# ESSENTIAL CENTRAL RANGE AND SELFADJOINT COMMUTATORS IN PROPERLY INFINITE VON NEUMANN ALGEBRAS([1])

BY

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ABSTRACT. The essential central range of an element $A$ of a von Neumann algebra with respect to a central ideal is characterized as those elements arbitrarily close to the compression of $A$ to a subspace large with respect to the ideal. The selfadjoint commutators in a properly infinite algebra are shown to be the elements whose essential central ranges with respect to the strong radical contain 0.

1. **Introduction.** Let $\mathcal{C}$ be a von Neumann algebra with center $\mathfrak{Z}$. Let $(\mathfrak{S})$ denote the set of nonzero projections in a subset $\mathfrak{S}$ of $\mathcal{C}$. A (closed two-sided) ideal $\mathfrak{I}$ of $\mathcal{C}$ is said to be a *central ideal* of $\mathcal{C}$ if given any bounded set $\{A_i\}$ in $\mathfrak{I}$ and a corresponding set $\{P_i\}$ of orthogonal projections in $(\mathfrak{Z})$ then the operator $\sum A_i P_i$ is in $\mathfrak{I}$. For an element $\zeta$ of the spectrum $Z$ of $\mathfrak{Z}$, let $[\zeta]$ denote the smallest ideal of $\mathcal{C}$ containing $\zeta$. Then the ideal $\mathfrak{I}$ is a central ideal if and only if the map

$$\zeta \to \|A(\mathfrak{I} + [\zeta])\| = \text{glb}\,\{\|A + B\|\,|\,B \in \mathfrak{I} + [\zeta]\}$$

is continuous on $Z$ for every $A$ in $\mathcal{C}$. If $P$ is a projection in $\mathfrak{Z}$ and if $E$ is a properly infinite projection in $\mathcal{C}P$ (by convention 0 is properly infinite), then the set of projections $F$ in $\mathcal{C}P$ such that $QF \succ EQ$ for some central projection $Q$ implies $EQ = 0$ is the set of all projections of a central ideal denoted by $\mathfrak{I}_P(E)$. All central ideals are of this form. If $\mathcal{C}P$ is the weak closure of the central ideal, then the form is *canonical* in the sense that $\mathfrak{I}_P(E) = \mathfrak{I}_Q(F)$ if and only if $P = Q$ and $E \sim F$ [12].

Now let $\mathfrak{I}$ be a central ideal of $\mathcal{C}$. If $A$ is an element in $\mathcal{C}$, there is a largest central projection $c_{\mathfrak{I}}(A)$ in $\mathcal{C}$ such that $c_{\mathfrak{I}}(A)A$ is in $\mathfrak{I}$. The projection $1 - c_{\mathfrak{I}}(1)$ is denoted by $P_{\mathfrak{I}}$. A projection $F$ in $\mathcal{C}$ is said to have *dimension greater* than the central ideal $\mathfrak{I} = \mathfrak{I}_P(E)$ (in symbols, dim $F$ > dim $\mathfrak{I}$) if $F$ has central support $P_{\mathfrak{I}}$ and if $F \succ EP_{\mathfrak{I}}$ or equivalently, if $F$ has central support $P_{\mathfrak{I}}$ and $c_{\mathfrak{I}}(F)$

$= 1 - P_\mathfrak{g}$. The *essential central spectrum* $\mathfrak{X} - \mathrm{Sp}_\mathfrak{g} A$ of an element $A$ in $\mathcal{C}$ modulo the central ideal $\mathfrak{g}$ is the set of all $C$ in $\mathfrak{X}$ such that $C\hat{\ }(\zeta)$ is in the spectrum of the image $A(\mathfrak{g} + [\zeta])$ of $A$ in $\mathcal{C}/(\mathfrak{g} + [\zeta])$ under the canonical homomorphism for every $\zeta$ in $Z$. Here $C\hat{\ }$ is the Gelfand transform of $C$. If $\mathfrak{g}$ is the ideal $(0)$, then the essential central spectrum of $A$ modulo $\mathfrak{g}$ is called the *central spectrum* of $A$ and is written $\mathfrak{X} - \mathrm{Sp}\, A$. The set $\mathfrak{X} - \mathrm{Sp}_\mathfrak{g} A$ is nonvoid and strongly closed. Let $\mathcal{C}^\sim$ be the Banach space of all bounded $\mathfrak{X}$-*module homomorphisms* of $\mathcal{C}$ into $\mathfrak{X}$ and let $\mathcal{C}^{\sim+}$ be the subset of all those $\phi$ in $\mathcal{C}^\sim$ that map $\mathcal{C}^+$ into $\mathfrak{X}^+$. An element $\phi$ in $\mathcal{C}^{\sim+}$ is called a *state* in $\mathcal{C}^\sim$ if $\phi(1) = 1$. For example, if $E$ is a *maximal abelian projection* (i.e., $E$ has central support 1) in the commutant $\mathfrak{X}'$ of $\mathfrak{X}$ and if $\tau_E(A)$ denotes the unique element in $\mathfrak{X}$ with $\tau_E(A)E = EAE$ for $A$ in $\mathcal{C}$, then the map $\tau_E$ is a state of $\mathcal{C}^\sim$. Let $E_a(\mathfrak{g})$ be the subset of $\mathcal{C}^{\sim+}$ given by

$$E_a(\mathfrak{g}) = \{\phi \in \mathcal{C}^{\sim+} | \phi(\mathfrak{g}) = 0, \phi(P_\mathfrak{g}) = P_\mathfrak{g}\}.$$

The *essential central range* $\mathcal{K}_\mathfrak{g}(A)$ of an element $A$ in $\mathcal{C}$ modulo the central ideal $\mathfrak{g}$ is defined to be the set

$$\mathcal{K}_\mathfrak{g}(A) = \{\phi(A) | \phi \in E_a(\mathfrak{g})\}.$$

The set $\mathcal{K}_\mathfrak{g}(A)$ is also equal to the set

$$\cap \{\text{unif clos } \{\tau_E(A + B) | E \text{ abelian of central support } P_\mathfrak{g}\} | B \in \mathfrak{g}\}$$

[12, 4.8], and is a weakly closed $\mathfrak{X}$-convex subset of $\mathfrak{X}$ containing $\mathfrak{X} - \mathrm{Sp}_\mathfrak{g} A$. Here a set $\mathcal{S}$ is $\mathfrak{X}$-*convex* if $CA_1 + (1 - C)A_2$ is in $\mathcal{S}$ whenever $C$ is in $\mathfrak{X}$ with $0 \leqslant C \leqslant 1$ and the $A_i$ are in $\mathcal{S}$ [12].

If $\mathcal{C}$ is a finite von Neumann algebra, then every central ideal is of the form $\mathcal{C}P$ for some central projection $P$. If $\mathcal{C}$ is a properly infinite von Neumann algebra, then the ideal $\mathfrak{g}_0$ generated by the finite projections is a central ideal and $\mathfrak{g}_0$ has canonical form

$$\mathfrak{g}_0 = \mathfrak{g}_1(E)$$

where $E$ is a projection of central support 1 for which there is an orthogonal set $\{P_i\}$ in $(\mathfrak{X})$ of sum 1 such that $EP_i$ is a $\sigma$-finite projection. If $\mathcal{C}$ is the set of all bounded operators on a Hilbert space, then $\mathfrak{g}_0$ is the ideal of compact operators. In general, we have that $\mathcal{K}_{\mathfrak{g}_0}(A)$ is equal to the intersection of $\mathfrak{X}$ with the weak closure of the convex hull of the set $UAU^*$ where $U$ runs through the set $U(\mathcal{C})$ of unitary operators in $\mathcal{C}$. The *strong radical* $\mathfrak{f}$ of $\mathcal{C}$ (i.e., the intersection of all maximal ideals of $(\mathcal{C})$) is also a central ideal of $\mathcal{C}$ and $\mathfrak{f}$ may be written as $\mathfrak{f} = \mathfrak{g}_1(1)$. In particular, the dimension of the projection $E$ is greater than that of $\mathfrak{f}$ if and only if $E \sim 1$. In this case the set $\mathcal{K}_\mathfrak{f}(A)$ is the

intersection of $\mathfrak{X}$ with the uniform closure of the convex hull of $\{UAU^*|U \in U(\mathcal{C})\}$. In the sequel, we write $\mathfrak{K}_{\mathcal{J}}(A)$ simply as $\mathfrak{K}(\mathcal{C})$ [10], [12].

In this note we show that $C$ is in the essential central range of $A$ modulo the central ideal $\mathcal{J}$ if and only if there is, for every $\varepsilon > 0$, a projection $E$ in $\mathcal{C}$ with $\dim E > \dim \mathcal{J}$ such that $\|E(C - A)E\| < \varepsilon$. For a properly infinite algebra $\mathcal{C}$, the operator $C$ is in $\mathfrak{K}_{\mathcal{J}}(A)$ if and only if there is a projection $E$ in $\mathcal{C}$ with $\dim E > \dim \mathcal{J}$ such that $CE$ is in the essential central spectrum of $EAE$ in the algebra $\mathcal{C}_E$ modulo the central ideal $\mathcal{J} \cap \mathcal{C}_E$. Here $\mathcal{C}_E$ is the von Neumann algebra $E\mathcal{C}E$ on the Hilbert space determined by $E$. Also the set $\mathfrak{K}_{\mathcal{J}_0}(A)$ is shown to be the intersection of the center with the weak closure of $\{UAU^*|U \in U(\mathcal{C})\}$. Using this characterization of the essential central range, we show by displaying a matrix form similar to that given by J. Anderson [1] that an element $A$ in a properly infinite von Neumann algebra $\mathcal{C}$ is a *selfadjoint commutator* in $\mathcal{C}$ (i.e., there are $B$ and $C$ in $\mathcal{C}$ with $A = BC - CB$ and $B = B^*$) if and only if 0 is in $\mathfrak{K}(A)$. We also characterize operators similar to selfadjoint commutators. Finally, we discuss the problem of characterizing commutators in properly infinite von Neumann algebras.

**2. Characterization of the essential central range.** In this section we show that a central element $C$ is in the essential range of $A$ modulo a central ideal $\mathcal{J}$ if and only if the compression of $C - A$ to a subspace that is large with respect to $\mathcal{J}$ is small.

LEMMA 2.1. *Let $C$ be in the essential central range of an element $A$ in a continuous von Neumann algebra $\mathcal{C}$ modulo the central ideal $\mathcal{J}$ of $\mathcal{C}$; then there is, for every $\varepsilon > 0$, an extreme point $\phi$ of $E_a(\mathcal{J})$ such that $\|\phi(A) - C\| < \varepsilon$ and such that the kernel of the canonical representation of the positive functional $\phi_\zeta$ given by $\phi_\zeta(B) = \phi(B)^\wedge(\zeta)$ is $\mathcal{J} + [\zeta]$ for every $\zeta$ in the spectrum $Z$ of the center $\mathfrak{X}$ of $\mathcal{C}$.*

PROOF. By performing a preliminary reduction, we may assume that $P_{\mathcal{J}} = 1$ so that the canonical representation $\mathcal{J}_P(E)$ of $\mathcal{J}$ has the property that the central support of $E$ is $P$ [11, 2.7]. The set $\mathfrak{S}$ of all positive $\mathfrak{X}$-module homomorphisms $\psi$ of $\mathcal{C}$ into $\mathfrak{X}$ such that $\psi(\mathcal{J}) = (0)$ and $\psi(E) = P$ is a nonvoid, $\mathfrak{X}$-convex subset of $\mathcal{C}^{\sim+}$ that is compact in the topology of pointwise convergence on $\mathcal{C}$ where $\mathfrak{X}$ is taken with the weak operator topology [12, 4.3]. If $\psi_1$ is an extreme point of $\mathfrak{S}$ and if $\psi_2$ is a pure (i.e., extreme) state of $\mathcal{C}^{\sim+}$ whose canonical representation is faithful [9, 4.6], then the $\mathfrak{X}$-module homomorphism $\phi = P\psi_1 + (1 - P)\psi_2$ is a pure state of $\mathcal{C}^{\sim+}$. We show that the kernel of the canonical representation $\pi_\zeta$ induced by the state $\varphi_\zeta$ is $\mathcal{J} + [\zeta]$. On the one hand, if $P^\wedge(\zeta) = 0$, then $\pi_\zeta(B) = 0$ if and only if

$$(\psi_2)_\zeta(D^*BD) = \phi_\zeta(D^*BD) = 0$$

for all $D$ in $\mathcal{C}$, or equivalently, if and only if $B$ is in the kernel $[\zeta] = \mathcal{I} + [\zeta]$ of the canonical representation of $(\psi_2)_\zeta$ [9, Theorem 4.7]. On the other hand, if $P^\wedge(\zeta) = 1$, then it is clear that the ideal $\mathcal{I} + [\zeta]$ is contained in the kernel of $\pi_\zeta$. Now supposing $B$ is not in $\mathcal{I} + [\zeta]$, we may find a spectral projection $F$ of $B^*B$ corresponding to an interval of the form $[\alpha, \|B\|^2]$ with $\alpha > 0$ such that $F$ is not in $\mathcal{I} + [\zeta]$. We have that $c_\mathcal{I}(F)^\wedge(\zeta) = 0$; otherwise, the projection

$$F = c_\mathcal{I}(F)F + (1 - c_\mathcal{I}(F))F$$

is in $\mathcal{I} + [\zeta]$. Because

$$(1 - c_\mathcal{I}(F))F \succ (1 - c_\mathcal{I}(F))E,$$

there is a partial isometry $U$ in $\mathcal{C}$ with $U^*U = (1 - c_\mathcal{I}(F))E$ and $UU^* \leqslant (1 - c_\mathcal{I}(F))F$. The fact that

$$\phi_\zeta(U^*U) = \phi_\zeta(E) = 1$$

implies that $\pi_\zeta(U) \neq 0$ and consequently that

$$\pi_\zeta(B^*B) \geqslant \alpha\pi_\zeta(F) \geqslant \alpha\pi_\zeta(UU^*) > 0.$$

This proves that the kernel of $\pi_\zeta$ is $\mathcal{I} + [\zeta]$.

