plane appears in [4]. Theorem 1 is proved in §3 below, using a preliminary result derived in §2.

If M has a given C^r -structure, and ϕ is C^r , then there is a (unique) C^r -structure on M/ϕ with respect to which π is C^r (see [5]). M/ϕ , with this C^r -structure and the order indicated above, completely determines the C^r -equivalence class of (M, ϕ) :

THEOREM 2. *Suppose ϕ, ϕ' are completely unstable C^r -flows on 2-manifolds M, M' respectively. Then ϕ and ϕ' are C^r -equivalent if and only if there is an order preserving C^r -diffeomorphism of M/ϕ onto M'/ϕ' .*

Comparing Theorems 1 and 2, we may obtain some information on the relation between topological and C^r -equivalence classes for completely unstable flows. We remark first that, if ϕ is any completely unstable continuous flow on a 2-manifold M , then ϕ is topologically equivalent to a C^∞ -flow on M (this is proved in [7]); i.e., any topological equivalence class contains nonempty C^r -equivalence classes, for $1 \leq r \leq \infty$. We show further that any C^r -structure on M/ϕ corresponds to a C^r -flow on M , which is topologically equivalent to ϕ . Then Theorem 2 may be given the following form:

THEOREM 3. *Suppose ϕ is a completely unstable continuous flow on the 2-manifold M and fix $r \in \{1, \dots, \infty\}$. Then the distinct C^r -equivalence classes contained in the topological equivalence class of ϕ are in 1-1 correspondence with the distinct C^r -structures on M/ϕ .*

On nonseparated 1-manifolds in a fairly large class, it is easy to construct a continuum of pairwise distinct C^r -structures. Thus we may apply Theorem 3 to prove, for example,

COROLLARY. *Suppose ϕ is a completely unstable continuous flow on the 2-manifold M and that ϕ has at most countably many separatrices but is not parallel. Then the topological equivalence class of (M, ϕ) contains a continuum of distinct C^r -equivalence classes, for any $r \in \{1, 2, \dots, \infty\}$.*

Here *separatrix* may be defined as follows (cf. [5]): for $p \in M/\phi$, the orbit $\pi^{-1}(p)$ is a *separatrix* if there is a point $q \in M/\phi$ such that p and q cannot be separated with disjoint open sets in M/ϕ . A completely unstable flow is *parallel* if it has no separatrices.

The proofs of Theorems 2 and 3 and the corollary are given in §4.

2. Subdivision of noncritical flows on 2-manifolds. Let ϕ be a continuous flow without critical points on the 2-manifold M .

DEFINITIONS. A *parallel neighborhood* (or *p-neighborhood*) of a point $m \in M$

is a closed 2-cell $N \subseteq M$, that is a neighborhood of m and that is homeomorphic with the rectangle

$$R = \{(x, t) \in \mathbb{R}^2 \mid |x|, |t| \leq 1\}$$

under a map that takes each orbit segment in N onto a vertical segment in R . The boundary ∂N of N then consists of two orbit segments of ϕ —the *edges* of N , and two arcs that are local sections of ϕ —the *ends* of N . It is known that every point of M has a p -neighborhood (cf. [1, Chapter IV, §2] and [3, Theorem 1]).

By a *subdivision* of (M, ϕ) we mean a cover of M by a locally finite collection $\{N_i\}$ of p -neighborhoods which intersect *properly*: viz., for each $i \neq j$, $N_i \cap N_j$ is either empty, a subarc of an edge of each or a subarc of an end of each.

LEMMA 1. Any noncritical flow (M, ϕ) admits a subdivision.

PROOF. There is a locally finite covering $\{N_i\}$ of M by p -neighborhoods. Note that we may assume that each N_i is contained in a slightly “longer” p -neighborhood $\tilde{N}_i$, corresponding to say $\tilde{R} = \{(x, t) \in \mathbb{R}^2 \mid |x| < 1, |t| < 1 + \varepsilon\}$ (that is, $(\tilde{N}_i, N_i)$ is fiberwise homeomorphic with $(\tilde{R}, R)$).

Let εN_i denote the union of the two ends of N_i . We first show that we may adjust the N_i so that no three of the εN_i intersect in a single point and so that $\varepsilon N_i \cap \varepsilon N_j$ is at most finite for each $i \neq j$. For, suppose that we have adjusted $N_1, \dots, N_{n-1}$ so that this is true for $i < j \leq n-1$. We may assume that no point of $\varepsilon N_i \cap \varepsilon N_j$ ($i < j \leq n-1$) lies on εN_n . Cover the ends of N_n with new p -neighborhoods $T_1, \dots, T_k$ which also miss $\varepsilon N_i \cap \varepsilon N_j$ ($i < j \leq n-1$). The interior of each εN_i ($i \leq n-1$) meets $\tilde{T}_1$ in at most countably many open arcs, only finitely many of which meet a “shorter” p -neighborhood $S_1 \subseteq T_1$ (see Figure 1). These finitely many, for each $i \leq n-1$, may be

FIGURE 1

replaced by new arcs with the same endpoints, that are again local sections of ϕ , but that now meet $\varepsilon N_n \cap T_1$ in at most finitely many points. We may do this, for example, with an isotopy of M that is the identity off T_1 , and that takes these arcs onto piecewise linear ones (in the piecewise linear structure

induced by some parametrization of T_1) in general position with respect to $\varepsilon N_n \cap T_1$. Note that then no new intersections $\varepsilon N_i \cap \varepsilon N_j$ ($i < j \leq n-1$) are introduced. Repeat with the adjusted arcs in $T_2, \dots, T_k$ in succession. Our assertion now follows by induction; note that, because of the local finiteness, we can insure that no εN_i is adjusted more than finitely many times.

