ULM'S THEOREM FOR PARTIALLY ORDERED STRUCTURES RELATED TO SIMPLY PRESENTED ABELIAN p -GROUPS

BY

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ABSTRACT. If we have an abelian p -group G , a multiplication by p for each element of G is defined by setting $px = x + x + \cdots + x$, where p is the number of terms in the sum. If we forget about the addition on G , and just keep the multiplication by p , we have the algebraic structure called a p -basic tree. A natural partial order can be defined, the graph of which is a tree with 0 as root. A p -basic tree generates a simply presented abelian p -group, and provides a natural direct sum decomposition for it. Ulm invariants may be defined directly for a p -basic tree so that they are equal to the Ulm invariants of the corresponding group. A central notion is that of a stripping function between two p -basic trees. Given a stripping function from X onto Y we can construct an isomorphism between the groups corresponding to X and Y ; in particular, X and Y have the same Ulm invariants. Conversely, if X and Y have the same Ulm invariants, then there is a map from X onto Y that is the composition of two stripping functions and two inverses of stripping functions. These results constitute Ulm's theorem for p -basic trees, and provide a new proof of Ulm's theorem for simply presented groups.

1. Introduction. In 1933, Ulm [7] characterized countable reduced abelian p -groups in terms of functions from ordinals to cardinals defined by $f(\alpha, G) = \dim p^\alpha G[p] / p^{\alpha+1} G[p]$, the Ulm invariants of G . This result was extended by Kolettis [6] in 1960 to direct sums of such groups, and by Hill [4], [5] in 1967 to the class of all totally projective groups. In 1969, Crawley and Hales [1] proved Ulm's theorem for reduced simply presented p -groups (T -groups, in their terminology), and showed that the class of reduced simply presented p -groups coincides with the class of all totally projective groups.

The principal object of study in this paper is the p -basic tree, which is a set upon which multiplication by a prime p is defined. A p -basic tree generates a simply presented p -group, and provides a natural direct sum decomposition for it. Conversely, any such group contains a p -basic tree which generates it. Ulm invariants can be defined directly for a p -basic tree so that they are equal to the Ulm invariants of the corresponding group.

A central notion introduced here is that of a stripping function between two p -basic trees. Given a stripping function, we can construct an isomorphism between the corresponding groups (Theorem 2). If a map between two

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p -basic trees exists which is the composition of a stripping function and the inverse of a stripping function (call such a map a *strip-graft*), then the natural direct sum decompositions of the corresponding groups have isomorphic refinements (Theorem 3). A principal result (Theorem 4) is that two p -basic trees generate isomorphic groups if and only if a map between them exists which is the composition of two strip-grafts. We finish with a version of Ulm's theorem for p -basic trees (Theorem 5), which gives a new proof of Ulm's theorem for simply presented groups.

2. Preliminaries. Throughout the discussion, the letter p shall denote a fixed but arbitrary prime, and "group" shall mean "abelian p -group". We shall conform to standard terminology and notation as used, for example, in [3].

If we have a group G , a multiplication by p for each element of G is defined by setting $px = x + x + \cdots + x$, where p is the number of terms in the sum. This is a most important structure on G ; in particular, it has the property that given g in G , there is an integer n with $p^n g = 0$. If we forget about the addition on G , and just keep the multiplication by p , we have the algebraic structure which we shall call a p -basic tree.

DEFINITION 1. A p -basic tree is a set X together with a multiplication by p for elements of X such that:

- (1) $px \in X$ for each x in X ,
- (2) there is an element 0 in X for which $p0 = 0$, and
- (3) for each nonzero element x in X , there is a positive integer n such that $p^n x = 0$.

