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THE BLUMBERG PROBLEM(1)

BY

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ABSTRACT. A compact Hausdorff space and a real-valued function on this space are constructed such that the function is not continuous on any dense subspace. This solves the Blumberg problem. Some related results are established.

In 1922 Henry Blumberg proved

PROPOSITION 1 **[B1]**. *If X is a separable complete metric space and f: $X \to \mathbf{R}$ is any real-valued function on X, then there exists a set D which is dense in X, such that the restriction of f to D is continuous.*

DEFINITION. Call a space *Blumberg* iff for any real-valued function f on X, there exists a dense subspace $D \subseteq X$ such that $f|D$ (the restriction of f to D) is continuous.

Recall that a space X is a *Baire* space iff no open subset of X is the union of countably many nowhere dense subsets. In 1960 J. C. Bradford and C. Goffman improved Blumberg's result, establishing

PROPOSITION 2 **[BG]**. *A metric space is Blumberg iff it is a Baire space.*

The question arose: "Which Baire spaces are Blumberg?" and in particular the Blumberg problem (probably due to Goffman): "Must every compact Hausdorff space be Blumberg?". Partial answers were given by R. Levy **[L1]**, **[L2]**, and H. E. White **[W2]**. In fact, there *is* a non-Blumberg compact Hausdorff space. An unusual feature of our example is that it is the disjoint union of two spaces, one or the other of which fails to be Blumberg, depending on whether or not the continuum hypothesis holds.

DEFINITION. A function f from a topological space X into a set Y is called a *δ-fine* function iff for each $y \in Y, f^{-1}(y)$ is nowhere dense in X.

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LEMMA 1. *Assume $2^{\aleph_0} = \aleph_1$. Suppose X is a topological space in which first category sets are nowhere dense. Let $\{f_\alpha: X \to [0,1]: \alpha < \omega_1\}$ be a family of $\aleph_1$ δ-fine functions on X. There exists a δ-fine function $g: X \to [0,1]$ such that for any $\alpha < \omega_1$, $\{x: f_\alpha(x) = g(x)\}$ is nowhere dense in X.*

PROOF. Let $\{y_\beta: 1 \leqslant \beta < \omega_1\}$ be a one-to-one enumeration of $[0,1]$. Define $E_\beta = \bigcup\{f_\alpha^{-1}(y_\beta): \alpha < \beta\}$. Each E_β is first category, hence nowhere dense. Define $g: X \to [0,1]$ as follows:

$$g^{-1}(y_\beta) = ((X - E_\beta) \cap \bigcup\{E_\alpha: \alpha < \beta\}) - \bigcup\{g^{-1}(y_\alpha): \alpha < \beta\}.$$

Clearly g is a well-defined function. We now show that $g(x)$ is defined for every $x \in X$.

Claim #1. $(\forall x \in X)(\exists \beta < \omega_1)(x \in E_\beta)$.

PROOF. If $x \in X$, let $f_0(x) = y_\beta$. Since $\beta \geqslant 1$, $x \in E_\beta$.

Claim #2. $(\forall x \in X)(\{\beta \in \omega_1: x \notin E_\beta\}$ is unbounded in $\omega_1)$.

PROOF. Suppose $\{\beta \in \omega_1: x \notin E_\beta\}$ is bounded by $\gamma < \omega_1$. Define a function $\psi: \omega_1 - \{0\} \to \omega_1$ as follows. If $0 < \beta \leqslant \gamma$, then let $\psi(\beta) = 0$. If $\beta > \gamma$, then $x \in E_\beta = \bigcup\{f_\alpha^{-1}(y_\beta): \alpha < \beta\}$; hence let $\psi(\beta) =$ least α such that $x \in f_\alpha^{-1}(y_\beta)$. ψ is pressing down (regressive), so by the pressing down lemma (see [**Ju**, p. 79]), there exists $\alpha_0 \in \omega_1$ such that $|\{\beta: \psi(\beta) = \alpha_0\}| = \aleph_1$. Thus we have $\beta_1 > \beta_2 > \gamma$ such that $\psi(\beta_1) = \alpha_0 = \psi(\beta_2)$. That is, $x \in f_{\alpha_0}^{-1}(y_{\beta_1})$ and $x \in f_{\alpha_0}^{-1}(y_{\beta_2})$ so that $y_{\beta_1} = f_{\alpha_0}(x) = y_{\beta_2}$. This contradicts that $\beta_1 \neq \beta_2$, since our enumeration of $[0,1]$ was one-to-one.