We next observe that by a slight modification of this argument we may assume that we have chosen $\tilde{N}_i \supseteq N_i$ (as above) so that no three of $\varepsilon \tilde{N}_i, \varepsilon \tilde{N}_j$ intersect and so that, for $i \neq j$, the various intersections $\varepsilon N_i \cap \varepsilon N_j, \varepsilon \tilde{N}_i \cap \varepsilon \tilde{N}_j, \varepsilon N_i \cap \varepsilon \tilde{N}_j$ are finite.

Finally, we obtain the desired subdivision by partitioning the N_i . Suppose we have determined p -neighborhoods $K_1, \dots, K_{m(n-1)}$ satisfying:

- (a) $K_1, \dots, K_{m(n-1)}$ intersect properly;
- (b) $N_1 \cup \dots \cup N_{n-1} \subseteq K_1 \cup \dots \cup K_{m(n-1)}$;
- (c) $\varepsilon K_i \cap \varepsilon N_j$ and $\varepsilon K_i \cap \varepsilon \tilde{N}_j$ are finite ($i \leq m(n-1), j \geq n$).

Let A denote the set of orbit segments in $\tilde{N}_n$ which pass through a point of $\varepsilon K_i \cap \varepsilon \tilde{N}_n$ or $\varepsilon K_i \cap \varepsilon N_n$ for some $i \leq m(n-1)$, or meet an edge of some K_j ($j \leq m(n-1)$) that lies in $\tilde{N}_n$ (cf. Figure 2). The arcs of A partition $\tilde{N}_n$ into

FIGURE 2

p -neighborhoods, which are further divided by the sections $\varepsilon K_1, \dots, \varepsilon K_{m(n-1)}, \varepsilon \tilde{N}_n$ (or εN_n) into the required p -neighborhoods $K_{m(n-1)+1}, \dots, K_{m(n)}$. It is not difficult to check that this may be done so that (a)–(c) are preserved. Hence our assertion follows by induction.

3. Topological equivalence of completely unstable flows. Let ϕ be a completely unstable flow on the 2-manifold M with orbit space M/ϕ .

DEFINITION. We define an order relation on certain points and subsets of M/ϕ as follows:

(1) Let $p, q \in M/\phi, p \neq q$. Define $p < q$ iff, for $x \in \pi^{-1}(p)$ and $y \in \pi^{-1}(q)$, we have $y \in J^+(x)$.

(2) Suppose $p, q \in M/\phi$ with both $p < q$ and $q < p$, and that there are

half-open intervals $I, J \subseteq M/\phi$ terminating at p, q respectively, with $I - \{p\} = J - \{q\}$. Choose disjoint sections $S, T \subseteq M$, lying over I, J respectively. Define $I < J$ iff orbits of ϕ meet S before T .

Throughout this section M/ϕ will denote the orbit space endowed with this order relation.

REMARKS. Note that two points of M/ϕ are related in this ordering exactly when they cannot be separated with disjoint open sets in M/ϕ . However there is very little restriction on the size or complexity of this set of *nonseparated* points. An example of Ważewski [8] shows that, even with $M = \mathbb{R}^2$ and $\phi \in C^\infty$, it may be all of M/ϕ .

In many cases, the orbit space with the order defined in (1) above is sufficient to determine the topological type of (M, ϕ) . It can be proved, for example, that two noncritical flows on the plane are topologically equivalent iff the corresponding orbit spaces are homeomorphic by a homeomorphism which preserves the order defined by (1). The following example shows that this is not true in the generality of Theorem 1.

EXAMPLE. Let $M = \mathbb{R}^2 - \{0\}$. Let $A, B \subseteq (0, \infty)$ be closed sets with $A \cup B = (0, \infty)$, and define $A' = \{(0, y) | y \in A\} \subseteq M$ and $B' = \{(0, y) | -y \in B\}$. Let $f: M \rightarrow [0, 1]$ be a C^∞ function that is 0 exactly on $A' \cup B'$, and let ϕ be the flow on M defined (in polar coordinates) by $\dot{r} = 0, \dot{\theta} = f(r, \theta)$. Now, let ϕ_1 be the flow obtained applying this construction with

$$A_1 = [1, \infty) \cup (0, \tfrac{1}{2}] \cup \bigcup_{n=1}^{\infty} \left[\frac{2n}{2n+1}, \frac{2n+1}{2n+2} \right]$$

and

$$B_1 = \{1\} \cup \bigcup_{n=1}^{\infty} \left[\frac{2n-1}{2n}, \frac{2n}{2n+1} \right]$$

(see Figure 3). Let ϕ_2 be the flow defined choosing

$$A_2 = \{1\} \cup (0, \tfrac{1}{2}] \cup \bigcup_{n=1}^{\infty} \left[\frac{2n}{2n+1}, \frac{2n+1}{2n+2} \right]$$

and

$$B_2 = [1, \infty) \cup \bigcup_{n=1}^{\infty} \left[\frac{2n-1}{2n}, \frac{2n}{2n+1} \right].$$

In either case the orbit space may be described as follows: it consists of two copies of $(0, \infty)$ (corresponding to the sections S_i, T_i of Figure 3) in which each pair of points corresponding to the same coordinate $x \in (0, \infty)$ is identified, except when $x = n/(n+1)$ ($n \in \mathbb{Z}^+$) or $x = 1$. Also, each pair

$\{p, q\}$ of nonseparated points (in either quotient space), according to (1) of the definition, satisfies both $p < q$ and $q < p$. However ϕ_1 and ϕ_2 are not topologically equivalent; in fact, it is easy to check that M/ϕ_1 and M/ϕ_2 are distinguished by the ordering defined in (2).