If $x \in X$, the least integer n such that $p^n x = 0$ is the *exponent* of x . The relation $>$ on X is defined as follows: for elements x, y of X , if there is a positive integer m such that $p^m x = y$, then $x > y$. Clearly, the relation $\geq$ is a partial order. The graph of X under this partial order is a tree with 0 as root. An *atom* of X is a nonzero element z with the property that, if $z \geq x$, then $x = z$ or $x = 0$. For each x in X , set $C(x) = \{y \in X: y \geq x \text{ or } y < x\}$. A subset S of X is a *subtree* of X if, for each s in S , the element ps is in S . A collection $\{S_i\}$ of subtrees of X is *disjoint* if $S_i \neq S_j$ implies $S_i \cap S_j = \{0\}$. If $\{S_i\}$ is a disjoint collection of subtrees of X such that $X = \bigcup S_i$, then X is the *direct sum* of its subtrees S_i and we write $X = \bigoplus S_i$. Note that if $x \in X$, then $C(x)$ is a subtree, and if A is the set of atoms of X , then $X = \bigoplus_{a \in A} C(a)$. The group whose generators are elements of X , subject to the relations $px = y$, is the *group generated by X* , and shall be denoted by $[X]$. The next definition is taken from [3].

DEFINITION 2. A group G is *simply presented* if it has a set X of generators and a set of relations such that:

- (1) for every x in X , $x \neq 0$ in G ,
- (2) if x, y are distinct elements in X , then $x \neq y$ in G , and
- (3) all relations are of the form $px = 0$ or $px = y$ where $x, y \in X$.

The set X is a p -basis of G . A p -basis can also be characterized in the following way [2], [8]: a subset X of G such that if $x \in X$ and $px \neq 0$, then $px \in X$, and every element in G can be written uniquely in the form $\sum_{x \in X} n_x x$ with $0 \leq n_x < p$. It is shown in [2] that a group is simply presented iff it has a p -basis. The representation of an element in the form $\sum n_x x$ will be referred to as the unique representation of the element as a *linear combination* of elements of X . It is clear that if X is a p -basic tree, then $[X]$ is a simply presented group with p -basis $X \sim \{0\}$; conversely, if a group is simply presented with a p -basis X , then $X \cup \{0\}$ is a p -basic tree.

Observe now that if X is a p -basic tree and $\{S_i\}$ a collection of subtrees such that $X = \bigoplus S_i$, then $[X] = \bigoplus [S_i]$. Consequently, if A is the set of atoms of X , the decomposition $[X] = \bigoplus_{a \in A} [C(a)]$ shall be referred to as the *natural* direct sum decomposition of $[X]$ given by X .

DEFINITION 3. Let X be a p -basic tree. Set $p^0 X = X$, and for any ordinal β , set $p^\beta X = \bigcap_{\alpha < \beta} p(p^\alpha X)$. An element x of X has *height* α if $x \in p^\alpha X \sim p^{\alpha+1} X$; we shall denote the height of x by hx . If $x \in p^\alpha X$ for every α , then we write $hx = \infty$, where ∞ is considered to be larger than any ordinal. There must be some ordinal λ for which $p^\lambda X = p^{\lambda+1} X$; the least such λ is the *length* of X . If $p^\lambda X = 0$, then X is *reduced*. Henceforth we shall assume that *all p -basic trees are reduced*. A useful property [1] of simply presented groups is that if $b = \sum_{i=1}^k r_i x_i$ is the unique representation of a nonzero element b as a linear combination of elements of X , then $b \in p^\alpha G$ iff $x_i \in p^\alpha G$ for $i = 1, \dots, k$. This can be strengthened.

PROPOSITION 1. Suppose X is a p -basic tree, and b is a nonzero element of $[X]$ whose representation as a linear combination of elements of X is $b = \sum_{i=1}^k r_i x_i$. If hx_i represents the height in X of x_i , then the height of b in the group $[X]$ is $\min\{hx_1, \dots, hx_k\}$.

PROOF. It suffices to verify that $p^\alpha X = X \cap p^\alpha [X]$ for any ordinal α ; this is accomplished by an induction on α . $\square$

DEFINITION 4. Two p -basic trees X and Y are *isomorphic* if there is a height-preserving bijection $\eta: X \rightarrow Y$ such that $\eta(px) = p\eta(x)$ for every x in X ; the map η is an *isomorphism*. If the groups $[X]$ and $[Y]$ are isomorphic, then X and Y are *equivalent*. Obviously, isomorphic p -basic trees are equivalent.