By Claim #1 and Claim #2, for each $x \in X$ there exists a least ordinal β_x such that $x \notin E_{\beta_x}$, but $x \in E_\alpha$ for some $\alpha < \beta_x$. Hence we have $x \in (X - E_{\beta_x}) \cap \bigcup\{E_\alpha: \alpha < \beta_x\}$. Therefore $g(x) = y_{\beta_x}$.

The function g is a δ-fine function since for each $\beta < \omega_1$, the set $\bigcup\{E_\alpha: \alpha < \beta\}$ is first category and hence nowhere dense. We now show that for each $\alpha < \omega_1$, $\{x \in X: f_\alpha(x) = g(x)\}$ is nowhere dense. If $x \in g^{-1}(y_\beta)$ then $x \in X - E_\beta$ so $x \notin \bigcup\{f_\alpha^{-1}(y_\beta): \alpha < \beta\}$. Therefore if $\beta > \alpha$, then $f_\alpha^{-1}(y_\beta) \cap g^{-1}(y_\beta) = \varnothing$. Let $I_\alpha = \{y \in [0,1]: f_\alpha^{-1}(y) \cap g^{-1}(y) \neq \varnothing\}$. I_α is countable. For each $\alpha < \omega_1$, $I_\alpha = \{y \in [0,1]: (\exists x \in X)(f_\alpha(x) = g(x) = y)\}$. So for any $\alpha < \omega_1$, $\{x \in X: f_\alpha(x) = g(x)\} \subseteq \{x \in X: (\exists y \in I_\alpha)(f_\alpha(x) = y)\} = \bigcup\{f^{-1}(y): y \in I_\alpha\}$ which is first category and hence nowhere dense.

LEMMA 2. *Suppose X is an extremally disconnected completely regular Blumberg space and $g: X \to [0,1]$ is a δ-fine function. There exists a continuous δ-fine function $f: X \to [0,1]$ and a set D dense in X such that $f|D = g|D$.*

PROOF. Suppose $g: X \to [0,1]$ is a δ-fine function. Since X is Blumberg, there is a dense subset $D \subseteq X$ such that $g|D$ is continuous. As is well known (see [**GJ**]), since X is extremally disconnected each dense subspace of X is C^*-

embedded. Hence there exists a continuous function $f\colon X \to [0,1]$ such that $g|D = f|D$.

We now show that f is a δ-fine function. Suppose there exists $y \in [0,1]$ such that $f^{-1}(y)$ is not nowhere dense. Since f is continuous, $f^{-1}(y)$ is closed and hence contains an open set $U \subseteq X$. Therefore $D \cap U$ is a dense subset of U on which f is constant. This contradicts that $f|D = g|D$ and g is a δ-fine function.

Recall that a *π-basis* $\mathfrak{U}$ for a space X is a collection of nonempty open subsets of X such that if V is any nonempty open subset of X then there exists $U \in \mathfrak{U}$ such that $U \subseteq V$. The *π-weight* of a space X is defined as $\pi(X) = \aleph_0 \cdot \min\{\kappa\colon$ there exists a π-basis $\mathfrak{U}$ for X and $|\mathfrak{U}| = \kappa\}$. The *cellularity* of a space X is defined as $c(X) = \aleph_0 \cdot \sup\{\kappa\colon \mathfrak{U}$ is a collection of pairwise disjoint open subsets of X and $|\mathfrak{U}| = \kappa\}$. Let $|C(X,Y)|$ denote the cardinality of the set of all continuous functions from X into Y. $|C(X)|$ represents the cardinality of the set of all real-valued continuous functions on X. The following lemma generalizes a result of [CH].