FIGURE 3

The main object of the present section is to prove that the ordered orbit space M/ϕ completely determines both M and ϕ . Thus let (M', ϕ') be another completely unstable flow and suppose that we are given an isomorphism (order preserving homeomorphism)

$$h: M'/\phi' \rightarrow M/\phi.$$

We will construct a topological equivalence k of ϕ' with ϕ that satisfies

$$h\pi' = \pi k.$$

Let $\{N'_i\}$ be a subdivision of (M', ϕ') . We will construct an isomorphic subdivision $\{N_i\}$ of (M, ϕ) , and use it to define k .

Let S'_i denote the "initial" end of N'_i and let

$$T'_i = \pi'(S'_i).$$

Note that we may assume no orbit of ϕ' crosses any S'_i more than once, so we may define $\alpha'_i = T'_i \rightarrow S'_i$ by

$$\pi' \circ \alpha'_i = 1_{T'_i}.$$

For $i \neq j$ let $A'_{ij} = T'_i \cap T'_j$ and define $f'_{ij}: A'_{ij} \rightarrow \mathbb{R}^1$ by

$$f'_{ij}(x) = t \quad \text{iff} \quad \alpha'_i(x) \cdot t = \alpha'_j(x);$$

thus f'_{ij} is the time along the orbit $\pi'^{-1}(x)$ from its intersection with S'_i to its intersection with S'_j . Each f'_{ij} is continuous on its domain and can be zero only at a common endpoint of T'_i and T'_j (i.e., exactly when S'_i and S'_j have a

common endpoint). However, as the examples above show, $\text{sgn}(f'_{ij})$ need not be constant on A'_{ij} .

In order to construct $\{N_i\}$ we wish to construct sections S_i of ϕ over $T_i = h(T'_i)$ which are related by ϕ in the same way the S'_i are related by ϕ' ; viz., for $i > j$ and $x \in A'_{ij}$, we want $\pi^{-1}(hx)$ to meet S_i before S_j if and only if $\pi'^{-1}(x)$ meets S'_i before S'_j . Equivalently, we require

$$\text{sgn } f_{ij}(hx) = \text{sgn } f'_{ij}(x) \quad (x \in A'_{ij}).$$

Suppose we have obtained such sections $\alpha_j: T_j \rightarrow S_j \subseteq M$ for $j \leq n-1$, so that

$$(a) \quad \text{sgn } f_{ij}(x) = \text{sgn } f'_{ij}(h^{-1}x) \quad (x \in A_{ij}, j < i \leq n-1).$$

Let $\alpha_n: t_n \rightarrow M$ denote any continuous section with $\pi\alpha_n = 1$ and let $S_n = \alpha_n(T_n)$. We do not assume that S_n is disjoint from $S_1, \dots, S_{n-1}$. We do assume in the following argument that S'_n does not have an endpoint in common with an S'_j ($j \leq n-1$). In the case there is such an endpoint, we may assume that S_n and the corresponding S_j have a common endpoint; the argument given is then easily adapted.

We will need to alter S_n a number of times; to simplify notation, each time an adjustment is made we agree to carry over the old notation to the adjusted section and time maps.

For $j \leq n-1$, let f_{nj} denote the time from S_n to S_j . We need to show that we can adjust S_n so that the resulting time maps f_{nj} satisfy

$$\text{sgn } f_{nj}(hx) = \text{sgn } f'_{nj}(x) \quad (x \in A'_{nj}; j \leq n-1).$$

LEMMA 2. *Suppose e is an endpoint of a component interval I of A_{nj} with $e \notin A_{nj}$, $e = he'$ and $I = h(I')$. Then*

$$\lim_{\substack{x \rightarrow e \\ x \in I}} f_{nj}(x) = \lim_{\substack{x \rightarrow e' \\ x \in I'}} f'_{nj}(x) = \pm \infty.$$

PROOF. First note that, since $e \notin A_{nj}$, we must have

$$(*) \quad \lim_{\substack{x \rightarrow e \\ x \in I}} f_{nj}(x) = +\infty \quad (\text{or } -\infty)$$

(similarly for f'_{nj}). We assume the limit in $(*)$ is $+\infty$; the argument in the remaining case is similar. Now $(*)$ holds if and only if

$$p = \lim_{\substack{x \rightarrow e \\ x \in I}} (\alpha_n(x) \cdot f_{nj}(x))$$

is in $J^+(\alpha_n(e))$. Let $d = \pi p$ and $d' = h^{-1}d$; then $e < d$. If we do not also have $d < e$, then, since h preserves order, we must have $e' < d'$, but not $d' < e'$, and it follows that $\lim_{x \rightarrow e'; x \in I'} f'_{nj}(x) = +\infty$ as asserted. If we do also have $d < e$, then by definition, $I \cup \{e\} < I \cup \{d\}$, so $I' \cup \{e'\} < I' \cup \{d'\}$ and we have the same result.