Now let $C$ be in $\mathcal{K}_\mathcal{I}(A)$. There is a $\psi$ in $E_a(\mathcal{I})$ with $\psi(A) = C$. For every $\zeta$ in $Z$, there is a state $\theta_\zeta$ of $\pi_\zeta(\mathcal{C})$ such that $\psi_\zeta = \theta_\zeta\pi_\zeta$. Since $\pi_\zeta$ is an irreducible representation of the continuous algebra $\mathcal{C}$ [9, Corollary, Theorem 4.3], there is a unitary operator $U$ in $\mathcal{C}$ such that

(1)          $|\theta_\zeta(\pi_\zeta(A)) - (\pi_\zeta(A)\pi_\zeta(U)x_\zeta, \pi_\zeta(U)x_\zeta)| < \varepsilon$

[8, Theorem 2]. Here $x_\zeta$ denotes a unit vector in the Hilbert space of $\pi_\zeta$ such that $(\pi_\zeta(B)x_\zeta, x_\zeta) = \phi_\zeta(B)$ for every $B$ in $\mathcal{C}$. Due to the continuity of the left hand side of (1) as a function of $\zeta$, we may use the fact that $Z$ is extremally disconnected to find a unitary $U$ in $\mathcal{C}$ such that $\|\psi(A) - \phi(U^*AU)\| < \varepsilon$. The state $B \to \phi(U^*BU)^\wedge(\zeta)$ of $\mathcal{C}$ is a pure state whose canonical representation has kernel $\mathcal{I} + [\zeta]$.   Q.E.D.

We now give a characterization of the essential central range. This was previously known only for selfadjoint elements [12, 4.9].

THEOREM 2.2. *In order that an element $C$ in the center $\mathcal{Z}$ of a von Neumann algebra $\mathcal{C}$ be in the essential central range of an element $A$ in $\mathcal{C}$ with respect to the central ideal $\mathcal{I}$, a necessary and sufficient condition is that $CP_\mathcal{I} = C$ and that, for every $\varepsilon > 0$, there is a projection $F$ in $\mathcal{C}$ with $\dim F > \dim \mathcal{I}$ such that $\|F(C - A)F\| < \varepsilon$.*

PROOF. Suppose first that $C$ is in $\mathfrak{K}_{\mathfrak{J}}(A)$. It is clear from the definition of $\mathfrak{K}_{\mathfrak{J}}(A)$ that $CP_{\mathfrak{J}} = C$. There is no loss of generality in assuming that $C = 0$ and $P_{\mathfrak{J}} = 1$, and that $\mathcal{Q}$ is either a continuous, or a discrete finite or properly infinite algebra.

First we assume $\mathcal{Q}$ is continuous. Given $\varepsilon > 0$, there is an extreme point $\phi$ of $E_a(\mathfrak{J})$ such that $\|\phi(A)\| < \varepsilon/3$, and such that the kernel of the canonical representation $\pi_{\zeta}$ induced by $\phi_{\zeta}$ on the Hilbert space $H_{\zeta}$ is $\mathfrak{J} + [\zeta]$ for every $\zeta$ in the spectrum $Z$ of $\mathfrak{Z}$ (Lemma 2.1). Let $D = A - \phi(A)$ and let $\{e_i | i \in I\}$ be a complete orthonormal base of $H_{\zeta}$ so that

$$\phi_{\zeta}(B) = (\pi_{\zeta}(B)e_1, e_1)$$

for every $B$ in $\mathcal{Q}$. Here we are assuming $1 \in I$. For every $S$ in the family $\mathfrak{F}(I)$ of all finite subsets of $I - \{1\}$ directed by the inclusion relation, we may find a projection $E(S)$ in $\mathcal{Q}$ such that

$$\pi_{\zeta}(E(S))e_1 = e_1 \quad \text{and} \quad \pi_{\zeta}(E(S))e_i = 0 \qquad (i \in S)$$

due to the fact $\pi_{\zeta}$ is irreducible (Lemma 2.1). Because the bounded net $\{\pi_{\zeta}(E(S)DE(S)) | S \in \mathfrak{F}(I)\}$ converges to 0 in the strong operator topology on $H_{\zeta}$, there is a set $S$ in $\mathfrak{F}(I)$ with

$$(2) \qquad \qquad \|\pi_{\zeta}(E(S)DE(S))e_1\| < \varepsilon/3.$$

The projection $c_{\mathfrak{J}}(E)$, where $E = E_{\zeta}$ is the spectral projection of $E(S)D^* E(S)DE(S)$ in the von Neumann algebra $\mathcal{Q}_{E(S)}$ corresponding to the interval $[0, \varepsilon^2/3]$, is in the ideal $\zeta$; otherwise, the projection $E$ is in the kernel $\mathfrak{J} + [\zeta]$ of $\pi_{\zeta}$ and a relation

$$\pi_{\zeta}(E(S)D^* E(S)DE(S))$$
$$= \pi_{\zeta}((E(S) - E)E(S)D^* E(S)DE(S)(E(S) - E))$$
$$\geqslant 3^{-1}\varepsilon^2 \pi_{\zeta}(E(S) - E) \geqslant 3^{-1}\varepsilon^2 \pi_{\zeta}(E(S))$$

incompatible with the relation (2) arises. Since the map

$$\xi \to \|\pi_{\xi}(EDE)\| = \|EDE(\mathfrak{J} + [\xi])\|$$

is continuous on $Z$, we can use the fact that $Z$ is extremally disconnected to find a projection $E$ in $\mathcal{Q}$ such that $c_{\mathfrak{J}}(E) = 0$ and such that

$$\|EDE(\mathfrak{J})\| = \text{lub} \{\|EDE(\mathfrak{J} + [\zeta])\| | \zeta \in Z\} < \varepsilon/3$$

(cf. [12, 3.1]). If $B$ is an element in $\mathfrak{J}$ with $EBE = B$ and

$$\|EDE - B\| < \varepsilon/3,$$

and if $F$ is a projection in $\mathfrak{g}$ majorized by $E$ such that

$$\|B(E - F)\| < \varepsilon/3,$$

then we have that $c_{\mathfrak{g}}(E - F) = 0$, or equivalently, that $\dim(E - F) > \dim \mathfrak{g}$, and that

$$\|(E - F)A(E - F)\| \leqslant \|EDE - B\| + \|B(E - F)\| + \|\phi(A)\| < \varepsilon.$$

Now assume that $\mathcal{C}$ is a type I algebra. There is no loss of generality in the assumption that the commutant $\mathcal{C}'$ of $\mathcal{C}$ is equal to $\mathfrak{L}$. If $\mathcal{C}$ is a finite algebra, the assumption that $P_{\mathfrak{g}} = 1$ implies $\mathfrak{g} = (0)$ [12, 2.3]. This means that there is a maximal abelian projection $E$ of $\mathcal{C}$ such that

$$\|EAE\| = \|\tau_E(A)E\| = \|\tau_E(A)\| < \varepsilon$$

[12, 4.8]. If $\mathcal{C}$ is properly infinite type I algebra, let $\{E_i\}$ be a maximal set of nonzero orthogonal abelian projections in $\mathcal{C}$ such that

$$\|E_i A E_j\| \leqslant 2^{-1}\delta_{ij}\varepsilon$$

for all $i, j$. Here $\delta_{ij}$ is the Kronecker delta. Setting $E = \sum E_i$, we get that

$$\|EAE\| = \mathrm{lub}\, \|E_i A E_i\| \leqslant \varepsilon/2.$$

We complete the first part of the proof by showing that $c_{\mathfrak{g}}(E) = 0$. On the contrary, if $P = c_{\mathfrak{g}}(E)$ is nonzero, then the least upper bound $F$ of the range projections of $EP$, $AEP$, and $A^* EP$ is in $\mathfrak{g}$ [7, III, 1, Proposition 2]. If $\phi$ is $E_a(\mathfrak{g})$ with $\phi(A) = 0$, then the map $\psi$ given by

$$\psi(B) = (P - F)\phi(B)$$

is a positive $\mathfrak{L}(P - F)$-module homomorphism of the von Neumann algebra $\mathcal{C}_{(P-F)}$ into its center $\mathfrak{L}(P - F)$ which vanishes on the central ideal $\mathfrak{g}_1 = \mathfrak{g} \cap \mathcal{C}_{(P-F)}$ of $\mathcal{C}_{(P-F)}$ and takes $P_{\mathfrak{g}_1} = P - F$ into $P - F$. There is a nonzero abelian projection $G$ in $\mathcal{C}_{(P-F)}$ such that

$$\|GAG\| \leqslant \varepsilon/2$$

due to the fact that

$$\psi((P - F)A(P - F)) = 0$$

implies that $0$ is in $\mathfrak{K}_{\mathfrak{g}_1}((P - F)A(P - F))$. Since $G$ is also a nonzero abelian projection in $\mathcal{C}$ and since

$$GE_i = GAE_i = E_iAG = 0$$

for all $i$, the set $\{E_i\}$ is not maximal. This is a contradiction. Hence, we have that $c_{\mathfrak{g}}(E) = 0$.

The converse is already known [12, 4.9].   Q.E.D.

REMARK. If $C$ is in the essential central range of an operator $A$ with respect to the strong radical in a properly infinite algebra $\mathfrak{A}$, then we have shown elsewhere [11, Proof of Theorem 4] that there is an invertible $B$ in $\mathfrak{A}$ and a projection $E$ equivalent to 1 such that $EB^{-1}ABE = CE$.

The next result answers a question of L. Zsidó [15].