LEMMA 3. *Let X be any topological space and let Y be a T_2 space. $|C(X,Y)| \leqslant \pi(X)^{c(X)\cdot w(Y)}$, where $w(Y)$ is the weight of Y.*

PROOF. Let $c(X) = \kappa$. Let $\mathfrak{B}$ be a π-basis for X of cardinality $\pi(X)$. Let $\mathcal{C}$ be the set of all collections of $\leqslant \kappa$ elements of $\mathfrak{B}$. Let $\mathfrak{U}$ be a basis for Y of cardinality $w(Y)$. Let $\mathfrak{F} = \{\theta\colon \theta$ is a function from $\mathfrak{U}$ into $\mathcal{C}\}$. Define a function $F\colon C(X,Y) \to \mathfrak{F}$ as follows. $F(g)$ is some function $\theta_g\colon \mathfrak{U} \to \mathcal{C}$ such that $\theta_g(U)$ is a maximal collection of pairwise disjoint elements of $\mathfrak{B}$ which are open subsets of $g^{-1}(U)$. Since $c(X) = \kappa$, $\theta_g \in \mathfrak{F}$ for each $g \in C(X,Y)$ and so F is well defined.

We now show that F is one-to-one. Suppose f and g are two distinct elements of $C(X,Y)$. There exists an $x \in X$ such that $f(x) \neq g(x)$. Since Y is T_2, there is an open $U \in \mathfrak{U}$ such that $g(x) \in U$ and $f(x) \notin \overline{U}$. It follows that $x \in \overline{\bigcup(\theta_g(U))}$ and $x \notin \overline{\bigcup(\theta_f(U))}$ so that $\theta_f \neq \theta_g$ and F is one-to-one. So we have $|C(X,Y)| \leqslant |\mathfrak{F}| = \pi(X)^{c(X)\cdot w(Y)}$.

COROLLARY [CH]. *If X is a regular space, then $|C(X)| \leqslant \pi(X)^{c(X)}$.*

THEOREM 1. *Assume $2^{\aleph_0} = \aleph_1$. Let X be a topological space such that*
(i) *X is completely regular and extremally disconnected,*
(ii) *$\pi(X) \leqslant \aleph_1$, and $c(X) = \aleph_0$,*
(iii) *first category subsets of X are nowhere dense,*
(iv) *there exists a real-valued δ-fine function on X.*
Then X is not Blumberg.

PROOF. If X were Blumberg, then by (i), (iv) and Lemma 2, there would exist a continuous δ-fine function $f\colon X \to [0,1]$. By Lemma 3 there are at most

$\aleph_1^{\aleph_0} = \aleph_1$ of these continuous δ-fine functions. By Lemma 1 there exists a δ-fine function $g: X \to [0, 1]$ such that if f is any continuous δ-fine function from X into $[0, 1]$, then $\{x: f(x) = g(x)\}$ is nowhere dense in X. This contradicts Lemma 2 so X is not Blumberg.

Let $\mathfrak{B}$ be the Boolean algebra of Lebesgue measurable subsets of $[0, 1]$, and let $\mathfrak{I}$ be the ideal of null sets of $\mathfrak{B}$. The reduced measure algebra is the complete Boolean algebra $\mathfrak{B}/\mathfrak{I}$. Let St $(\mathfrak{B}/\mathfrak{I})$ be the Stone space of the reduced measure algebra. As is shown in **[Ha]**, St $(\mathfrak{B}/\mathfrak{I})$ is a compact T_2, extremally disconnected space in which first category sets are nowhere dense and which has weight $\leqslant 2^{\aleph_0}$ and cellularity $\aleph_0$.