(Note that we may identify $\pi^{-1}(T_n)$ with $T_n \times \mathbf{R}^1$, since the map $\phi(\alpha_n(x), t) \leftrightarrow (x, t)$ ($x \in T_n, t \in \mathbf{R}^1$) is a homeomorphism. We may then think of the intersections of the S_j ($j \leq n-1$) with $\pi^{-1}(T_n)$ as the graphs of the functions f_{nj} .)

Now define functions $u_n, l_n: T_n \rightarrow [-\infty, \infty]$ as follows:

$$u_n(x) = \begin{cases} \min f_{nj}(x) & \text{all } j \leq n-1 \text{ with } \operatorname{sgn} f'_{nj} h^{-1}(x) \geq 0, \\ \infty & \text{if no such } j \text{ exist;} \end{cases}$$

$$l_n(x) = \begin{cases} \max f_{nj}(x) & \text{all } j \leq n-1 \text{ with } \operatorname{sgn} f'_{nj} h^{-1}(x) \leq 0, \\ -\infty & \text{if no such } j \text{ exist.} \end{cases}$$

Then we have

LEMMA 3. u_n is continuous, except possibly at an endpoint e of T_j with $e \in A_{nj}$; at such a point e both one-sided limits exist (one may be $+\infty$), and

$$\lim_{\substack{x \rightarrow e \\ x \in A_{nj}}} u_n(x) = u_n(e).$$

PROOF. First note that if $x \in T_n$ is in no A_{ni} ($i \leq n-1$), so that $u_n(x) = \infty$, and $\{x_k\}$ is a sequence of points of T_n converging to x , then

$$\lim_{k \rightarrow \infty} u_n(x_k) = \infty.$$

For suppose not. We may assume that $u_n(x_k) = f_{nj}(x_k)$ for some fixed $j \leq n-1$. By Lemma 2 we may assume that the x_k are in distinct component intervals of A_{nj} . Suppose that $f_{nj}(x_k) \rightarrow s$, s finite or $-\infty$. Then we can pick y_k in the component of A_{nj} containing x_k , with

$$f_{nj}(y_k) = \begin{cases} s+1, & s \text{ finite,} \\ 0, & s = -\infty. \end{cases}$$

But then $\alpha_n(y_k) \cdot f_{nj}(y_k) \in S_j$ and, hence,

$$\lim_{k \rightarrow \infty} (\alpha_n(y_k) \cdot f_{nj}(y_k)) = \begin{cases} \alpha_n(x) \cdot (s+1), & s \text{ finite,} \\ \alpha_n(x), & s = -\infty, \end{cases}$$

is in S_j , contrary to our assumption that $x \notin A_{nj}$.

Next suppose that $x \in A_{nj_1} \cap \dots \cap A_{nj_l}$ (and no other A_{ni}), but that x is not an endpoint of any T_{j_i} ($i = 1, \dots, l$). If x is adherent to A_{ni} , but $x \notin A_{ni}$, then, as above, $f_{ni}(y) \rightarrow \infty$ as $y \rightarrow x$ (with $y \in A_{ni}$, $\operatorname{sgn} f'_{ni} h^{-1}(y) = 1$). Thus there is a neighborhood N of x on which $u_n = \min(f_{nj_1}, \dots, f_{nj_l})$, and this function is continuous at x .

Finally, if $x \in A_{nj}$ is an endpoint of T_j , the assertions of the lemma are clear.

Similarly, we may prove

LEMMA 4. l_n is continuous except possibly at an endpoint $e \in A_{nj}$ of T_j ; in this case, both one-sided limits at e exist in $[-\infty, \infty)$, and

$$\lim_{\substack{x \rightarrow e \\ x \in A_{nj}}} l_n(x) = l_n(e).$$

LEMMA 5. $l_n(x) < u_n(x)$ for all $x \in T_n$.

PROOF. Suppose x is a point at which both l_n and u_n are finite. Then $l_n(x) = f_{ni}(x)$ and $u_n(x) = f_{nj}(x)$ for some $i, j \leq n-1$. Note that $f'_{ij}h^{-1}(x) > 0$. Hence, also

$$u_n(x) - l_n(x) = f_{nj}(x) - f_{ni}(x) = f_{ij}(x) > 0.$$

Now let U denote the subset of $\pi^{-1}(T_n)$ that corresponds to $\{(x, t) \in T_n \times \mathbf{R}^1 | l_n(x) < t < u_n(x)\}$. It follows from the preceding lemmas that U is fiberwise homeomorphic with $T_n \times \mathbf{R}^1$. Thus we may assume that the section S_n is chosen to lie in U and is therefore related in the desired way to the S_j ($j \leq n-1$).