COROLLARY 2.3. *Let $A$ be an element in a properly infinite von Neumann algebra $\mathfrak{A}$ with center $\mathfrak{X}$; then the intersection of $\mathfrak{X}$ with the sets*

$$\mathfrak{S}_1 = weak\ closure\ [convex\ \{UAU^* | U \in U(\mathfrak{A})\}]$$

*and*

$$\mathfrak{S}_2 = weak\ closure\ \{UAU^* | U \in U(\mathfrak{A})\}$$

*coincide. Here $U(\mathfrak{A})$ is the set of unitary operators of $\mathfrak{A}$.*

PROOF. The ideal $\mathfrak{g}_0$ of $\mathfrak{A}$ generated by the finite projections of $\mathfrak{A}$ is a central ideal and $\mathfrak{S}_1 \cap \mathfrak{X}$ is equal to $\mathfrak{K}_{\mathfrak{g}_0}(A)$ [12, 4.16]. If $C$ is in $\mathfrak{K}_{\mathfrak{g}_0}(A)$, then there is, for every $n = 1, 2, \ldots$, a projection $E_n$ with $\dim E_n > \dim \mathfrak{g}_0$ such that

$$\|E_n(C - A)E_n\| < n^{-1}$$

(Theorem 2.2). Each projection $E_n$ is properly infinite with central support 1. We may assume that, for each $n$, there is a set $\{P_i | i \in I\}$ orthogonal central projections of sum 1 such that each $P_i$ is the sum of an infinite set $\{E_{ij} | j \in J_i\}$ equivalent orthogonal projections such that $E_{ij} = E_n P_i$ for some $j$ in $J_i$ [7, III, 8, Theorem 1, Corollary 2 and III, 1, Theorem 1, Corollary 1]. Let $\mathfrak{X}$ be a finite set of unit vectors in the Hilbert space of $\mathfrak{A}$. There is a finite subset $S$ of $I$, and for each $i \in S$, a finite subset $S_i$ of $J_i$ such that

$$\|(1 - \sum \{P_i E_{ij} | j \in S_i, i \in S\})x\| \leqslant (2(\|C - A\| + 1)n)^{-1}$$

for all $x$ in $\mathfrak{X}$. There is, for every $i$ in $S$, a unitary operator $U_i$ in $\mathfrak{A} P_i$ that carries $F_i = E_n P_i$ onto $G_i = \sum \{P_i E_{ij} | j \in S_i\}$ and $P_i - F_i$ onto $P_i - G_i$ because $F_i \sim G_i$ and $P_i - F_i \sim P_i - G_i$. Thus there is a unitary operator $U = U(\mathfrak{X}, n)$ in $\mathfrak{A}$ such that $UP_i = U_i$ for $i \in S$. Setting $G = \sum \{G_i | i \in S\}$ and $P = \sum \{P_i | i \in S\}$, we get that

$$|((C - A)Ux, Ux)|$$

$$\leqslant 2\|(C - A)\| \, \|(1 - G)x\| + |((C - A)E_n PUx, E_n PUx)|$$

$$\leqslant n^{-1} + n^{-1} = 2n^{-1}$$

for every $x \in \mathfrak{X}$. This proves that $C$ is in the set $\mathfrak{S}_2 \cap \mathfrak{X}$. Since the relation $\mathfrak{S}_2 \cap \mathfrak{X} \subset \mathfrak{S}_1 \cap \mathfrak{X}$ is clear, we see that $\mathfrak{S}_1 \cap \mathfrak{X} = \mathfrak{S}_2 \cap \mathfrak{X}$.　　Q.E.D.

We now prove a stronger version of Theorem 2.2 that will be needed to characterize selfadjoint commutators. For this we need to study the spectral resolution of normal operators in a von Neumann algebra.

DEFINITION 2.4. Let $\Omega$ be the spectrum of the abelian von Neumann algebra $\mathfrak{B}$ generated by the center $\mathfrak{X}$ of a von Neumann algebra $\mathcal{C}$ and a selfadjoint element $A$ in $\mathcal{C}$; for every element $C$ in the set $\mathfrak{X}_h$ of selfadjoint elements of $\mathfrak{X}$, let $E_A(C)$ denote the projection in $\mathfrak{B}$ whose Gelfand transform is the characteristic function of the open and closed subset closure $\{\omega \in \Omega | \hat{A}(\omega) < \hat{C}(\omega)\}$ of $\Omega$.

We notice that $\{E_A(\lambda) | -\infty < \lambda < \infty\}$ is the spectral resolution of $A$.

PROPOSITION 2.5. *Let $A$ be a selfadjoint operator in the von Neumann algebra $\mathcal{C}$ with center $\mathfrak{X}$ and let $\mathfrak{I}$ be a central ideal of $\mathcal{C}$. The map*

$$C \to E_A(C) = E(C)$$

*of $\mathfrak{X}_h$ into the lattice of projections of the von Neumann algebra $\mathfrak{B}$ generated by $\mathfrak{X}$ and $A$ satisfies the following properties*:

(i) $PE(C) = PE(CP)$ *for every $C$ in $Z_h$ and every projection $P$ in $\mathfrak{X}$*;

(ii) $E(\text{lub } \mathfrak{X}) = \text{lub } \{E(C) | C \in \mathfrak{X}\}$ *for every subset $\mathfrak{X}$ of $\mathfrak{X}_h$ that is bounded above*;

(iii) $(A - C)(\text{glb } \{E(C') | C' > C\} - E(C)) = 0$ *for every $C$ in $\mathfrak{X}_h$; and*

(iv) $c_{\mathfrak{I}}(E(C + D) - E(C - D)) = 0$ *for every $D > 0$ in $\mathfrak{X}_h$ if and only if $C \in \mathfrak{X} - \text{Sp}_{\mathfrak{I}}A$.*

*Here $C > D$ means that the set $\{\omega \in \Omega | \hat{C}(\omega) > \hat{D}(\omega)\}$ is dense in $\Omega$.*

PROOF. (i) Clear.

(ii) This is clear if $\mathfrak{X}$ is a monotonely increasing net in $\mathfrak{X}$. In general, for any finite subset $C_1, \ldots, C_n$ of $\mathfrak{X}$, there are projections $P_1, \ldots, P_n$ in $\mathfrak{X}$ of sum 1 such that $\text{lub } C_i = \sum P_i C_i$. Using part (i) and the fact that $E(C) \leqslant E(D)$ for $C \leqslant D$ in $\mathfrak{X}_h$, we have that

$$\text{lub } E(C_i) \leqslant E(\text{lub } C_i) = \sum P_i E(C_i) \leqslant \text{lub } E(C_i).$$

This means that

$$E(\text{lub } \mathfrak{X}) = E(\text{lub } \{\text{lub } C_i | \{C_i\} \text{ is a finite subset of } \mathfrak{X}\})$$

$$= \text{lub } \{E(\text{lub } C_i) | \{C_i\} \text{ is a finite subset of } \mathfrak{X}\}$$

$$= \text{lub } \{E(C) | C \in \mathfrak{X}\}.$$

(iii) Let $C' = C + n^{-1}$. Then we have that

$$0 \leqslant (A - C)(E(C') - E(C)) \leqslant n^{-1}.$$

Thus we have

$$(A - C) \text{ glb } \{E(C') - E(C) | C' > C\} = 0.$$

(iv) Let $C$ be in $\mathfrak{X} - \text{Sp}_{\mathfrak{g}} A$ and let $\{P_n\}$ be a sequence of orthogonal projections in $\mathfrak{X}$ of sum 1 such that $DP_n \geqslant n^{-1}P_n$ for every $n = 1, 2, \ldots$. Because

$$P(E(C + D) - E(C - D)) \geqslant P(E(P(C + n^{-1})) - E(P(C - n^{-1})))$$

for every central projection $P$ majorized by $P_n$ and because $CP$ is in $\mathfrak{X}P - \text{Sp}_{\mathfrak{g}P} AP$ [12, 3.10], it is sufficient to show that the assumption that $(E(C + n^{-1}) - E(C - n^{-1})) = E$ is in $\mathfrak{g}$ leads to a contradiction. In fact, we have that $(C - A)(1 - E)$ is invertible in $\mathfrak{B}(1 - E)$ since $(C - A)\hat{}(\omega) \neq 0$ for every $\omega$ in $\Omega$ with $(1 - E)\hat{}(\omega) = 1$. If $B$ is the inverse of $(C - A)(1 - E)$ in $\mathfrak{B}(1 - E)$, then $B$ is the inverse of $C - A$ modulo $\mathfrak{g}$. This contradicts the definition of the essential central spectrum. Thus, we have that $c_{\mathfrak{g}}(E) = 0$.

Conversely, let $C$ in $\mathfrak{X}_h$ be such that $c_{\mathfrak{g}}(E(C + D) - E(C - D)) = 0$ for every $D$ in $\mathfrak{X}_h$ with $D > 0$. For every $\zeta$ in the spectrum of $\mathfrak{X}$ and every $n = 1, 2, \ldots$, the projection $E(C + n^{-1}) - E(C - n^{-1})$ is not in $\mathfrak{g} + [\zeta]$ [12, 3.1] and

$$\|(C\hat{}(\zeta) - A)(E(C + n^{-1}) - E(C - n^{-1}))(\mathfrak{g} + [\zeta])\| \leqslant n^{-1}.$$

Hence, each $C\hat{}(\zeta)$ is in the spectrum of the canonical image of $A$ in $\mathfrak{A}/(\mathfrak{g} + [\zeta])$.  Q.E.D.

COROLLARY 2.6. *Let $A$ be a normal element in a von Neumann algebra $\mathfrak{A}$ with center $\mathfrak{X}$ and let $A_1$ (resp. $A_2$) be $2^{-1}(A + A^*)$ (resp. $(2i)^{-1}(A - A^*)$). Then an element $C$ in $\mathfrak{X}$ is in the essential central spectrum of $A$ modulo the central ideal $\mathfrak{g}$ if and only if*

$$c_{\mathfrak{g}}((E_{A_1}(C_1 + D_1) - E_{A_1}(C_1 - D_1))(E_{A_2}(C_2 + D_2) - E_{A_2}(C_2 - D_2))) = 0$$

*for every $D_1, D_2 > 0$ in $\mathfrak{X}$. Here*

$$C_1 = 2^{-1}(C + C^*) \quad and \quad C_2 = (2i)^{-1}(C - C^*).$$

The next result is needed in §3.

COROLLARY 2.7. *If $A$ is a selfadjoint operator in a type* $II_1$ *von Neumann algebra* $\mathcal{C}$, *then there are two equivalent orthogonal projections of sum 1 that commute with* $A$.

PROOF. The set $\mathfrak{X}$ of all selfadjoint elements $C$ in the center $\mathfrak{Z}$ of $\mathcal{C}$ such that the value of the canonical operator-valued trace $\Phi$ of $\mathcal{C}$ (cf. [7, III, 4, Definition 2]) does not exceed $\frac{1}{2}$ at the point $E(C) = E_A(C)$ is a bounded monotonely increasing net in $\mathfrak{Z}$ (Proposition 2.5(i) and (ii)). We get an element $C_0 = \text{lub } \mathfrak{X}$ such that $\Phi(E(C_0)) \leqslant \frac{1}{2}$ and such that $\Phi(E(C)) > \frac{1}{2}$ for every $C > C_0$ (Proposition 2.5(ii)). There is a projection $E$ majorized by

$$F = \text{glb } \{E(C) - E(C_0)|C > C_0\}$$

such that

$$\Phi(E) = \frac{1}{2} - \Phi(E(C_0))$$

because $\mathcal{C}$ is a continuous algebra. Since $E$ commutes with $A$ due to the fact $AF = C_0 F$ (Proposition 2.5(iii)), the projections $E(C_0) + E$ and its orthogonal complement are equivalent orthogonal projections that commute with $A$. Q.E.D.

The next two lemmas are phrased in sufficient generality so that they will be applicable to the matrix decomposition in §3.

LEMMA 2.8. *Let $E_i$ $(i = 1, 2)$ be projections of central support 1 in the continuous von Neumann algebra $\mathcal{C}$ and let $\phi$ be a $\sigma$-weakly continuous state of $\mathcal{C}^{\sim}$ such that $\phi(E_1) \geqslant \alpha > 0$. Then for any $\beta$ in the interval $]0, \alpha[$ and any $A$ in $\mathcal{C}$, there are projections $F_i$ $(i = 1, 2)$ of central support 1 majorized by the $E_i$ respectively such that $\phi(F_1) \geqslant \beta$ and $F_2 A F_1 = 0$.*

PROOF. Let $\mathfrak{Z}$ be the center of $\mathcal{C}$. Given any $P$ in $(\mathfrak{Z})$ it is sufficient to show that there is a $Q$ in $(\mathfrak{Z}P)$ and projections $F_i$ of central support $Q$ majorized by the respective $E_i$ such that $\phi(F_1) \geqslant \beta Q$ and $F_2 A F_1 = 0$. There is no loss of generality in the assumption that $P = 1$.

We may assume that $E_2 A E_1$ is not 0. Let $U$ be a partial isometry and $B$ a positive operator in $\mathcal{C}$ so that $UB$ is the polar decomposition of $E_2 A E_1$. We may assume that the central support of the range projection $F$ of $B$ is 1 and that the central support $P$ of

$$E = \text{glb } \{E_B(C') - E_B(C)|C' > C\},$$

where $C$ is in $\mathfrak{Z}F - \text{Sp } B$ in $\mathcal{C}_F$, is either 0 or 1. On the one hand, if $P = 0$, then the strong limit of the monotonely decreasing sequence

$$G_n = E_B(C + n^{-1}) - E_B(C - n^{-1}) \qquad (n = 1, 2, \dots)$$

is 0 and so we can find a number $n$ and a $Q$ in $(\mathfrak{Z})$ such that

$$Q\phi(G_n) \leqslant (\alpha - \beta)Q,$$

and such that $FQ - G_n Q$ has central support $Q$. But the projection $G_n Q$ also has central support $Q$ (Proposition 2.5(iv)). Hence the projections

$$F_1 = Q(E_1 - G_n) \quad \text{and} \quad F_2 = UG_n U^* Q$$

are projections of central support $Q$ such that $\phi(F_1) \geqslant \beta Q$ and $F_2 A F_1 = 0$. On the other hand, if $P = 1$, then we may write $E$ as the sum of an infinite sequence $\{G'_n\}$ of orthogonal projections of central support 1 which commutes with $B$ (Proposition 2.5(iii)). There is a $Q$ in $(\mathfrak{Z})$ with $\phi(G'_n)Q \leqslant (\alpha - \beta)Q$. Setting $F_1 = (E_1 - G'_n)Q$ and $F_2 = UG'_n U^*$, we get two projections with $\phi(F_1) \geqslant \beta Q$ and $F_2 A F_1 = 0$.   Q.E.D.