3. Ulm invariants and stripping functions. Recall that, for a group G and an ordinal α , the α th Ulm invariant of G is the cardinal number

$$f(\alpha, G) = \dim p^\alpha G[p] / p^{\alpha+1} G[p].$$

The corresponding notion for p -basic trees is the following.

DEFINITION 5. Let X be a p -basic tree of length λ , and α an ordinal such that $\alpha < \lambda$. Define $U(\alpha, X) = \{x \in X: hx = \alpha \text{ and } h(px) > \alpha + 1\}$. De-

note by A the set of elements of X with height $\alpha + 1$, and suppose $z \in A$. Choose an element in $p^{-1}z$ with height α and denote it by z' . Set $D(z) = \{y \in p^{-1}z: hy = \alpha \text{ and } y \neq z'\}$. We now define the α th *Ulm invariant* of X to be the cardinal number

$$f(\alpha, X) = |U(\alpha, X)| + \left| \bigcup_{z \in A} D(z) \right|.$$

PROPOSITION 2. *Let X be a p -basic tree of length λ , and α an ordinal such that $\alpha < \lambda$. Then $f(\alpha, X) = f(\alpha, [X])$.*

PROOF. Denote the group $p^\alpha[X][p]$ by G , and its subgroup $p^{\alpha+1}[X][p]$ by H . For each x in $U(\alpha, X)$, there is an element u in $p^\alpha X$ with $p^2u = px$, so $x - pu + H$ is a nonzero element of G/H . Let $B_1 = \{x - pu + H: x \in U(\alpha, X), u \in p^\alpha X, \text{ and } p^2u = px\}$. For z in A (where A is the set of elements of height $\alpha + 1$), and y in $D(z)$, the element $y - z' + H$ is a nonzero element of G/H . Let $B_2 = \{y - z' + H: z \in A, y \in D(z)\}$. We proceed to show that $B_1 \cup B_2$ is a basis of G/H , considered as a vector space over $Z(p)$.

To see that $B_1 \cup B_2$ is independent, suppose

$$\begin{aligned} \sum_{i=1}^n r_i(x_i - pu_i + H) + \sum_{i=1}^k t_{i1}(y_{i1} - z'_1 + H) \\ + \cdots + \sum_{i=1}^{k'} t_{im}(y_{im} - z'_m + H) = 0 \end{aligned}$$

in G/H , where each r_i and each t_{ij} is a nonnegative integer less than p . This implies that in $[X]$, the element

$$\sum_{i=1}^n r_i x_i + \sum_{i=1}^k t_{i1} y_{i1} + \cdots + \sum_{i=1}^{k'} t_{im} y_{im} - \left(\sum_{i=1}^k t_{i1} \right) z'_1 - \cdots - \left(\sum_{i=1}^{k'} t_{im} \right) z'_m$$

has height at least $\alpha + 1$. Since all elements from X in this expression are distinct, each $r_i = t_{ij} = 0$.

To see that $B_1 \cup B_2$ spans G/H , suppose $v + H$ is a nonzero element of G/H , and $v = \sum_{i=1}^n r_i x_i$ is its representation in $[X]$ as a linear combination of elements of X . Assume that any element in G whose representation contains less than n terms is in the linear span S of $B_1 \cup B_2$ in G/H . In the expression for v , if any $px_i = 0$, then $v - r_i x_i$ satisfies the induction hypothesis, and $x_i + H$ is equal to an element of B_1 , so $v + H \in S$. If no $px_i = 0$, then in the set $\{x_1, \dots, x_n\}$ there must be at least two elements x_i and x_j for which $px_i = px_j$; say $px_1 = \cdots = px_k = z$. If $hz > \alpha + 1$, there is u in $p^\alpha X$ such that $p^2u = z$, and so

$$v = r_1(x_1 - pu) + \cdots + r_k(x_k - pu) + \sum_{i=1}^k r_i(pu) + \sum_{i=k+1}^n r_i x_i.$$

Then $v - \sum_{i=1}^k r_i(x_i - pu)$ satisfies the induction hypothesis, and each $x_i - pu$

$+ H$ is equal to an element of B_1 , so $v + H \in S$. If $hz = \alpha + 1$, write

$$v = r_1(x_1 - z') + \cdots + r_k(x_k - z') + \left(\sum_{i=1}^k r_i \right) z' + \sum_{i=k+1}^n r_i x_i.$$

Since $x_i - z' + H \in B_2$, a similar argument shows that $v + H \in S$. $\square$

DEFINITION 6. A p -basic tree is *fully stripped* if, whenever there are distinct elements x and y with $px = py$, then $h(px)$ is a limit ordinal or ∞ .