By induction, we can then obtain a collection $\{S_n\}$ of sections related appropriately by ϕ . However we need, in addition, to insure that the resulting collection is locally finite at points of M . To do this we strengthen the induction as follows: Let a_k, b_k denote the endpoints of T_k and a'_k, b'_k the endpoints of T'_k . For $k \leq n-1$, let $u_k, l_k: T_k \rightarrow [-\infty, \infty]$ and $u'_k, l'_k: T'_k \rightarrow [-\infty, \infty]$ denote the analogues of u_n, l_n above. Assume by induction that, in addition to (a), the sections $S_1, \dots, S_{n-1}$ satisfy

- (b) for each $k \leq n-1$, either
 - (b1) there exist $i, j \leq n-1$ such that $a_k \in \dot{T}_i \cap \dot{T}_j$ and $\alpha_k(a_k)$ separates $\alpha_i(a_k)$ from $\alpha_j(a_k)$, or
 - (b2) $|f_{kj}(a_k)| > |f'_{kj}(a'_k)|$ for all $j \leq n-1$;
(and similarly for $b_k, k \leq n-1$).
- (c) for each $k \leq n-1$,
 - (c1) if $x \in T_k$ and $l_k(x) = -\infty$, then $u_k(x) > \min(u'_k(h^{-1}x), 1)$, and
 - (c2) if $x \in T_k$ and $u_k(x) = \infty$, then $l_k(x) < \max(l'_k(h^{-1}x), -1)$.

(These conditions say essentially that as each new section is added to M , it is added, whenever possible, at least as far from the preceding sections as the distance between the corresponding sections in M' .)

We have seen that we can add an n th section S_n so that (a) holds with n in place of $n-1$; this we assume done. It remains to check that S_n may be adjusted so that then (b) and (c) hold for $k \leq n$.

LEMMA 6. S_n may be adjusted so that (a) and (b) hold with n in place of $n-1$.

PROOF. Case 1. Fix $x \in \dot{T}_n$. Let $p_1, \dots, p_r$ denote the liftings of $\{a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}\}$ that lie on $\pi^{-1}(x)$ and do not separate liftings of interior points of $T_1, \dots, T_{n-1}$; suppose $p_i = \alpha_{k_i}(a_{k_i})$ (or b_{k_i}), $i = 1, \dots, r$. We may assume $r \geq 1$. First consider the case in which at least some of the p_i , say

$p_i, \dots, p_s$ ($s \geq 1$), lie above $\alpha_n(x)$. If no lifting of x lies below $\alpha_n(x)$, then we can clearly adjust S_n so that (b2) holds at x , for $k \leq n-1$ and the resulting section. On the other hand, if $\alpha_j(x)$ is any lifting of x below $\alpha_n(x)$ (possibly one of the p_i) then, by induction, each $p_i \cdot f'_{k,n}(h^{-1}x)$ ($i = 1, \dots, s$) lies above $\alpha_j(x) \cdot f'_{j,n}(h^{-1}x)$. It follows that there is an open interval of possible positions of $\alpha_n(x)$ for which (b2) holds. In the contrary case (all of $p_1, \dots, p_r$ lie below $\alpha_n(x)$) an analogous argument shows that (b2) can be recovered.

Case 2. Now suppose that x is an endpoint of T_n . If there are no liftings of x above $\alpha_n(x)$ (or none below) then we can clearly place $\alpha_n(x)$ so that (b2) is satisfied. Thus we assume there are liftings of x both above and below $\alpha_n(x)$. If there are liftings of interior points both above and below, then the argument of Case 1 applies. Hence we may assume that all liftings of interior points lie above (say). Now suppose $\alpha_j(x)$ is any lifting of x above $\alpha_n(x)$, and $\alpha_k(x)$ any below. Then x is an endpoint of T_k and (b2) is satisfied by induction. Thus $\alpha_k(x) \cdot f'_{k,n}(h^{-1}x)$ is below $\alpha_j(x) \cdot f'_{j,n}(h^{-1}x)$. Since this is true for each such pair, there is an open interval of satisfactory positions for the adjusted $\alpha_n(x)$.

Thus we now assume that S_n has been chosen so that both (a) and (b) hold with n in place of $n-1$.

LEMMA 7. S_n may be adjusted so that (a)–(c) are satisfied for $S_1, \dots, S_{n-1}, S_n$.

PROOF. Define $I_- = \{x \in T_n | l_n(x) = -\infty\}$, $I_+ = \{x \in T_n | u_n(x) = \infty\}$. Let E_- (E_+) denote the set of those endpoints of the T_k ($k \leq n-1$) which are adherent to I_- (I_+), but not in I_- (I_+). By Lemmas 3 and 4, $I_- \cup E_-$ and $I_+ \cup E_+$ are closed.

If $e \in E_-$ then for some $k \leq n-1$ we have $e = a_k$ (or b_k) and $l_n(e) = f_{nk}(e)$ (see Figure 4). Since (b) holds we have $f_{kj}(e) > f'_{kj}(h^{-1}e)$ for all $j \leq n$ with $e \in T_j$. Hence we may adjust S_n so that on an open interval U_e containing e we have

$$(*) \quad \begin{cases} u_n(x) > u'_n(h^{-1}x) & (\text{or } u_n(x) = u'_n(h^{-1}x) = \infty), \\ l_n(x) < l'_n(h^{-1}x) & (\text{or } l_n(x) = l'_n(h^{-1}x) = -\infty) \end{cases}$$

and so that (a), (b) continue to hold. Let $U = \bigcup_{e \in E_-} U_e$. Then $I_- - U$ is compact. Hence by Lemma 4 we may choose finitely many open intervals W_i ($i = 1, \dots, s$), that cover $I_- - U$ and such that $l_n(x) < -2$ ($x \in W = \bigcup_{i=1}^s W_i$). Choose open intervals V_i with $\bar{V}_i \subseteq W_i$ and $I_- - U \subseteq V = \bigcup_{i=1}^s V_i$. Let $\delta: T_n \rightarrow [-1, 0]$ be a continuous function that is 0 off W and -1 on each $\bar{V}_i$. Replace S_n with the section defined by $\alpha_n(x) \cdot \delta(x)$ ($x \in T_n$). Note that we may assume that W contains no a_k (or b_k) with $\alpha_k(a_k)$ (or $\alpha_k(b_k)$) below S_n , so that (b) is not affected by this adjustment. The adjusted

section satisfies (c1) for all $x \in T_n$, and also (c2), at least for $x \in O = V \cup W$.