LEMMA 2.9. *For any finite subset $\mathfrak{X}$ of a properly infinite von Neumann algebra $\mathfrak{A}$ and any sequence $\{E_n\}$ of properly infinite projections of central support 1, there is a sequence $\{E'_n\}$ of equivalent projections of central support 1 such that $E'_m \leqslant E_m$ and $E'_n A E'_m = 0$ for every $1 \leqslant m < n < \infty$ and $A$ in $\mathfrak{X}$.*

PROOF. We may assume that $\mathfrak{A}$ is either purely infinite or semifinite. First assume $\mathfrak{A}$ is purely infinite. By passing to a direct summand of $\mathfrak{A}$ and to subprojections of the $E_n$ if necessary, we may assume that each $E_n$ is $\sigma$-finite [7, III, 1, Lemma 7]. Since the least upper bound of the $E_n$ is $\sigma$-finite, we may assume $\mathfrak{A}$ is $\sigma$-finite by reducing to $\mathfrak{A}_E$ if necessary. Applying Lemma 2.7 recursively, we may find projections

$$\{E_{nm} | m \geqslant n\} \qquad (n = 1, 2, \dots)$$

in $\mathfrak{A}$ and corresponding maximal abelian projections $\{F_n\}$ in the commutant of the center $\mathfrak{Z}$ of $\mathfrak{A}$ such that

$$\begin{aligned}
E_{nm} \leqslant E_{np} \leqslant E_n \qquad & \text{for } n \leqslant p \leqslant m; \\
E_{nn} A E_{mn} = 0 \qquad & \text{for } 1 \leqslant m < n \text{ and } A \text{ in } \mathfrak{X};
\end{aligned}$$

and

$$\tau_n(E_{n\,n+k}) \geqslant 2 - \sum \{3^{-j} | 0 \leqslant j \leqslant k\} \text{ for } k = 0, 1, \dots.$$

Here $\tau_n = \tau_{F_n}$. Setting $E'_n = \text{glb}_m E_{nm}$, we get that

$$\tau_n(E'_n) = \text{glb}\,\tau_n(E_{nm}) \geqslant 2^{-1}$$

so that $E'_n$ is equivalent to 1 [7, III, 8, Theorem 1, Corollary 5] and that $E'_n A E'_m = 0$ for $1 \leqslant m < n < \infty$ and $A$ in $\mathfrak{X}$.

Now let $\mathcal{A}$ be semifinite. Suppose we have found equivalent finite projections $E'_1, E'_2, \ldots, E'_n$ of central support 1 such that $E'_m \leqslant E_m$ for $m = 1, 2, \ldots, n$ and $E'_p A E'_m = 0$ for $1 \leqslant m < p \leqslant n$ and $A$ in $\mathcal{X}$. The least upper bound $F$ of the range projections of $E_{n+1} A E'_m$ for $1 \leqslant m \leqslant n$ and $A$ in $\mathcal{X}$ is a finite projection, and so there is a projection $E'_{n+1}$ with $E'_n \sim E'_{n+1} \leqslant E_{n+1} - F$. By induction the required sequence $\{E'_n\}$ exists.   Q.E.D.

We now obtain the desired extension of Theorem 2.2.

THEOREM 2.10. *Let $\mathcal{A}$ be a von Neumann algebra with center $\mathcal{Z}$, and let $\mathcal{J} = \mathcal{J}_Q(G)$ be a central ideal of $\mathcal{A}$ in canonical form. Suppose that $\mathcal{A}(1 - Q)$ is a continuous von Neumann algebra. Let $0$ be in the essential central spectrum modulo $\mathcal{J}$ of an element $A$ in $\mathcal{A}$. For any sequence $\{\alpha_n\}$ of positive numbers and any finite subset $\mathcal{X}$ of $\mathcal{A}$, there is a sequence $\{E_n\}$ of projections in $\mathcal{A}$ such that*

(1) $\dim E_n > \dim \mathcal{J}$;

(2) $E_n B E_m = 0$ *for all* $1 \leqslant m < n < \infty$ *and $B$ in $\mathcal{X}$; and*

(3) $\|E_n A E_n\| \leqslant \alpha_n$ *for all $n$.*

*In particular, if $1$ is in $\mathcal{X}$, then the projections $\{E_n\}$ are orthogonal.*

REMARK. For the applications in §3 it is important to notice that the strong radical $\mathcal{J}$ of a properly infinite algebra $\mathcal{A}$ satisfies the hypothesis of Theorem 2.10. We recall, in fact, that $\mathcal{A}Q$ is the weak closure of the central ideal $\mathcal{J}_Q(G)$ (cf. Introduction) and so $\mathcal{A}(1 - Q)$ is of type III or (0) if $\mathcal{J}_Q(G) = \mathcal{J}$ because $\mathcal{J}$ contains all finite projections.

If $\mathcal{A}$ has a nonzero type I direct summand $\mathcal{A}P$ with $\mathcal{J}P = (0)$ and $E$ is an abelian projection of central support $P$, there is no orthogonal sequence $\{E_n\}$ of projections of central support $P$ that satisfies (3) for $\alpha_n = 2^{-1}$ ($n = 1, 2, \ldots$) and $A = 1 - E$.

PROOF. We may assume that $P_{\mathcal{J}} = 1$, and by passing to direct summands, that $\mathcal{A}$ is continuous and $\mathcal{J} = (0)$ or that the support of $\mathcal{J}$ is 1.

If $\mathcal{A}$ is continuous and $\mathcal{J} = (0)$, then there is a sequence $\{F_n\}$ of projections with central support 1 such that $\|F_n A F_n\| \leqslant \alpha_n$. The desired sequence $\{E_n\}$ may be constructed in a recursive fashion from Lemma 2.8 (cf. Proof, Lemma 2.9).

Now suppose the support of $\mathcal{J}$ is 1. We also suppose that $\mathcal{X}$ contains the identity 1. There is a sequence $\{F_n\}$ of projections in $\mathcal{A}$ with $\dim F_n > \dim \mathcal{J}$ and $\|F_n A F_n\| \leqslant \alpha_n$ for every $n = 1, 2, \ldots$ (Theorem 2.2). The projections $F_n$ must be properly infinite projections of central support 1 since every finite projection is in $\mathcal{J}$ (12, 2.2). Let $\mathbb{S}$ be the set of all families

$$S = \{E_{ni} | (n, i) \in N \times I\}$$

of nonzero projections in $\mathcal{A}$ with the following properties

(i) $E_{mi} \sim E_{ni}$ for all $m, n$ in $N$ and $i \in I$;

(ii) $E_{nj}BE_{mi} = 0$ for all distinct pairs $(m,i)$, $(n,j)$ in $N \times I$ and $B$ in $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}^* \cup \{A, A^*\}$; and

(iii) $\|E_{ni}AE_{ni}\| \leqslant \alpha_n$ for $n$ in $N$ and $i \in I$.

Here $\mathcal{X}^* = \{B^* | B \in \mathcal{X}\}$ and $N$ is the set of natural numbers. The set $\mathcal{S}$ is nonvoid, and in fact, for every nonzero central projection $P$, there is a sequence $\{E_n'\}$ of equivalent orthogonal projections of central support $P$ such that $E_n' \leqslant F_n$ and $E_n'BE_m' = 0$ for all $1 \leqslant m < n < \infty$ and $B$ in $\mathcal{Y}$ (Lemma 2.9). By Zorn's lemma we may find a maximal set $S = \{E_{ni} | (n,i) \in N \times I\}$ in the partial ordering on $\mathcal{S}$ given by the relation

$$(3) \qquad \{E_{ni}' | (n,i) \in N \times I'\} < \{E_{ni}'' | (n,i) \in N \times I''\}$$

if and only if $I' \subset I''$ and $E_{ni}' = E_{ni}''$ for $i \in I'$. For convenience, let $k$ be a point not in $I$. Let $\{N_m | m \in N\}$ be a partition of the natural numbers into countably many disjoint infinite sets such that the smallest number in the set $N_m$ is not smaller than $m$, and let

$$E_m = \sum \{E_{ni} | (n,i) \in N_m \times I\}$$

for $m = 1, 2, \ldots$. We show that each projection $E_n$ has dimension greater than that of $\mathcal{G}$ by showing $c_{\mathcal{G}}(E_n) = 0$. First we notice that each projection $E_n' = \sum \{E_{ni} | i \in I\}$ has central support $P$ equal to 1; otherwise, there would be a sequence $\{E_{nk}\}$ of central support $1 - P$ such that

$$\{E_{ni} | (n,i) \in N \times (I \cup \{k\})\}$$

lies in $\mathcal{S}$. Since the projections $E_n'$ are equivalent, the projections $E_n$ are equivalent, properly infinite, orthogonal projections of central support 1 [7, III 2, Proposition 10]. We show that $c_{\mathcal{G}}(E_1) = P$ is zero. On the contrary, assume $P$ is nonzero. We may write $E_1$ as the sum of an infinite sequence $\{G_n\}$ of orthogonal projections each equivalent to $E_1$ and thus to each $E_m$. Because

$$P \sum E_n \prec P \sum G_n = PE_1,$$

we get that $P \sum E_n$ is in $\mathcal{G}$. The least upper bound $E$ of the range supports of the finite set of elements $\{PB \sum E_n | B \in \mathcal{Y}\}$ is also a projection in $\mathcal{G}$. This means that the algebra $\mathcal{C} = \mathcal{Q}_{(P-E)}$ is properly infinite and that 0 is in the essential range of the element $C = (P - E)A(P - E)$ in $\mathcal{C}$ modulo the central ideal $\mathcal{G}' = \mathcal{G} \cap \mathcal{C}$ of $\mathcal{C}$. Indeed, if $\phi$ is a positive $\mathcal{X}$-module homomorphism of the algebra $\mathcal{Q}$ into $\mathcal{X}$ with $\phi(1) = 1$, $\phi(A) = 0$ and $\phi(\mathcal{G}) = (0)$, then the function $\psi$ on $\mathcal{C}$ given by

$$\psi(D) = (P - E)\phi(D)$$

is a positive module homomorphism of $\mathcal{C}$ into its center $\mathfrak{X}(P - E)$ with $\psi(P - E) = P - E$, $\psi(C) = 0$, and $\psi(\mathcal{G}') = (0)$. There is a sequence $\{F_n'\}$ of projections in $\mathcal{C}$ with $\dim F_n' > \dim \mathcal{G}'$ such that $\|F_n' A F_n'\| \leqslant \alpha_n$. Since the weak closure of $\mathcal{G}'$ is $\mathcal{C}$ and since the orthogonal complement of the largest central projection in $\mathcal{C}$ that is in $\mathcal{G}'$ is 0, the projections $F_n'$ are properly infinite and of central support $P - E$ in $\mathcal{C}$. There are equivalent projections $\{E_{nk} | n \in N\}$ of central support 1 such that $E_{mk} \leqslant F_m'$ and $E_{nk} B E_{mk} = 0$ for $1 \leqslant m < n < \infty$ and $B$ in the set $\{(P - E)D(P - E) | D \in \mathfrak{Y}\}$. Then the set

$$\{E_{ni} | (n, i) \in N \times (I \cup \{k\})\}$$

is in $\mathfrak{S}$. This is a contradiction. Hence, we have that $c_\mathcal{G}(E_1) = 0$ and that $\dim E_n > \dim \mathcal{G}$. Thus $\{E_n\}$ is a sequence of projections satisyfing property (1). We show that $\{E_n\}$ also satisfies (2) and (3). In fact, for $1 \leqslant m < n < \infty$ and $B$ in $\mathfrak{X}$, we have that

$$\|E_n B E_m\| = \left\| \sum \{E_{pi} B E_{qj} | (p, q, i, j) \in N_n \times N_m \times I \times I\} \right\| = 0$$

from (ii), and that

$$\|E_m A E_m\| = \left\| \sum \{E_{pi} A e_{qj} | (p, q, i, j) \in N_m \times N_m \times I \times I\} \right\|$$

$$= \left\| \sum \{E_{pi} A E_{pi} | (p, i) \in N_m \times I\} \right\|$$

$$= \mathrm{lub} \{\|E_{pi} A E_{pi}\| | (p, i) \in N_m \times I\} \leqslant \alpha_m$$

from (ii) and (iii).    Q.E.D.