DEFINITION 7. Let X and Y be p -basic trees and $\sigma: X \rightarrow Y$ a bijection. Then σ is a *stripping function* if

- (1) σ preserves heights, and
- (2) if $\sigma(px) \neq p\sigma(x)$, then $p\sigma(x) = 0$. A bijection ξ is a *strip-graft* if there is a p -basic tree Z , a stripping function $\sigma: X \rightarrow Z$, and a stripping function $\psi: Y \rightarrow Z$ such that $\xi = \psi^{-1}\sigma$. An easy consequence of the definition is the following.

PROPOSITION 3. Suppose X and Y are p -basic trees and $\sigma: X \rightarrow Y$ is a stripping function. Let a be an atom of X , and set $N(a) = \{x \in C(a): p\sigma(x) \neq \sigma(px)\}$. Then σa is an atom of Y , and σx is an atom of Y for each x in $N(a)$. Furthermore, the image of the subtree $C(a)$ is

$$\sigma[C(a)] = \bigoplus_{x \in N(a)} C(\sigma x) \oplus C(\sigma a).$$

DEFINITION 8. Let X be a p -basic tree and z a nonzero element of X . A subset D of $p^{-1}z$ is *dense* in $p^{-1}z$ if, given an ordinal $\alpha < hz$, there is an element y in D with $\alpha \leq hy$.

LEMMA 1. Let X be a p -basic tree, and σ a bijection of X onto a set Y . Suppose multiplication by p is defined on Y so that either $p\sigma(x) = \sigma(px)$, or $p\sigma(x) = \sigma(0)$. Then Y is a p -basic tree, and σ is a stripping function iff for every nonzero element z in X , the set $\{y \in p^{-1}z: p\sigma(y) = \sigma(py)\}$ is dense in $p^{-1}z$.

PROPOSITION 4. Let X be a p -basic tree. Then there is a fully stripped p -basic tree X' and a stripping function $\sigma: X \rightarrow X'$.

PROOF. Define $*$, a new multiplication by p for X , as follows: $p * x = px$, if $h(px)$ is a limit ordinal or if there is some z in X for which $x = z'$, and $p * x = 0$ otherwise. With this multiplication, X is a fully stripped p -basic tree, which shall be denoted X' . Lemma 1 shows that the identity function σ is a stripping function. $\square$

THEOREM 1. If X and Y are p -basic trees and $\sigma: X \rightarrow Y$ is a stripping function, then X and Y have the same Ulm invariants.

PROOF. Let α denote an arbitrary ordinal, and A the set of elements of X with height $\alpha + 1$. The set A' of elements of Y with height $\alpha + 1$ coincides with $\sigma[A]$. The result follows from the fact that there is a one-to-one

correspondence between the sets $U(\alpha, X) \cup \bigcup_{z \in A} D(z)$ and $U(\alpha, Y) \cup \bigcup_{z \in A} D(z)$. $\square$

In view of Theorem 1, it is a consequence of Ulm's theorem for simply presented groups that $[X]$ and $[Y]$ are isomorphic. However, we can prove this result directly in a more general setting, and use it to prove Ulm's theorem.