FIGURE 4

Now let $e \in E_+$, but $e \notin O$. Note that we may assume that O is chosen so $e \notin \bar{O}$. As above, we may adjust S_n on a neighborhood of e so that, afterward, (*) holds on an open interval B_e about e , and so that (a), (b) are not affected. Let $B = \bigcup_{e \in E_+ - O} B_e$. Then $I_+ - (B \cup O)$ is compact; the argument of the preceding paragraph allows us to adjust S_n on a sufficiently small neighborhood of this set, so that the resulting section satisfies (c).

Hence, by induction, we may construct a sequence $\{S_j\}$ of sections of ϕ satisfying (a)–(c) for all n . It is easy to check that $\{S_j\}$ is locally finite.

We can now obtain an equivalence $k: M' \rightarrow M$ of ϕ' with ϕ as follows. For each N'_i in the given subdivision of (M', ϕ') , let N_i denote the p -neighborhood in M bounded by the S_j that correspond to the S'_j in the ends of N'_i , and the orbit segments of ϕ determined by the endpoints of the initial end. Define k by taking each N'_i homeomorphically fiberwise onto N_i , and so that $\pi k = h\pi'$ holds on each N'_i . Then k is an embedding since both $\{N_i\}$ and $\{N'_i\}$ are locally finite. Finally, since condition (c) holds, it is easily checked that k must be onto. Thus we have proved a slight strengthening of Theorem 1:

THEOREM 1'. *Suppose ϕ, ϕ' are completely unstable continuous flows on*

2-manifolds M, M' respectively, and that $h: M/\phi \rightarrow M'/\phi'$ is an isomorphism of the corresponding orbit spaces. Then there is an equivalence k of ϕ with ϕ' that satisfies $\pi'k = k\pi$.

REMARK. We may determine which ordered spaces can appear as the orbit space of a completely unstable flow. To do this we must first define order on a nonseparated 1-manifold, in a way which depends only on the 1-manifold (and not on its representation as the quotient space of a flow).

DEFINITION. Let K be a nonseparated 1-manifold. An *order* on K consists of a (countable) atlas $\{T_n\}$ of open subsets of K , each homeomorphic with $(0, 1)$, and a collection $\{\sigma_{ij}\}$ of continuous functions $\sigma_{ij}: T_i \cap T_j \rightarrow \{-1, 1\}$ satisfying

- (1) $\sigma_{ji} = -\sigma_{ij}$,
- (2) if $\sigma_{ij}(x) = +1$ (-1) and $\sigma_{jk}(x) = +1$ (-1) then $\sigma_{ik}(x) = +1$ (-1),
- (3) if $e \in T_i$ is an endpoint of a component interval I of $T_i \cap T_j$ and $e \in T_i \cap T_k$, then, for $x \in I \cap T_k$, $\sigma_{ij}(x)\sigma_{jk}(x) = -1$.

If ϕ is a completely unstable flow on the 2-manifold M , then M/ϕ may be ordered in this sense: choose disjoint sections $S_n \subseteq M$ of ϕ , each homeomorphic with an open interval, and so that $\{T_n = \pi(S_n)\}$ covers M/ϕ ; define $f_{ij}: T_i \cap T_j \rightarrow \mathbf{R}^1$ as in the proof of Theorem 1 and set $\sigma_{ij} = \text{sgn } f_{ij}$. It follows from the proof of Theorem 1 that this ordered space completely determines (M, ϕ) . One may also prove the following:

Any nonseparated 1-manifold ordered in this sense can be realized as the ordered orbit space of a completely unstable flow (M, ϕ) , where M is a Hausdorff 2-manifold.

4. C^r -equivalence of completely unstable flows. We can now give the proof of Theorem 2. We will need the following lemma; the proof is straightforward and so we omit it. As above, S_ϵ denotes $\{(x, y) \in \mathbf{R}^2 \mid |x|, |y| \leq 1 + \epsilon\}$. A homeomorphism $f = (f_1, f_2): S_\epsilon \rightarrow \mathbf{R}^2$ (into) is said to be *fiberwise* if $f_1(x, y)$ is independent of y .

LEMMA 8. Suppose $f = (f_1, f_2): S_\epsilon \rightarrow \mathbf{R}^2$ is a fiberwise homeomorphism, that is a C^r -diffeomorphism on an open set $U \subseteq S_\epsilon$, that f_1 is C^r on all of S_ϵ and that f_{1x} is nonzero on S_ϵ . Fix $\eta > 0$. Then there is a fiberwise homeomorphism $g: S_\epsilon \rightarrow \mathbf{R}^2$ satisfying

- (a) $g_1(x, y) = f_1(x, y)$ for all $(x, y) \in S_\epsilon$,
- (b) $|f(x, y) - g(x, y)| < \eta$ for all $(x, y) \in S_\epsilon$,
- (c) g is a C^r -diffeomorphism on $U \cup \dot{S}_{\epsilon/4}$, and
- (d) $g(x, y) = f(x, y)$ for all $(x, y) \in S_\epsilon - S_{3\epsilon/4}$.