We now examine the relation of the essential central spectrum and essential central range.

COROLLARY 2.11. *Let $\mathcal{Q}$ be a von Neumann algebra and let $\mathcal{G} = \mathcal{G}_Q(G)$ be a central ideal of $\mathcal{Q}$ in canonical form. Suppose that $\mathcal{Q}(1 - Q)$ is a continuous von Neumann algebra. Then an element $C$ in $\mathfrak{X}$ is in the essential central range of an element $A$ in $\mathcal{Q}$ modulo the central ideal $\mathcal{G}$ if and only if there is a projection $E$ in $\mathcal{Q}$ with $\dim E > \dim \mathcal{G}$ such that $CE$ is in the essential central spectrum of $EAE$ in $\mathcal{Q}_E$ modulo the central ideal $\mathcal{G} \cap E\mathcal{Q}E$.*

PROOF. There is no loss of generality in that assumption that $P_\mathcal{G} = 1$ and $C = 0$. If 0 is in $\mathfrak{X}_\mathcal{G}(A)$, there is a sequence $\{E_n\}$ of projections in $\mathcal{Q}$ with $\dim E_n > \dim \mathcal{G}$, $\|E_m A E_m\| \leqslant m^{-1}$, and $E_n B E_m = 0$ for $1 \leqslant m < n < \infty$ and $B$ in the set $\{1, A, A^*\}$ (Theorem 2.10). Let $E$ be the projection $E = \sum E_n$. Each $E_m$ has dimension greater than that of $\mathcal{G}' = \mathcal{G} \cap E\mathcal{Q}E$ since

(4)                                        $c_{\mathfrak{z}'}(E_m) = c_{\mathfrak{z}}(E_m)E = 0.$

Since

$$\lim \|EAEE_m\| = \lim \|E_m AE_m\| = 0,$$

we get that 0 is in $\mathfrak{X}E - \mathrm{Sp}_{\mathfrak{z}'} EAE$.

Conversely, let $E$ be a projection in $\mathcal{R}$ with $\dim E > \dim \mathfrak{z}$ such that 0 is in the essential central spectrum of $EAE$ modulo $\mathfrak{z}' = \mathfrak{z} \cap \mathcal{R}_E$. For every $m = 1, 2, \ldots$, there is an $E_m$ in $\mathcal{R}_E$ with $\dim E_m > \dim \mathfrak{z}'$ such that $\|E_m EAEE_m\| \leqslant m^{-1}$ (cf. [12, 3.16]). Because $E$ has central support 1, we get that $\dim E_m > \dim \mathfrak{z}$ from relation (4). We conclude from Theorem 2.2 that $\mathcal{K}_{\mathfrak{z}}(A)$ contains 0.   Q.E.D.

**3. Selfadjoint commutators.** In this section the selfadjoint commutators in properly infinite von Neumann algebras are characterized as those elements that contain 0 in their essential central range modulo the strong radical. A matrix form similar to that used for bounded operators on a separable space in [1] is found for such operators.

DEFINITION 3.1. A sequence $\{E_n | 1 \leqslant n < m\}$ (where $m$ is a finite or countable cardinal) of orthogonal projections in a properly infinite von Neumann algebra is said to be a *matrix base* if $1 \sim E_1 \sim E_2 \sim \cdots$ and $\sum E_n = 1$.

Let $\{E_n | 1 \leqslant n \leqslant m\}$ be a matrix base for the properly infinite algebra $\mathcal{R}$ and let $U_n$ be partial isometries of $\mathcal{R}$ with domain support $E_n$ and range support 1. Then each element $A$ in $\mathcal{R}$ can be written as

$$A = \sum E_n AE_m = \sum U_n^* \cdot U_n AU_m^* \cdot U_m.$$

The $m \times m$-matrix $(U_n AU_m^*)$ is said to be a *matrix with regard to the base* $\{E_n\}$. The matrix depends on the choice of the partial isometries; however, in the sequel the choice of the matrix base is critical and the choice of the partial isometries is immaterial.

We now begin a series of matrix reductions. The first extends the result in Corollary 2.7.

LEMMA 3.2. *For every selfadjoint operator $A$ in a properly infinite von Neumann algebra $\mathcal{R}$, there is a matrix base $\{E_i | 1 \leqslant i \leqslant 2\}$ for $\mathcal{R}$ that commutes with $A$.*

PROOF. Let $\mathfrak{z}$ be the strong radical of $\mathcal{R}$, let $\mathfrak{X}$ be the center of $\mathcal{R}$, and let $C_u$ and $C_l$ be the least upper bound and greatest lower bound respectively of $\mathfrak{X} - \mathrm{Sp}_{\mathfrak{z}}A$ [12, 3.11]. Since there is a projection $P$ in $\mathfrak{X}$ with $PC_u > PC_l$ and $(1 - P)C_u = (1 - P)C_l$, we may assume that either $C_u > C_l$ or that $C_u = C_l$.

If $C_u > C_l$, then the projections $E = E_A(C_u + D) - E_A(C_u - D)$ and $1 - E$ are equivalent to 1 whenever $0 < D \leqslant 2^{-1}(C_u - C_l)$ due to Proposition 2.5(iv). Hence, we may assume $C_u = C = C_l$.

Let $E$ denote the range projection of the selfadjoint operator $C - A$ in $\mathfrak{G}$; then the projection $c_{\mathfrak{G}}(E)(1 - E)$, which annihilates $C - A$, is equivalent to $c(E) = c_{\mathfrak{G}}(E)$ and can be written as the sum of two orthogonal projections $E_1$, $E_2$ both equivalent to $c(E)$ [7, III, 8, Theorem 1, Corollary 2]. The projections $E_1 + c(E)E$ and $E_2$ form a matrix base for $\mathfrak{C}c(E)$ that commute with $c(E)(C - A)$ and thus with $c(E)A$. So we may assume that $c(E) = 0$, or equivalently, that $E \sim 1$.

It is sufficient to show that given any selfadjoint operator $A$ in $\mathfrak{G}$ with range support equivalent to 1, there are equivalent orthogonal projections of sum 1 that commute with $A$. We prove first that there are at least two nonzero equivalent orthogonal projections majorized by the range support $E$ of $A$ that commute with $A$. Assume for the present that $E = 1$. Let $C_1, C_2$ be in $\mathfrak{X}$ with $C_1 > C_2 > 0$ and let $F(C_1, C_2)$ be the projection in $\mathfrak{G}$ given by

$$F(C_1, C_2) = (E(C_1) - E(C_2)) + (E(-C_2) - E(-C_1))$$

where $E(C_2) = E_A(C_2)$ (cf. Definition 2.4). By decomposing into the product of two algebras, we may suppose that either there is a $C_1$ in $\mathfrak{X}$ with

$$\|A\| + 1 = C_0 > C_1 > 0$$

such that $F(C_0, C_1)$ is properly infinite of central support 1 or that $F(C_0, C_1)$ is finite or zero for every $C_0 > C_1 > 0$ (Proposition 2.5(i)).

First assume $F(C_0, C_1)$ is properly infinite of central support 1. We show that there is a $C_2$ in $\mathfrak{X}$ with $C_1 > C_2 > 0$ such that

$$F(C_1, C_2) \succ F(C_0, C_1).$$

In fact, we show more: for any $C$, $C'$ in $\mathfrak{X}$ with $C > C' > 0$ so that $F(C, C')$ is properly infinite of central support 1, there is a $C''$ in $\mathfrak{X}$ with $C' > C'' > 0$ such that

$$F(C', C'') \succ F(C, C').$$

Since we have that

$$PE(D) = PE(PD)$$

for every selfadjoint $D$ in $\mathfrak{X}$ (Proposition 2.5(i)), it is sufficient to show that for any $P$ in $(\mathfrak{X})$ there is a $Q$ in $(\mathfrak{X}P)$ and a $C''$ in $ZQ$ with $C'Q > C''Q > 0$ and

$$QF(C'Q, C''Q) \succ QF(CQ, C'Q).$$

There is no loss of generality in the assumption that $P = 1$. We show that the relation

$$F(C, C') \succ F(C', C'')$$

for all $C''$ with $C' > C'' > 0$ leads to a contradiction. In fact let $\{F_n\}$ be a sequence of orthogonal projections of sum $F(C, C')$ such that $F_n \sim F(C, C')$ for every $n$ [7, III, 8, Theorem 1, Corollary 2]. We have that

$$1 \sim 1 - F(C_0, C') = \sum F(m^{-1}C', (m+1)^{-1}C') \prec \sum F_n \leqslant F(C, C')$$

because the range projection of $A$ given by $1 - E(0+) + E(0)$ is equal to 1. This is a contradiction due to the fact $F(C, C')$ is in $\mathfrak{I}$. So there is a $C''$ in $\mathfrak{X}$ with $C' > C'' > 0$ such that

$$F(C', C'') \succ F(C, C').$$

Noting that $F(C', C'')$ in the previous expression is properly infinite of central support 1, we may find a strictly decreasing sequence $\{C_n\}$ in $\mathfrak{X}^+$ such that $F(C_0, C_1) \prec F(C_1, C_2) \prec \cdots$. Setting

$$E_1 = \sum F(C_{2n}, C_{2n+1}) \quad \text{and} \quad E_2 = \sum F(C_{2n-1}, C_{2n}),$$

we obtain two orthogonal projections $E_1$ and $E_2$ of sum 1 such that

$$E_2 \prec E_1 \prec F(C_0, C_1) + E_2.$$

Since $F(C_0, C_1)$ is in the strong radical, we get that $E_1 \sim E_2 \sim 1$.

Now we assume that $F(C_0, n^{-1})$ is a finite or zero projection for every $n = 1, 2, \ldots$. If $\mathcal{C}$ is a continuous algebra, then there are, for each $n = 1, 2, \ldots$, two equivalent orthogonal projections of sum $F(C_0, n^{-1})$ which commute with $AF(C_0, n^{-1})$ and consequently with $A$ (Corollary 2.7). Since $F(C_0, n^{-1})$ is nonzero for some $n$ (Proposition 2.5(iii)), we obtain nonzero equivalent orthogonal projections commuting with $A$. Therefore, we may assume that $\mathcal{C}$ is of type I. There is an $n$ such that $F(C_0, n^{-1})$ is not an abelian projection; otherwise, the properly infinite projection 1 can be expressed as the least upper bound of a monotonely increasing sequence $\{F(C_0, n^{-1})\}$ of abelian projections which is also an abelian projection. This means that there is an $n$ and a $Q$ in $(\mathfrak{X})$ such that $QF(C_0, n^{-1})$ is the sum of $m$ ($m \geqslant 2$) orthogonal abelian projections of central support $Q$. Because $AF(C_0, n^{-1})Q$ can be represented as a diagonal $m \times m$ operator matrix over $\mathfrak{X}F(C_0, n^{-1})Q$ based on matrix units arising from the $m$ equivalent orthogonal abelian projections of sum $F(C_0, n^{-1})Q$ [6], we may find two nonzero orthogonal equivalent projections (actually abelian projections) which commute with $AF(C_0, n^{-1})Q$ and thus with $A$.

In every case we have found two nonzero equivalent orthogonal projections that commute with $A$. We now remove the assumption that the range projection $E$ is equal to 1. Let $V$ be an isometry in $\mathcal{C}$ whose range support is $E$. The element $V^*AV$ is a selfadjoint element in the strong radical whose range support is 1. As was just shown, there are nonzero orthogonal equivalent projections $E_1$ and $E_2$ that commute with $VAV^*$. The projections $VE_1V^*$ and $VE_2V^*$ are nonzero equivalent orthogonal projections majorized by $E$ that commute with $A$.