THEOREM 2. *Let X and Y be p -basic trees and suppose $\sigma: X \rightarrow Y$ is a stripping function. Then X and Y are equivalent.*

PROOF. Define the function d on X as follows: set $d(0) = 0$, and if the exponent of x is $k + 1$, set $d(x) = d(px)$ if $p\sigma(x) = \sigma(px)$, and set $d(x) = d(px) + 1$ if $p\sigma(x) \neq \sigma(px)$. We use d to define inductively a function $\pi: X \rightarrow X$ such that for each x in X we have $p\pi(x) = \pi(px)$ iff $p\sigma x = \sigma(px)$, and such that: (1) $h(\pi x) \geq hx$, (2) there is a nonnegative integer n for which $\pi^n(x) = 0$, (3) either $p\pi x = \pi px$ or $p\pi x = px$, and (4) if $\pi x \neq 0$, then $d(\pi x) < d(x)$. The construction of π is as follows. If $d(x) = 0$ or if $px = 0$, set $\pi x = 0$. Assume π has been defined to satisfy (1)–(4) for all y such that $d(y) < k$, and for all y such that $d(y) = k$ and $p^{m-1}y = 0$, where k and m are positive integers. Suppose x is an element of X whose exponent is m , and for which $d(x) = k$. If $p\sigma x \neq \sigma px$, then by Lemma 1 there is an element y of X with $py = px$, $hy \geq hx$, and $\sigma py = p\sigma y$. Since $d(y) = d(py) = d(px) < d(x)$, πy has been defined. Set $\pi x = y$. If $p\sigma x = \sigma px$, then $d(x) = d(px)$, but the exponent of px is $m - 1$; hence $\pi(px)$ has been defined. If $\pi(px) = 0$ set $\pi x = 0$. Otherwise, note $h(\pi(px)) \geq h(px) > hx$, so there is an element u of X with $pu = \pi(px)$ and $hu \geq hx$. If $p\sigma u = \sigma pu$, then πu has been defined; set $\pi x = u$. On the other hand, if $p\sigma u \neq \sigma pu$, then by Lemma 1 there is an element w of X with $pw = pu$, $hw \geq hu$, and $\sigma(pw) = p\sigma w$. Set $\pi x = w$.

We use π to define yet another function $\pi': X \rightarrow [X]$ by setting $\pi'(x) = x - \pi x$. Denote the image of X in $[X]$ by X' . Then X' is a p -basic tree, and $\pi': X \rightarrow X'$ is a stripping function. Furthermore, $X' \sim \{0\}$ is a p -basis of $[X]$ (to see this, note if $g = \sum_{i=1}^k r_i x_i$ in $[X]$, with each x_i in X , a representation of g as a linear combination of elements of X' is

$$g = \sum_{i=1}^k [r_i(x_i - \pi x_i) + \cdots + r_i(\pi^{m-1}x_i - \pi^m x_i)],$$

where m is large enough so that $\pi^m x_i = 0$ for all $i = 1, \dots, k$).

Finally, observe that Y and X' are isomorphic under the map $\eta: Y \rightarrow X'$ defined by $\eta(\sigma x) = \pi'(x)$. $\square$

THEOREM 3. *Let X and Y be p -basic trees and $\xi: X \rightarrow Y$ a strip-graft. Then the natural direct sum decompositions of $[X]$ and $[Y]$ have isomorphic refinements.*

PROOF. Let Z be the p -basic tree with $\sigma: X \rightarrow Z$ and $\psi: Y \rightarrow Z$ the stripping functions such that $\xi = \psi^{-1}\sigma$. Then the set T of atoms of Z is $T = \{\sigma a: a \in A\} \cup \{\sigma x: p\sigma x \neq \sigma p x\}$, where A is the set of atoms of X . Consequently, $[Z] = \bigoplus_{t \in T} [C(t)]$ is isomorphic to a refinement of $[X] = \bigoplus_{a \in A} [C(a)]$. Similarly, $[Z]$ is isomorphic to a refinement of $[Y]$.

4. p -basic trees with the same Ulm invariants. In what follows, we shall be interested in p -basic trees with the same Ulm invariants, and because of Theorem 1 and Proposition 4, we may restrict our attention to fully stripped p -basic trees.

DEFINITION 9. Suppose X and Y are p -basic trees and $\theta: X \rightarrow Y$ is a height-preserving bijection which has the property that $\theta(px) = p\theta x$, except possibly when both $h(px)$ and $h(p\theta x)$ are limit ordinals or ∞ . Then we shall say that θ is a T -function.

Observe that the composition of two T -functions is a T -function; the inverse of a T -function is a T -function; and if both X and Y are fully stripped, and $\sigma: X \rightarrow Y$ is a stripping function, then both σ and σ^{-1} are T -functions.