PROOF OF THEOREM 2. Let ϕ (ϕ') denote a completely unstable C^r -flow on the 2-manifold M (M'). Let M/ϕ (M'/ϕ') denote the corresponding orbit space, endowed with the order relation defined in §3 and the (unique)

C' -structure with respect to which the natural projection $\pi: M \rightarrow M/\phi$ ($\pi': M' \rightarrow M'/\phi'$) is C' (cf. §1). It is easy to check that a C' -equivalence of ϕ with ϕ' induces a C' -diffeomorphism of M/ϕ onto M'/ϕ' that preserves order. We prove the converse. Thus suppose we are given an order preserving C' -diffeomorphism h of M/ϕ onto M'/ϕ' . By Theorem 1, there is a topological equivalence k of ϕ with ϕ' that makes the accompanying diagram commute. We wish to smooth k to a C' -diffeomorphism, maintaining the commutativity. The preceding lemma will allow us to do this locally.

$$\begin{array}{ccc} M & \xrightarrow{k} & M' \\ \pi \downarrow & & \downarrow \pi' \\ M/\phi & \xrightarrow{h} & M'/\phi' \end{array}$$

Let $\{U_n\}$ ($n \in \mathbb{Z}^+$) be a cover of M by open sets of the form $S_n \cdot \mathbb{R}^1$, where $S_n \cong (0, 1)$ is a C' -section of ϕ . For each $n \in \mathbb{Z}^+$, define $W_n = U_1 \cup \dots \cup U_n$. Suppose by induction that we have obtained an equivalence k_n , that covers h , and that is a C' -diffeomorphism on W_n .

In W_{n+1} , $U_{n+1} - W_n$ and $\partial U_{n+1} \cap W_n$ are disjoint closed sets; hence, let G_{n+1} and H_{n+1} be disjoint open sets in W_{n+1} , with $U_{n+1} - W_n \subseteq G_{n+1}$, $\partial U_{n+1} \cap W_n \subseteq H_{n+1}$. We may also assume that

$$G_{n+1} \subseteq \{x \in M \mid \rho(x, U_{n+1} - W_n) < 1/(n+1)\},$$

where ρ is the given metric on M . We will replace k_n with an equivalence k_{n+1} , that differs from k_n only on G_{n+1} , and that is C' on W_{n+1} .

Let $\{(N_i, O_i)\}$ ($i \in \mathbb{Z}^+$) be a collection of p -neighborhood pairs in G_{n+1} , with $(N_i, O_i) \cong (S_\varepsilon, S_0)$ (some $\varepsilon > 0$), such that $\{\dot{O}_i\}$ is a cover of $U_{n+1} - W_n$ and $\{N_i\}$ is locally finite at points of U_{n+1} . We now apply Lemma 8 to each N_i in succession. Assume by induction that we have adjusted $k_n = k_{n,0}$ to $k_{n,j}$ so that:

- (i) $k_{n,j}$ is a C' -diffeomorphism on $W_n \cup O_1 \cup \dots \cup O_j$,
- (ii) for $i \leq j$, $\rho'(k_{n,i-1}, k_{n,i}) < 2^{-i}$,
- (iii) for $i \leq j$, $k_i = k_{i-1}$ off N_i .

Apply Lemma 8 to $k_{n,j}$ and N_{j+1} , with $U = W_n \cup O_1 \cup \dots \cup O_j$ and $\eta = 2^{-(j+1)}$, to obtain $k_{n,j+1}$. By induction we obtain a sequence $\{k_{n,i}\}$ of equivalences, satisfying (i)–(iii) for all $j \in \mathbb{Z}^+$. Since $\{N_i\}$ is locally finite at points of U_{n+1} , the map l defined by

$$l(x) = \lim_{i \rightarrow \infty} k_{n,i}(x) \quad \text{for } x \in U_{n+1},$$

is a fiberwise diffeomorphism of U_{n+1} onto $k_n(U_{n+1})$ that covers $h|_{\pi(U_{n+1})}$. We must check that the extension of this map by k_n on $M - U_{n+1}$ remains an

equivalence. Thus fix $x \in \partial U_{n+1}$. If $x \in W_n$, then k_n has not been adjusted on a neighborhood (H_{n+1}) of x . Hence, we may assume $x \in \partial U_{n+1} - W_n$. Fix $\delta > 0$. Pick j_0 so that $2^{-j_0+1} < \delta$. There is a neighborhood O of x that misses $N_1 \cup \dots \cup N_{j_0}$, so if $y \in O \cap U_{n+1}$ we must have $\rho'(k_n(x), l(x)) < \delta$. Since δ was arbitrary, it follows that, if $\{y_i\} \subseteq U_{n+1}$ and $y_i \rightarrow x$ ($i \rightarrow \infty$), then $l(y_i) \rightarrow k_n(x)$ ($i \rightarrow \infty$). Thus k_n extends l continuously as desired.

Now denote by k_{n+1} the equivalence defined by

$$k_{n+1}(x) = \begin{cases} l(x), & x \in U_{n+1}, \\ k_n(x), & x \in M - U_{n+1}. \end{cases}$$

We have shown that k_{n+1} is an equivalence of ϕ with ϕ' that is a C' -diffeomorphism on W_{n+1} and still covers h . By induction we obtain a sequence $\{k_n\}$ ($n \in \mathbb{Z}^+$) of such equivalences, where, for each n , k_{n+1} differs from k_n at most on $G_{n+1} \subseteq \{x \in M \mid \rho(x, U_{n+1} - W_n) < 1/(n+1)\}$.