We now construct the desired projections by a maximality argument. Let $\{F_i, G_i | i \in I\}$ be a maximal set of nonzero mutually orthogonal projections majorized by $E$ that commute with $A$ such that $F_i \sim G_i$ for every $i$. The projections $F = \sum F_i$ and $G = \sum G_i$ are equivalent orthogonal projections majorized by $E$ that commute with $A$. The projection

$$E' = c(F)(E - (F + G))$$

is equivalent to $c(F)$ since $c(F) = c(G)$, and the range support $E'$ of $AE'$ in the strong radical of the properly infinite von Neumann algebra $\mathcal{C}c(F)$ is equivalent to $c(F)$. This means $c(F) = 0$; otherwise, the arguments of the preceding paragraphs can be applied to show that $\{F_i, G_i\}$ is not maximal. Thus we have that the projections $F, 1 - F$ form a matrix base that commutes with $A$.   Q.E.D.

REMARK. Combining the results of Corollary 2.7, Lemma 3.2, and [6], we obtain that, for every selfadjoint operator $A$ in a von Neumann algebra with no type $I_n$ ($n$ odd) part, there are two equivalent orthogonal projections of sum 1 commuting with $A$.

The next lemma extends the results of Anderson and Stampfli [2].

LEMMA 3.3. *Let $A$ be an element in a properly infinite von Neumann algebra $\mathcal{C}$; then there is a matrix base $E_1$, $E_2$ (resp. $F_1$, $F_2$) of $\mathcal{C}$ such that $F_1AE_2 = F_2AE_1 = 0$.*

PROOF. Let $B$ be a positive selfadjoint operator in $\mathcal{C}$ and let $U$ be a partial isometry of $\mathcal{C}$ so that $A = UB$ is the polar decomposition of $A$. The projections $c(U^*U)(1 - U^*U)$ and $c(U^*U)(1 - UU^*)$ may be written as the sum of two equivalent orthogonal projections $E$, $E'$ and $F$, $F'$ respectively. Here $c(U^*U) = c_{\mathcal{J}}(U^*U)$ where $\mathcal{J}$ is the strong radical of $\mathcal{C}$. The projections

$$E_1 = E + c(U^*U)U^*U, \qquad E_2 = E'$$

and

$$F_1 = F + c(U^*U)UU^*, \qquad F_2 = F'$$

are equivalent orthogonal projections of sum $c(U^*U)$ respectively with

$$F_1 A E_2 = F_2 A E_1 = 0.$$

So we may assume that $c(U^* U) = 0$. There are equivalent orthogonal projections $E$, $E'$ of sum $U^* U$ that commute with $B$ due to the fact that $U^* U$ is properly infinite (Lemma 3.2). The projections $UEU^*$ and $UE'U^*$ are orthogonal equivalent projections of sum $UU^*$. Setting

$$E_1 = E + (1 - U^* U), \qquad E_2 = E'$$

and

$$F_1 = UEU^* + (1 - UU^*), \qquad F_2 = UE'U^*,$$

we get matrix bases that satisfy the desired relations.   Q.E.D.

We now are ready to characterize a selfadjoint commutator. We treat the product of $\sigma$-finite type I algebras separately. This case is slightly anomalous and in fact follows from the preceding lemmas via the analysis of J. Anderson [1]. In the sequel let $\alpha_n = ((n + 1)!)^{-1}$ for $n = 1, 2, \ldots$ .

THEOREM 3.4. *Let $\mathcal{C}$ be the product of $\sigma$-finite properly infinite type I von Neumann algebras. Then an element $A$ in $\mathcal{C}$ is a selfadjoint commutator in $\mathcal{C}$ if and only if $0$ is in the essential central range of $A$ modulo the strong radical $\mathcal{G}$ of $\mathcal{C}$.*

PROOF. First let $A$ have the matrix representation

$$(5) \qquad \qquad \begin{pmatrix} * & 0 \\ 0 & B \end{pmatrix},$$

where $B \in \mathcal{G}$. Since $\mathfrak{K}(B^* B + BB^*) = \{0\}$, we may write $B$ as the infinite matrix operator $B = (B_{ij})$ over $\mathcal{C}$ with $\|B_{ij}\| \leqslant 2\|B\| \min\{\alpha_i, \alpha_j\}$. In fact, by applying Theorem 2.10 to $B^* B + BB^*$, we may find a sequence $\{E_n\}$ of orthogonal projections in $\mathcal{C}$ such that dim $E_n >$ dim $\mathcal{G}$ and

$$\|E_n(B^* B + BB^*)E_n\| \leqslant 4\alpha_n^2 \|B\|^2$$

for all $1 \leqslant n < \infty$. Replacing $E_1$ by $1 - \sum \{E_n | n \geqslant 2\}$, we get a matrix base $\{E_n\}$ of $\mathcal{C}$ such that

$$\|E_n B E_m\| \leqslant 2\|B\| \min\{\alpha_m, \alpha_n\}.$$

The calculation of M. David [5, Theorem 3] now shows that $A$ is a selfadjoint commutator.

Now let $A$ be an operator in $\mathcal{C}$ such that $\mathfrak{K}(A)$ contains 0. Applying Theorem 2.10, we may find an orthogonal sequence $\{F_n\}$ of maximal abelian projections of $\mathcal{C}$ of sum $F$ such that $1 - F \sim F \sim 1$ and such that

$$\|F_m A F_n\| \leqslant n^{-1}\delta_{mn}$$

for all $m, n = 1, 2, \ldots$ [7, III, 8, Theorem 1, Corollary 5]. Here $\delta$ is the Kronecker delta. We have that $FAF$ is in $\mathfrak{g}$ since $G = \sum \{F_n | 1 \leqslant n \leqslant m\}$ is in $\mathfrak{g}$ and

$$\|FAF(1 - G)\| = \mathrm{lub}\ \{\|F_n A F_n\| \ |n > m\} \leqslant (m + 1)^{-1}.$$

The 2,2 term of the matrix of $A$ induced by the matrix base $1 - F$, $F$ is in $\mathfrak{g}$. Using Lemma 3.3, we can follow the analysis of J. Anderson [1] to write $A$ as an $8 \times 8$-matrix over $\mathfrak{A}$ such that each of the four $2 \times 2$ diagonal blocks is of the form (5). Thus $A$ is a selfadjoint commutator.

Conversely, let $A = BC - CB$ with $B = B^*$ and $C$ in $\mathfrak{A}$. We may assume that 0 is in $\mathfrak{X} - \mathrm{Sp}_{\mathfrak{g}} B$. Given $\varepsilon > 0$, there is a projection $E$ in $\mathfrak{A}$ with $\dim E > \dim \mathfrak{g}$ and with $\|BE\| = \|EB\| < \varepsilon$ (Proposition 2.5(iv)). We have that $\|EAE\| \leqslant 2\varepsilon\|C\|$, and thus that $0 \in \mathfrak{K}_{\mathfrak{g}}(A)$.   Q.E.D.

Now we can restrict our attention to properly infinite algebras with no $\sigma$-finite type I direct summands.

LEMMA 3.5. *Let $A$ be a selfadjoint operator in a properly infinite von Neumann algebra $\mathfrak{A}$ with no $\sigma$-finite type* I *direct summands and let $\{E_n\}$ be a sequence of projections in $\mathfrak{A}$ which commute with $A$; then for every $n = 1, 2, \ldots$, there are equivalent orthogonal projections $E_{n1}$ and $E_{n2}$ commuting with $A$ and majorized by $E_n$ such that $\mathrm{lub}\ (E_n - (E_{n1} + E_{n2}))$ is in the strong radical of $\mathfrak{A}$.*

PROOF. We may assume that either $\mathfrak{A}$ is a continuous $\sigma$-finite algebra or that $\mathfrak{A}$ has no $\sigma$-finite direct summand. In case $\mathfrak{A}$ is a continuous $\sigma$-finite algebra, we show that $E_n$ may in fact be written as the sum of two equivalent orthogonal projections commuting with $A$. Let $P_n$ be the largest central projection in $\mathfrak{A}$ such that $E_n P_n$ is finite. Then $E_n P_n$ (resp. $E_n(1 - P_n)$) may be written as the sum of two equivalent orthogonal projections commuting with $AE_n P_n$ (resp. $AE_n(1 - P_n)$) due to Corollary 2.7 (resp. Lemma 3.2). So $E_n$ may be written as the sum of two equivalent orthogonal projections commuting with $A$.

Now suppose $\mathfrak{A}$ has no $\sigma$-finite direct summands. For every $n = 1, 2, \ldots$, we may use the reasoning of the preceding paragraph to find two equivalent orthogonal projections $E_{n1}, E_{n2}$ commuting with $A$ and majorized by $E_n$ so that $E_n - (E_{n1} + E_{n2})$ is a finite projection. To complete the proof it is sufficient to show that the least upper bound $F$ of a monotonely increasing sequence $\{F_n\}$ of finite projections in a von Neumann algebra with no $\sigma$-finite direct summands is in the strong radical [7, III, 2, Proposition 5]. Since the strong radical is a central ideal, and since any direct summand of a finite projection is finite, it is sufficient to show that there is a nonzero central projection $Q$ such that $FQ$ is in the strong radical. Let $x$ be a unit vector in

the Hilbert space of $\mathcal{C}$. For each $n = 1, 2, \ldots$, there is an orthogonal set $\{Q_i\}$ central projections of sum 1 such that $F_n Q_i$ is $\sigma$-finite [7, I, 6, Proposition 9(iii)]. There is a finite subset of the $Q_i$ of sum $Q_n$ such that $\|x - Q_n x\| \leqslant 2^{-n-1}$. Setting $Q$ equal to the greatest lower bound of the monotonely decreasing sequence $\{Q_1 Q_2 \cdots Q_n\}$ of central projections, we obtain the required nonzero projection because $\|x - Qx\| \leqslant 2^{-1}$ and because $FQ$ is $\sigma$-finite and consequently in the strong radical.   Q.E.D.

If 0 is in the essential central range of $A$ modulo the strong radical of the properly infinite algebra $\mathcal{C}$, the projections $\{E_n\}$ of Theorem 2.10 induce the matrix decomposition $A = (C_{ij})$ over $\mathcal{C}$ with $\|C_{mn}\| \leqslant \alpha_n \delta_{mn}$ for $m, n \geqslant 2$. Here we replace $E_1$ of Theorem 2.10 by $1 - \sum \{E_n | n \geqslant 2\}$, which majorizes $E_1$ and is thus equivalent to 1. We may also choose the $E_n$ so that $E_m B E_n = 0$ for $2 \leqslant m < n < \infty$ and $B$ in a preassigned finite subset of $\mathcal{C}$. We now obtain a second matrix form for $A$.

LEMMA 3.6. *Let $A$ be an element in a properly infinite von Neumann algebra $\mathcal{C}$ with no $\sigma$-finite type* I *direct summands and let $\{E_n\}$ be an infinite matrix base of $\mathcal{C}$ such that $\|E_n B E_m\| = 0$ for $2 \leqslant m < n < \infty$ and $B$ in the set $\mathcal{X} = \{A^*, A, A^* A, AA^*\}$. Then there are orthogonal equivalent projections $E_{n1}$ and $E_{n2}$ of sum $E_n$ such that*

$$E_{11} A \sum \{E_{n2} | n \geqslant 2\} = E_{12} A \sum \{E_{n1} | n \geqslant 2\} = 0.$$

PROOF. Let $U$ be a partial isometry and $B$ a positive element of $\mathcal{C}$ so that $UB$ is the polar decomposition of $E_1 A \sum \{E_n | n \geqslant 2\}$. Because each $E_n$ $(n \geqslant 2)$ commutes with $B$ due to the validity of the relation

$$E_m B^* B E_n = E_m A^* (E_1 + E_m)(E_n + E_1) A E_n = E_m A^* A E_n = 0$$

for $2 \leqslant m < n < \infty$, each $E_n$ commutes with the range projection $E$ of $B$. There are equivalent orthogonal projections $F_{n1}$ and $F_{n2}$ $(n \geqslant 2)$ commuting with $B$ and majorized by $EE_n$ such that

$$F = \sum \{EE_n - (F_{n1} + F_{n2}) | n \geqslant 2\}$$

is in the strong radical $\mathcal{J}$ of $\mathcal{C}$ (Lemma 3.5). The projection $E_n - E_n E$ $(n \geqslant 2)$ (resp. $E_1 - UU^*$) is also the sum of orthogonal projections $G_{n1}, G_{n2}, G_{n3}$ such that $G_{n1} \sim G_{n2}$ and $G_{n3}$ is in $\mathcal{J}$. Then we let

$$E_{n1} = (EE_n - F_{n2}) + G_{n1} + G_{n3}$$

and

$$E_{n2} = F_{n2} + G_{n2}$$

for $n \geqslant 2$ and

$$E_{11} = U \sum \{EE_n - F_{n2} | n \geqslant 2\}U^* + G_{11} + G_{13}$$

and

$$E_{12} = U \sum \{F_{n2} | n \geqslant 2\}U^* + G_{12}.$$

We have that $c(E_{n1}) = c(E_{n2}) = 0$ for $n \geqslant 2$ and so $E_{n1} \sim 1 \sim E_{n2}$. Here $c = c_q$. Also we have that

$$c(E_{11}) = c\left(U\left(F + \sum \{F_{n1} | n \geqslant 2\}\right)U^* + G_{11} + G_{13}\right)$$

$$= c\left(U\left(\sum \{F_{n1} | n \geqslant 2\}\right)U^* + G_{11}\right)$$

$$= c\left(U\left(\sum \{F_{n2} | n \geqslant 2\}\right)U^* + G_{12}\right) = c(E_{12}),$$

and thus $E_{11} \sim 1 \sim E_{12}$. We also have that

$$E_{11} A \sum \{E_{n2} | n \geqslant 2\} = U\left(E - \sum \{F_{n2} | n \geqslant 2\}\right) \sum \{F_{n2} | n \geqslant 2\} B = 0$$

and

$$E_{12} A \sum \{E_{n1} | n \geqslant 2\} = U \sum \{F_{n2} | n \geqslant 2\}\left(E - \sum \{F_{n2} | n \geqslant 2\}\right) B = 0.$$

Q.E.D.