For the purposes of the next several definitions and propositions, we shall assume that X and Y are fully stripped p -basic trees, and the symbol θ shall be reserved for a T -function.

PROPOSITION 5. The p -basic trees X and Y have the same Ulm invariants iff there is a T -function $\theta: X \rightarrow Y$.

PROOF. Suppose that X and Y have the same Ulm invariants; this is equivalent to the possession of bijections $\eta_\alpha: U(\alpha, X) \rightarrow U(\alpha, Y)$ for each ordinal α . Note if x is a nonzero element of X with height α and $x \notin U(\alpha, X)$, then there is a least positive integer n such that $p^n x \in U(\xi, X)$ for the ordinal $\xi = \alpha + n$. Thus in Y , there is an element y such that $p^n y = \eta(p^n x)$; since Y is fully stripped, y is unique. The T -function θ is defined as follows: set $\theta(0) = 0$, set $\theta(x) = \eta_\alpha(x)$ if $x \in U(\alpha, X)$, and set $\theta(x) = y$ if $x \notin U(\alpha, X)$ and y is as described above. The converse is easily verified. $\square$

Our immediate goal is to describe a T -function in terms of stripping functions. A certain partition of X will be necessary.

DEFINITION 10. Let $E = \{x \in X: p\theta x = \theta p x\}$. Suppose S is a subset of X which is the disjoint union of subsets K , L , $S \cap E$, and A , and e is a function from S to the nonnegative integers such that $e(0) = 0$, and for nonzero x in S ,

- (1) if $x \in K$ then $px \in S$ and $e(x) = e(px) + 1$,
- (2) if $x \in L$ then $\theta^{-1}(p\theta x) \in S$ and $e(x) = e(\theta^{-1}(p\theta x)) + 1$,
- (3) if $x \in S \cap E$ then $px \in S$ and $e(x) = e(px) + 1$, and
- (4) if $x \in A$ then $e(x) = 1$.

Under these conditions, we shall say that (S, e, K, L, A) is a *suitably partitioned subset of X* (or simply that S is suitably partitioned). If (S, e, K, L, A) and (S', e', K', L', A') are two suitably partitioned subsets of X , we shall say $S \leq S'$ if $S \subset S'$, $K \subset K'$, $L \subset L'$, $A \subset A'$, and e' is an extension of e . Clearly, $\leq$ is a partial order.

DEFINITION 11. If (S, e, K, L, A) is a suitably partitioned subset of X , we may construct a larger suitably partitioned subset by adjoining elements x in E for which $px \in S$, and then defining $e(x) = e(px) + 1$ for such elements. Denote by S^* the largest superset of S which can be obtained in this way, and let e^* denote the corresponding extension of e . Then (S^*, e^*, K, L, A) is a suitably partitioned subset of X .

LEMMA 2. Let (S, e, K, L, A) be a suitably partitioned subset of X , and F a finite subset of X which is the disjoint union of sets K_0 , L_0 , and A_0 , and for which $F \cap (S \cup E) = \emptyset$. Then we can construct a suitably partitioned subset $(S', e', K', L', A') \geq (S, e, K, L, A)$ such that $S \cup F \subset S'$, $K \cup K_0 = K'$, $L \cup L_0 = L'$, $A \cup A_0 \subset A'$, $S' \sim S$ is finite, and $S' \sim S \subset F \cup E$.

PROOF. By induction on the size of F , we may assume it consists of a single element x . If $x \in A_0$, set $e'(x) = 1$, $A' = A \cup A_0$, $K' = K$, $L' = L$, and $S' = S \cup F$. If $x \in K_0$, let m be the least positive integer such that $p^m x = 0$ or $p^m x \notin E$. If $p^m x \in S$, put $x, px, \dots, p^{m-1}x$ in S' and define $e'(p^i x) = e(p^m x) + m - i$ for $0 \leq i \leq m$. If $p^m x \notin S$, put $p^m x$ in A' , put $p^i x$ in S' for $0 \leq i \leq m$, and define $e'(p^i x) = 1 + m - i$. Define S' (and A') to include any elements we have added, in addition to S (to A). Define $K' = K \cup K_0$ and $L' = L \cup L_0$. If $x \in L_0$, let m be the least positive integer such that $\theta^{-1}(p^m \theta x) = 0$ or $\theta^{-1}(p^m \theta x) \notin E$. Proceed as for K_0 , considering $\theta^{-1}(p^i \theta x)$ instead of $p^i x$ for $0 \leq i \leq m$. $\square$