Finally, we show that, defining

$$k'(x) = \lim_{n \rightarrow \infty} k_n(x) \quad (x \in M),$$

we obtain an equivalence covering h that is a C' -diffeomorphism of M onto M' . It suffices to prove that, for any $x \in M$, there is a neighborhood O of x and an index n_0 , such that $k_n(y) = k_{n_0}(y)$ for any $y \in O$ and all $n \geq n_0$. Fix $x \in M$. Let m be the least index with $x \in U_m$ and let $d = \rho(x, \partial U_m)$. Pick n_0 so that $1/n_0 < d/2$ and let O be the disc of radius $d/2$ centered at x . Then for any $n > n_0$ we see that $O \cap G_n = \emptyset$ and hence that k_n agrees with k_{n_0} on O , as asserted.

PROOF OF THEOREM 3. Fix $r \in \{1, 2, \dots, \infty\}$ and let ϕ denote a completely unstable (not necessarily C') flow on the 2-manifold M . It is proved in [7] that M/ϕ admits at least one C' -structure and that any C' -structure on M/ϕ lifts to a C' -structure on M , with respect to which ϕ is C' . By Munkres' theorem [6], the C' -structure on a 2-manifold is unique; hence, there is a C' -diffeomorphism h , of M with the C' -structure lifted from M/ϕ , onto M with its given (original) C' -structure. If we let ψ denote the flow induced by ϕ under h (viz., $\psi(x, t) = h\phi(h^{-1}x, t)$), then ψ is C' in the given structure on M , and is topologically equivalent (under h^{-1}) to ϕ . Thus to any C' -structure on M/ϕ we may assign a C' -flow on M that is in the topological equivalence class of ϕ . It then follows from Theorem 2 that this assignment defines a 1-1 correspondence between the C' -equivalence classes contained in the topological class of ϕ and the C' -structures on M/ϕ .

PROOF OF COROLLARY. Fix $r \in \{1, 2, \dots, \infty\}$. By Theorem 3 it suffices to construct a continuum of distinct C' -structures on M/ϕ . Since there is at least one such structure, we may assume that we are given a C' -atlas $\mathfrak{A} = \{(T_i, \alpha_i)\}$ ($i \in \mathbb{Z}^+$) on M/ϕ . Here $\{T_i\}$ is an open cover of M/ϕ and each α_i is a homeomorphism of $(-1, 1)$, say, onto T_i . Fix $n, m \in \mathbb{Z}^+$ with

$T_n \cap T_m \neq \emptyset$. Let e_n be an endpoint of the component interval I of $T_n \cap T_m$, and let e_m be the corresponding point of T_m (see Figure 5; of course $T_n \cap T_m$ may be much more complicated than is indicated here). We may assume that e_n is in no T_i except T_n and that both α_n and α_m map $(0, \epsilon)$ onto I , with $e_m = \alpha_m(0)$ and $e_n = \alpha_n(0)$.

FIGURE 5

Now suppose that f is an orientation preserving homeomorphism of $(-1, 1)$ onto itself that fixes 0 and is a C^r -diffeomorphism in the complement of $\{0\}$. Note that if we alter the atlas $\mathcal{U}$ only by replacing α_n with $\alpha_n \circ f$, then we obtain a new C^r -atlas on M/ϕ . For each real number $p > 1$, we construct in this manner an atlas $\mathcal{U}_p$, using the homeomorphism f_p defined by

$$f_p(x) = \begin{cases} l(x^p), & x \in (0, 1), \\ x, & x \in (-1, 0], \end{cases}$$

where l is a fixed C^r -diffeomorphism of $(0, 1)$ onto itself that agrees with $\alpha_n^{-1}\alpha_m$ on some interval $(0, \delta)$ ($0 < \delta < \epsilon$). It is not difficult to check that no pair of the C^r (nonseparated) manifolds $\{(M, \mathcal{U}_p)\}$ are even C^1 -diffeomorphic, by a diffeomorphism which fixes e_n and e_m . For if $d: (M, \mathcal{U}_p) \rightarrow (M, \mathcal{U}_q)$ is such a diffeomorphism, let $h, k: (-1, 1) \rightarrow (-1, 1)$ denote the diffeomorphisms defined by $\alpha_m^{-1}d\alpha_n, f_q^{-1}\alpha_n^{-1}d\alpha_n f_p$ respectively. Then both

$$h'(0) = \lim_{x \rightarrow 0^+} \frac{h(x)}{x}$$

and

$$k'(0) = \lim_{x \rightarrow 0^+} \frac{k(x)}{x} = \lim_{x \rightarrow 0^+} \frac{(h(x^p))^{1/q}}{x} = \lim_{x \rightarrow 0^+} \left(\frac{h(x^p)}{x^p} \right)^{1/q} \cdot x^{(p-q)/q}$$

are finite and nonzero, an impossibility if $p \neq q$. But then, since the images of e_n and e_m under any diffeomorphism of M/ϕ must lie in the set of nonseparated points, which is countable by assumption, it follows that there are uncountably many distinct C^r -structures in the collection $\{\mathcal{U}_p\}$.

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DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403