Now if 0 is in the essential central spectrum of $A$ in the properly infinite von Neumann algebra $\mathcal{C}$, then we may employ Theorem 2.10 and Lemma 3.6 (as applied to $A^*$) to write $A$ as a $4 \times 4$-matrix $(A_{ij})$ over $\mathcal{C}$ such that $A_{41} = A_{32} = 0$ and such that $A_{33}$ and $A_{44}$ can be further decomposed to infinite operator matrices $(C_{ij})$ over $\mathcal{C}$ of the form $\|C_{ij}\| \leqslant \alpha_i \delta_{ij}$. To apply Lemma 3.6 to this new matrix we need the next lemma.

LEMMA 3.7. *Let $A$ be an element of a properly infinite von Neumann algebra $\mathcal{C}$ and let $\{F_n\}$ be an infinite matrix base for $\mathcal{C}$ such that $F_1 A F_m = 0$ for $m \geqslant 2$ and $\|F_n A F_m\| \leqslant \alpha_n \delta_{mn}$ for $m, n \geqslant 2$; then there is an infinite matrix base $\{E_n\}$ for $\mathcal{C}$ such that $E_n B E_m = 0$ for $2 \leqslant m < n < \infty$ and $B$ in the set $\mathcal{X} = \{1, A, A^*, AA^*, A^*A\}$, and $\|E_1 A E_m\| \leqslant \alpha_m$, $\|E_m A E_m\| \leqslant \alpha_m$ for $m \geqslant 2$.*

PROOF. We use the same arguments as in the proof of Theorem 2.10. Let $\mathcal{S}$ be the family of all sets

$$S = \{E_{ni} | (n, i) \in N \times I\}$$

of nonzero projections in $\mathcal{C}$ such that

(i) $E_{mi} \sim E_{ni}$ for all $m,n$ in $N$ and $i \in I$;

(ii) $E_{nj} B E_{mi} = 0$ for all $B$ in $\mathfrak{X}$ and distinct pairs $(m,i)$, $(n,j)$ in $N \times I$; and

(iii) $E_{ni} \leqslant F_n$ for all $n$ in $N$ and $i$ in $I$.

From Lemma 2.9, we conclude that $\mathfrak{S}$ is nonvoid, and from Zorn's lemma we find a maximal set $S = \{E_{ni} | (n,i) \in N \times I\}$ in $\mathfrak{S}$ in the partial ordering (3). Let $\{N_n | n \in N\}$ be a partition of $N$ into a sequence of infinite sets such that the smallest element of $N_n$ is not less than $n$. As in Lemma 2.9, we can show that each of the projections

$$E_n = \sum \{E_{mi} | (m,i) \in N_n \times I\}$$

is a properly infinite projection of central support 1. We show that each $E_n$ is equivalent to 1. If $\mathcal{Q}$ is the product of $\sigma$-finite algebras, then each $E_n$ is equivalent to 1 since each $E_n$ is properly infinite. So we may assume that $\mathcal{Q}$ has no $\sigma$-finite direct summands. We show that $E_n \sim 1$ by showing that the largest central projection $P$ such that $PE_n$ is in the strong radical $\mathfrak{J}$ is $P = 0$. We argue by contradiction. As in Theorem 2.10, we have that $P \sum E_n$ is in $\mathfrak{J}$. For every $n = 1, 2, \ldots$, the orthogonal complement $G_n$ in $PF_n$ of the least upper bound of the range supports of the elements $F_n BP \sum E_n$ for $B$ in $\mathfrak{X}$ is equivalent to $P$ due to the fact this least upper bound is in $\mathfrak{J}$. Setting $G = \sum G_n$ and applying Lemma 2.9 to the element $GAG$ in the properly infinite algebra $\mathcal{Q}_G$ and to the sequence $\{G_n\}$ of projections, we may find a sequence $\{G_{nk}\}$ of projections in $\mathcal{Q}_G$ so that

$$\{E_{ni} | (n,i) \in N \times (I \cup \{k\})\}$$

is greater than $S$ in the partial ordering on $\mathfrak{S}$. This is a contradiction. So $E_n \sim 1$.

Replacing $E_1$ by $1 - \sum \{E_n | n \geqslant 2\}$ if necessary, we obtain a matrix base $\{E_n\}$ that satisfies the required conditions. On the one hand, we have

$$\|E_m A E_m\| = \text{lub} \{\|E_{ni} A E_{ni}\| \,|\, (n,i) \in N_m \times I\} \leqslant \alpha_m$$

for $m \geqslant 2$, and

$$\|E_m B E_n\| = \text{lub} \{\|E_{pi} B E_{qj}\| \,|\, (p,q,i,j) \in N_m \times N_n \times I \times I\} = 0$$

for $m, n \geqslant 2$ and $B$ in $\mathfrak{X}$. On the other hand, we have that

$$\|E_1 A E_m\| \leqslant \left\| A \sum \{E_{ni} | (n,i) \in N_m \times I\} \right\|$$

$$\leqslant \left\| \sum \left\{ A \sum \{E_{ni} | i \in I\} | n \in N_m \right\} \right\| \leqslant \left\| \sum \{A F_n | n \in N_m\} \right\|$$

$$\leqslant \left\| \sum \{F_n A F_n | n \in N_m\} \right\| \leqslant \mathrm{lub}\, \{\|F_n A F_n\| \,| n \in N_m\} \leqslant \alpha_m$$

for $m \geqslant 2$. Q.E.D.

Now we can prove our main theorem.

THEOREM 3.8. *An element $A$ in a properly infinite von Neumann algebra $\mathcal{C}$ is a selfadjoint commutator if and only if 0 is in the essential central range of $A$ modulo the strong radical of $\mathcal{C}$.*

PROOF. If $A$ is a selfadjoint commutator, then the essential central spectrum $\mathcal{K}(A)$ of $\mathcal{C}$ modulo the strong radical $\mathcal{J}$ contains 0. The proof presented in Theorem 3.4 is sufficient.

Conversely, we may assume $\mathcal{C}$ has no $\sigma$-finite type I direct summands (Theorem 3.4). Let $A$ be in $\mathcal{C}$ and let $0 \in \mathcal{K}(A)$. We have already indicated after Lemma 3.5 that with regard to a matrix base $E_i$ ($1 \leqslant i \leqslant 4$) the operator $A$ can be written as a $4 \times 4$-matrix $(A_{ij})$ over $\mathcal{C}$ such that $A_{41} = A_{32} = 0$, and such that $A_{33}$ and $A_{44}$ can be written with respect to an infinite matrix base as matrices over $\mathcal{C}$ of the form $(C_{ij})$, where $\|C_{ij}\| \leqslant \alpha_j \delta_{ij}$ for $i,j = 1, 2, \ldots$ . Let $A'$ (resp. $A''$) be given by

$$A' = \begin{pmatrix} A_{11} & A_{14} \\ 0 & A_{44} \end{pmatrix} \left( \text{resp. } A'' = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \right).$$

Setting $C$ equal to $A'$ (resp. $A''$) and applying Lemma 3.7 to $C^*$, we get a matrix base $\{F_n | 1 \leqslant n < \infty\}$ for $\mathcal{C}$ such that $F_n B F_m = 0$ for all $2 \leqslant m < n < \infty$ and $B$ in the set $\mathcal{X} = \{1, C, C^*, C^*C, CC^*\}$, and $\|F_m C F_1\| \leqslant \alpha_m$, $\|F_m C F_m\| \leqslant \alpha_m$ for $m \geqslant 2$. From Lemma 3.6 applied to the matrix base $\{F_n\}$, we see that $C$ may be written as a $4 \times 4$-matrix over $\mathcal{C}$ of the form Figure 1, where $\|A_i\|, \|B_i\|, \|C_i\|, \|D_i\|$ are majorized by $\alpha_i$. Actually, the entries (*) of the 3,1 and 4,2 (resp. 3,4 and 4,3) blocks have the same form as the 4,1 (resp. 4,4) block. However, we do not need this information here. Reassembling the matrix $A$ from the various parts and applying J. Anderson's rearrangement [1], viz. $1 \to 7 \to 6 \to 4 \to 1, 2 \to 5 \to 2, 3 \to 3, 8 \to 8$, we obtain an $8 \times 8$-matrix over $\mathcal{C}$ whose four $2 \times 2$-diagonal blocks have the form Figure 2, where $\|B_i\| \leqslant \alpha_i$, $\|C_i\| \leqslant \alpha_i$. From the calculation of M. David [5], we get that each of these four diagonal blocks is a selfadjoint commutator in $\mathcal{C}$ and thus that $A$ itself is a selfadjoint commutator (cf. [1], [11, Proof, 4]).   Q.E.D.
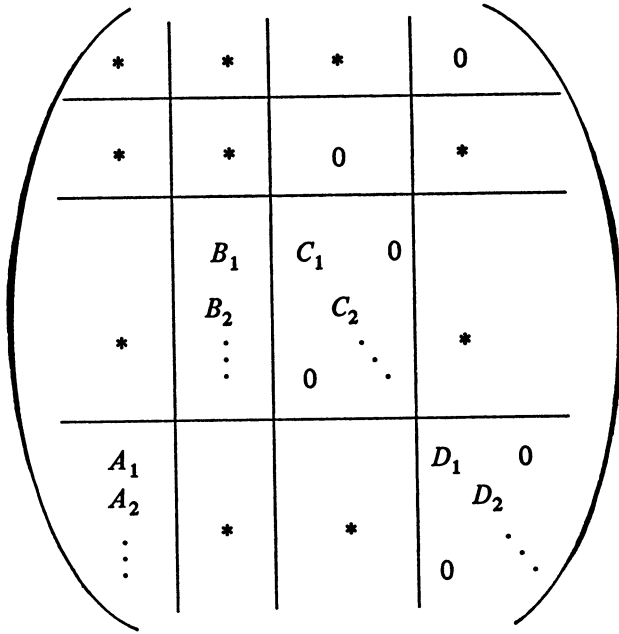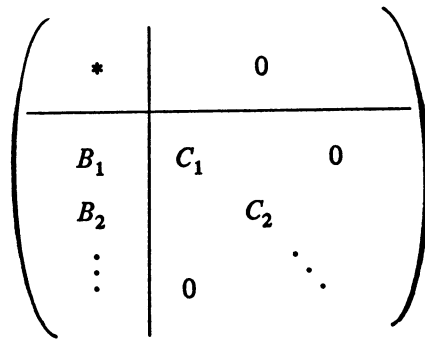
FIGURE 1



FIGURE 2

An operator $A$ in a von Neumann algebra $\mathcal{C}$ is said to be *similar* to an operator $B$ in $\mathcal{C}$ if there is an invertible $C$ in $\mathcal{C}$ with $CAC^{-1} = B$. We can now characterize those operators in a properly infinite algebra that are similar to a selfadjoint commutator.