PROPOSITION 6. Suppose (S, e, K, L, A) is a suitably partitioned subset of X with these additional properties:

- (1) if $x \in L \cup A$ and $px \neq 0$, then there is an element u in $K \cup E$ such that $px = pu$ and $hu > hx$, and
- (2) if $x \in K \cup A$ and $p\theta x \neq 0$, then there is an element v in $L \cup E$ such that $p\theta x = p\theta v$ and $h\theta v > h\theta x$.

If $z \notin S$, we can construct a suitably partitioned subset $(S'', e'', K'', L'', A'') \geq (S, e, K, L, A)$ which contains z and for which (1) and (2) hold (with K'' , L'' , A'' replacing K , L , A).

PROOF. We shall construct increasing sequences $\{K_i\}_{i=0}^\infty$, $\{L_i\}_{i=0}^\infty$, $\{A_i\}_{i=0}^\infty$ of subsets of X , such that for each i , $F_i = K_i \cup L_i \cup A_i$ is a disjoint union and $F_i \cap (S \cup E) = \emptyset$. In addition:

(1') when i is odd and $x \in L_i \cup A_i$ with $px \neq 0$, there shall be an element u in $K_{i+1} \cup K \cup E$ so that $pu = px$ and $hu > hx$;

(2') when i is even and $x \in K_i \cup A_i$ with $p\theta x \neq 0$, there shall be an element v in $L_{i+1} \cup L \cup E$ so that $p\theta v = p\theta x$ and $h\theta v > h\theta x$.

Finally, for each $i > 0$ we shall have a suitably partitioned subset

$$(S_i, e_i, K \cup K_i, L \cup L_i, A \cup A_i) \\ \leq (S_{i+1}, e_{i+1}, K \cup K_{i+1}, L \cup L_{i+1}, A \cup A_{i+1}).$$

Since S^* satisfies (1) and (2) if S does, we shall assume $S = S^*$, and thus by induction on the exponent of z , we may assume $z \notin E$. Thus both hpz and $h(p\theta z)$ are limit ordinals or ∞ .

To begin, set $F_0 = \{z\}$ and apply Lemma 2, with $e_0 = e$, $A_0 = \{z\}$, and $K_0 = L_0 = \emptyset$. Then define $S_1 = S'$, $A_1 = A' \sim A$, $K_1 = K_0$, $L_1 = L_0$, and e_1 to be the extension of e_0 . Given $(S_i, e_i, K_i, L_i, A_i)$, to define F_{i+1} we proceed as follows. If i is odd, consider x in $L_i \cup A_i$ with $px \neq 0$; note then $h(px)$ is a limit ordinal. We can find an element u for which $px = pu$ and $hu > hx$, and since F_i is finite we may assume $u \notin F_i$. If $u \in S \sim (K \cup E)$ then $u \in L \cup A$ and so by (1), there is u' in $K \cup E$ with $pu' = pu$ and $hu' > hu$; thus (1') holds. If $u \notin S \cup E$, put u in K_{i+1} so that (1') will hold in any case. Define K_{i+1} to include all elements u added in this way, along with K_i . Define $L_{i+1} = L_i$. To define A_{i+1} , apply Lemma 2 to $F_i \cup K_{i+1}$; this gives a set A' for which $A' \sim A$ is finite; set $A_{i+1} = A_i \cup (A' \sim A)$. Now that K_{i+1} , L_{i+1} , and A_{i+1} are defined, set F_{i+1} to be their union. Apply Lemma 2 to obtain S_{i+1} and e_{i+1} .