PROPOSITION 3.9. *An operator $A$ in a properly infinite von Neumann algebra $\mathcal{Q}$ with strong radical $\mathcal{J}$ is similar to a selfadjoint commutator in $\mathcal{Q}$ if and only if there is a $\kappa > 0$ such that*

$$\|(\alpha - A)(\mathcal{J} + [\zeta])\| \geqslant \kappa|\alpha|$$

*for every complex $\alpha$ and every $\zeta$ in the spectrum $Z$ of the center $\mathcal{Z}$ of $\mathcal{Q}$, or equivalently, such that*

$$\|(\alpha - A)(\mathfrak{M})\| \geqslant \kappa|\alpha|$$

*for every $\alpha$ and maximal ideal $\mathfrak{M}$ in $\mathcal{Q}$.*

PROOF. Suppose there exists such a $\kappa > 0$. For convenience suppose that $\|A\| = 1$. There is a $C$ in $\mathcal{Z}$ such that

$$C^{\hat{}}(\zeta) = \|A(\mathcal{J} + [\zeta])\|$$

[12, 3.2]. There is a projection $P$ in $\mathcal{Z}$ such that $CP > 0$ (i.e. $\{\zeta \in Z | C^{\hat{}}(\zeta) > 0\}$ is dense in $\{\zeta \in Z | P^{\hat{}}(\zeta) = 1\}$) and $C(1 - P) = 0$. We have that $A(1 - P)$ is in the strong radical of $\mathcal{Q}_{(1-P)}$ [12, 3.1], and thus $A(1 - P)$ is a selfadjoint commutator in $\mathcal{Q}_{(1-P)}$. By passing to the direct summand $\mathcal{Q}_P$, we may assume that $C > 0$. There is an orthogonal sequence $\{P_n\}$ of projections in $\mathcal{Z}$ of sum 1 such that $C^{\hat{}}(\zeta)$ is in the interval $[(n + 1)^{-1}, n^{-1}]$ whenever $P_n^{\hat{}}(\zeta) = 1$. For every $n = 1, 2, \ldots$ there is a $B_n$ in $\mathcal{J}$ such that

$$\|(A + B_n)([\zeta])\|$$

is in the interval $[(n + 1)^{-1}, 2n^{-1}]$ for every $\zeta$ with $P_n^{\hat{}}(\zeta) = 1$. The sum $\sum (n + 1)(A + B_n)P_n$ converges in the strong topology to an element $B$ in $\mathcal{Q}$. We have that

$$\|B(\mathcal{J} + [\zeta])\| \geqslant 1$$

and

$$\|(\alpha - B)(\mathcal{J} + [\zeta])\| \geqslant \kappa|\alpha|$$

for every complex $\alpha$ and every $\zeta$ in the union of the sets $\{\zeta \in Z | P_n^{\hat{}}(\zeta) = 1\}$ and thus for every $\zeta$ in $Z$. This means that the canonical image of $B$ in the algebra $\mathcal{Q}/(\mathcal{J} + [\zeta])$ is never a scalar multiple of the identity. There is an invertible $C$ in $\mathcal{Q}$ and a projection $F$ equivalent to 1 such that $FCBC^{-1}F = 0$ [11, Proof, Theorem 4]. For every $\varepsilon > 0$ and every $n = 1, 2, \ldots$, there is a projection $E_n$ in $\mathcal{J}$ such that

$$\|B_n(1 - E_n)\| \leqslant \varepsilon/(\|C\| \|C^{-1}\|).$$

The range projection $F_n$ of $F(C^{-1})^* E_n$ is in $\mathcal{J}$ and thus $G_n = F - F_n$ is equivalent to 1. Setting $G = \sum G_n P_n$, we get a projection $G$ equivalent to 1 such that

$$\|GCAC^{-1}G\| \leqslant \|FCBC^{-1}F\| + \|C\| \, \|C^{-1}\| \operatorname{lub}_n \|B(1 - E_n)\| \leqslant \varepsilon.$$

We conclude that $\mathcal{K}(CAC^{-1})$ contains 0 (Theorem 2.2), and consequently that $CAC^{-1}$ is a selfadjoint commutator in $\mathcal{C}$ (Theorems 3.4 and 3.8).

Conversely, let $A$ be a selfadjoint commutator in $\mathcal{C}$. We have that

$$(6) \qquad\qquad \|(\alpha - A)(\mathcal{J} + [\zeta])\| \geqslant |\alpha|$$

for every scalar $\alpha$ and every $\zeta \in Z$; otherwise,

$$|\alpha| = |\phi(\alpha - A)\hat{\ }(\zeta)| \leqslant \|(\alpha - A)(\mathcal{J} + [\zeta])\| < |\alpha|$$

for some $\alpha$ and $\zeta$. Here $\phi$ is a state of $A^\sim$ which vanishes on $\mathcal{J} \cup \{A\}$ (Theorems 3.4 and 3.8). If $C$ is an invertible element in $\mathcal{C}$, then we have that

$$\|(\alpha - CAC^{-1})(\mathcal{J} + [\zeta])\| \geqslant (\|C\| \, \|C^{-1}\|)^{-1} |\alpha|$$

for every $\alpha$ and $\zeta$.   Q.E.D.

REMARK. The preceding proposition shows that the *operators of class* $(F')$ in a properly infinite algebra, viz. those operators which are not equal to a scalar multiple of the identity (zero included) modulo any maximal ideal, that were studied in [10] are contained in the set of operators similar to a selfadjoint commutator.

COROLLARY 3.10. *For any operator $A$ and any normal operator $B$ in a properly infinite von Neumann algebra the operator $BA - AB$ is a selfadjoint commutator.*

PROOF. We may assume that 0 is in the essential central spectrum of $B$ modulo the strong radical [12, 3.5]. For every $\varepsilon > 0$, there is a projection $E$ in $\mathcal{C}$ equivalent to the identity with

$$\|EB\| = \|BE\| \leqslant \varepsilon$$

(Corollary 2.6). Hence, the essential central spectrum of $BA - AB$ modulo the strong radical contains 0 (Theorem 2.2) and so $BA - AB$ is a selfadjoint commutator (Theorem 3.8).   Q.E.D.

REMARK. Actually, for any $A$ and any seminormal $B$ (i.e. either $\pm(BB^* - B^*B)$ is positive) in a properly infinite algebra, the operator $AB - BA$ is a selfadjoint commutator (cf. [11, Note "added in proof"]).

The next proposition extends the results of H. Radjavi [13] for the algebra of bounded operators on a separable Hilbert space.

PROPOSITION 3.11. *A selfadjoint operator in a properly infinite von Neumann algebra is a selfadjoint commutator if and only if its essential central spectrum modulo the strong radical contains a nonpositive and a nonnegative element.*

PROOF. The essential central range of a selfadjoint operator modulo the strong radical is the smallest central convex set containing the essential central spectrum modulo the strong radical [12, 4.4].   Q.E.D.

The next result extends that of J. Williams [14] for the algebra of all bounded operators on a separable Hilbert space.

PROPOSITION 3.12. *Every commutator of a properly infinite factor von Neumann algebra is similar to a selfadjoint commutator.*

PROOF. An operator $A$ is a commutator in a properly infinite factor $\mathcal{C}$ if and only if there is an invertible $B$ in $\mathcal{C}$ and a projection $E$ in $\mathcal{C}$ with $E \sim 1$ such that $EBAB^{-1}E = 0$ [3], [4], [10]. Thus, every commutator of $\mathcal{C}$ is similar to a selfadjoint commutator (Theorems 3.4 and 3.8).

One may also use the arguments of J. Williams [14] together with Theorems 3.4 and 3.8 to avoid the computations of [3], [4], [10].   Q.E.D.

One might conjecture that an operator is a commutator if and only if it is not equal to a nonzero scalar multiple of the identity modulo every maximal ideal (i.e. if and only if it is an operator of *class* (*F*)). If the center is large, this turns out to be false.

EXAMPLE 3.13. In every properly infinite von Neumann algebra $\mathcal{C}$ not equal to a finite product of factors, there is an operator of class (*F*) which is not a commutator. Indeed, there is for every natural number $m$ a neighborhood $V = V_m$ of 1 contained in the sphere of radius one about 1 such that $AB - BA$ in $V_m$ and $\|A\| \leqslant 1$ implies $\|B\| \geqslant m$. This holds even for factors. Notice that every neighborhood of the identity contains commutators [10, 3.10]. On the contrary, there would be bounded sequences $\{A_n\}$ and $\{B_n\}$ in $\mathcal{C}$ with $\{A_n B_n - B_n A_n\}$ converging to the identity. This is impossible [16, Problem 183]. So such a neighborhood $V_m$ exists.

Now let $\{P_m\}$ be an orthogonal sequence of nonzero central projections of $\mathcal{C}$ of sum 1. There is a commutator $C_m$ in $\mathcal{C}P_m$ such that $\|C_m - 1\| \leqslant 1$ and $C_m = AB - BA$ for $\|A\| \leqslant 1$ implies $\|B\| \geqslant m^2$. Then the operator $C = \sum m^{-1} C_m$ is in class (*F*). In fact, let $\zeta$ be in the spectrum of the center. Either $P_m^{\,\hat{}}(\zeta) = 1$ for some $m$ or $P_m^{\,\hat{}}(\zeta) = 0$ for all $m$. In the first case

$$C(\mathcal{J} + [\zeta]) = (BA - AB)(\mathcal{J} + [\zeta])$$

for some $A$ and $B$ in $\mathcal{C}P_m$. Here $\mathcal{J}$ is the strong radical. Hence $C$ is not a nonzero scalar multiple of the identity modulo $\mathcal{J} + [\zeta]$. In the second case we have that

$$\|C(\mathcal{J} + [\zeta])\| \leqslant \left\| C\left(1 - \sum \{P_n | n \geqslant m\}\right)([\zeta])\right\| \leqslant 2(m+1)^{-1}$$

and so $C(\mathcal{J} + [\zeta]) = 0$. Now suppose $C = AB - BA$ for $A$, $B$ in $\mathcal{C}$. We may assume that $\|A\| \leqslant 1$. This means that $\|mBP_m\| \geqslant m^2$. This is impossible. So $C$ is not a commutator.

In contradistinction to factors we show that not every commutator is similar to a selfadjoint commutator when the center is sufficiently large.

PROPOSITION 3.14. *Let $A$ be an operator of class $(F)$ in a properly infinite von Neumann algebra $\mathcal{C}$; then there is a sequence $\{\beta_n\}$ of strictly positive numbers and a sequence $\{P_n\}$ of orthogonal central projections of sum $1$ such that $\sum \beta_n A P_n$ converges in the strong topology to a commutator in $\mathcal{C}$.*

PROOF. Let $\mathcal{J}$ be the strong radical of $\mathcal{C}$ and let $P_0 = c_{\mathcal{J}}(A)$. There is a sequence $\{P_n | n \geqslant 1\}$ of orthogonal central projections of sum $1 - P_0$ such that $\|A(\mathcal{J} + [\zeta])\| \geqslant n^{-1}$ for every $\zeta$ in the spectrum of the center of $\mathcal{C} P_n$ [12, 3.1]. Therefore, the operators $A P_n$ ($n \geqslant 1$) are of class $(F')$ in $\mathcal{C} P_n$. There is $B_n$ and $C_n$ in $\mathcal{C} P_n$ with

$$A P_n = B_n C_n - C_n B_n$$

for all $n \geqslant 0$ [10]. Setting $\beta_n$ equal to the inverse of $\|B_n\| \|C_n\| + 1$, we see that $\sum \beta_n A P_n$ converges in the strong topology and that it is a commutator in $\mathcal{C}$. Q.E.D.

EXAMPLE 3.15. Let $\mathcal{C}$ be a properly infinite von Neumann algebra. Suppose there is a sequence $\{P_n\}$ of nonzero orthogonal central projections of sum $1$. Let $E_n$ be a projection in $\mathcal{C} P_n$ such that $E_n \sim P_n - E_n \sim P_n$ and let

$$A = \sum (n^{-1} P_n + n^{-2} E_n).$$

Then $A$ is in class $(F)$ and so there is a sequence $\{\beta_n\}$ of strictly positive numbers and a sequence $\{Q_n\}$ of orthogonal central projections of sum $1$ such that $B = \sum \beta_n A Q_n$ is a commutator in $\mathcal{C}$ (Proposition 3.14). There is no state $\phi$ in $\mathcal{C}^\sim$ and invertible $C$ in $\mathcal{C}$ with $\phi(\mathcal{J}) = (0)$ and $\phi(CBC^{-1}) = 0$; otherwise, the state $\phi$ would vanish on $CAC^{-1}$ or equivalently $\mathcal{K}(CAC^{-1})$ would contain $0$. This is known to be impossible (cf. [11, p. 63ff.]). Hence, the operator $B$ is not similar to a selfadjoint commutator.

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