If i is even, consider $K_i \cup A_i$ and proceed in a similar manner so that (2') will hold. It is clear that with $S'' = \bigcup S_i$, $K'' = \bigcup K_i$, $L'' = \bigcup L_i$, $A'' = \bigcup A_i$ and e'' defined in the obvious way, the result follows. $\square$

A Zorn's Lemma argument yields the next result.

PROPOSITION 7. *There are subsets K, L, A of X and a function e such that (X, e, K, L, A) is suitably partitioned. In addition,*

(1) *if $x \in L \cup A$ with $px \neq 0$, then there is an element u in $K \cup E$ such that $px = pu$ and $hu > hx$, and*

(2) *if $x \in K \cup A$ with $p\theta x \neq 0$, then there is an element v in $L \cup E$ such that $p\theta x = p\theta v$ and $h\theta v > h\theta x$.*

PROPOSITION 8. *The T -function θ is the composition of two strip-grafts.*

PROOF. Let e, K, L, A be such that (X, e, K, L, A) is suitably partitioned and (1) and (2) of Proposition 7 hold. For each x in X , let $x' = x$ and set $X' = \{x': x \in X\}$, with multiplication by p on X' defined by $p(x') = (px)$ if $x \in K \cup E$, $p(x') = 0$ if $x \in L \cup A$. With $\sigma_1: X \rightarrow X'$ the identity function,

Lemma 1 shows that X' is a p -basic tree. In view of (1), the set $\{x \in p^{-1}z: p\sigma_1(x) = \sigma_1(px)\}$ is dense for a nonzero z in X . Thus, Lemma 1 shows that σ_1 is a stripping function.

For each y in Y , let $y' = y$ and set $Y' = \{y': y \in Y\}$, with multiplication by p on Y' defined by $p(y') = (py)'$ if $\theta^{-1}y \in L \cup E$, $p(y') = 0$ if $\theta^{-1}y \in K \cup A$. Then Y' is a p -basic tree and the identity function $\sigma_2: Y \rightarrow Y'$ is a stripping function.

Now define $Z = \{(x, y) \in X \times Y: y = \theta x\}$, with multiplication by p in Z defined as $p(x, y) = (px, \theta(px))$ if $x \in K \cup E$; $p(x, y) = (\theta^{-1}py, py)$ if $x \in L \cup E$, and $p(x, y) = (0, 0)$ if $x \in A$. An easy induction shows that $e(x)$ is the exponent of the element (x, y) in Z , so that Z is a p -basic tree. If α is the height of (x, y) in Z , an induction on α shows that α is the common value $hx = hy$.

Let $\sigma_3: Z \rightarrow X'$ be defined by $\sigma_3(x, y) = x'$, and $\sigma_4: Z \rightarrow Y'$ be defined by $\sigma_4(x, y) = y'$. Note $p\sigma_3(x, y) = \sigma_3(p(x, y))$ for x in $K \cup E$, whereas $p\sigma_3(x, y) = 0$ for x in $L \cup A$. In view of this and (1), the set $\{(u, v) \in p^{-1}(x, y): p\sigma_3(u, v) = \sigma_3(p(u, v))\}$ is dense in $p^{-1}(x, y)$ for nonzero (x, y) . Consequently, σ_3 is a stripping function; similarly, σ_4 is a stripping function. Setting $\xi_2 = \sigma_3^{-1}\sigma_1$ and $\xi_1 = \sigma_2^{-1}\sigma_4$, we observe that $\theta = \xi_1\xi_2$, which completes the proof. $\square$

We now drop the assumption that X and Y are fully stripped. The remaining results follow easily from the preceding discussion.

THEOREM 4. *The p -basic trees X and Y are equivalent iff there is a p -basic tree Z such that:*

- (1) *the natural direct sum decompositions of $[X]$ and $[Z]$ have isomorphic refinements, and*
- (2) *the natural direct sum decompositions of $[Y]$ and $[Z]$ have isomorphic refinements.*

THEOREM 5 (ULM'S THEOREM FOR P -BASIC TREES). *Two p -basic trees are equivalent if and only if they have the same Ulm invariants.*

COROLLARY (ULM'S THEOREM FOR SIMPLY PRESENTED GROUPS). *Two simply presented groups are isomorphic if and only if they have the same Ulm invariants.*

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