For each $B \in \mathfrak{B}$, let $[B]$ denote the equivalence class of B mod $\mathfrak{I}$. Let m denote Lebesgue measure. Let $\mathfrak{B}_n = \{[B] \in \mathfrak{B}/\mathfrak{I}: m(B) = 1/n\}$. For each $[B] \in \mathfrak{B}/\mathfrak{I}$, let $[B]^*$ be the basic open subset of St $(\mathfrak{B}/\mathfrak{I})$ associated with $[B]$. For each $f \in {}^{\omega}\omega$, let $N(f) = \bigcap\{[B]^*_{f(n)}: n < \omega$ and $[B]_{f(n)} \in \mathfrak{B}_n\}$. Let $\mathfrak{F} = \{N(f): f \in {}^{\omega}\omega\}$. It is easy to see that $|\mathfrak{F}| = 2^{\aleph_0}$ and $\bigcup \mathfrak{F} =$ St $(\mathfrak{B}/\mathfrak{I})$. Let $\{F_\alpha: \alpha < 2^{\aleph_0}\}$ enumerate $\mathfrak{F}$. For each $\alpha < 2^{\aleph_0}$, let $F'_\alpha = F_\alpha - \bigcup\{F_\beta: \beta < \alpha\}$. If $\{y_\alpha: \alpha < 2^{\aleph_0}\}$ is a one-to-one enumeration of $[0, 1]$, then g: St $(\mathfrak{B}/\mathfrak{I}) \to [0, 1]$ is a δ-fine function, where $g''(F'_\alpha) = y_\alpha$. Thus we get the following corollary to Theorem 1.

COROLLARY. *Assume* $2^{\aleph_0} = \aleph_1$. St$(\mathfrak{B}/\mathfrak{I})$ *is a compact* T_2 *space which is not Blumberg.*

We now turn our attention to LOTS (linearly ordered topological spaces).

DEFINITION. A collection $\mathcal{C}$ of open subsets of a space X is *oblivious* iff there exists an open subset V of X such that $(\forall x \in V)(\forall \mathcal{C}' \subseteq \mathcal{C})(x \in \bigcap \mathcal{C}'$ implies $(\exists$ open $W \subseteq V)(\varnothing \neq W \subseteq \bigcap \mathcal{C}'))$.

THEOREM 2. *If X is a Baire LOTS, then X is not Blumberg iff there exists an open $U \subseteq X$ such that*

(i) *U is the union of $\leqslant 2^{\aleph_0}$ nowhere dense sets, and*

(ii) *every countable collection of open subsets of U is oblivious.*

PROOF. We shall first show necessity. Let X be a space and let U be an open subset of X satisfying (i) and (ii).

By (i) we can assume that U is the union of κ pairwise disjoint nowhere dense sets, where $\kappa \leqslant 2^{\aleph_0}$; let $U = \bigcup\{E_\alpha: \alpha < \kappa\}$.Let $\{y_\alpha: \alpha < \kappa\}$ enumerate some distinct elements of $\mathbf{R} - \{0\}$. Define $f: X \to \mathbf{R}$ as follows: if $x \in X - U, f(x) = 0$; if $x \in E_\alpha, f(x) = y_\alpha$.

We now show that X is not Blumberg by showing that f is not continuous on any dense subset of X. Suppose $D \subseteq X$ is dense and $f|D$ is continuous. Let $\mathcal{G}$ be the usual countable basis for $\mathbf{R}$. Let $\mathfrak{U} = \{f^{-1}(G) \cap D: G \in \mathcal{G}\}$. Since f is continuous on D, $\mathfrak{U}$ is a countable open cover of D; hence for each $f^{-1}(G) \cap D \in \mathfrak{U}$, there exists V_G open in X such that $V_G \cap D = f^{-1}(G)$

$\cap D$. $\{V_G: G \in \mathcal{G}\}$ is a countable collection of open subsets of X, so let $\mathcal{C} = \{V_G \cap U: G \in \mathcal{G}\}$ be a countable collection of open subsets of U.

By (ii) $\mathcal{C}$ is oblivious on U, hence there is an open $U' \subseteq U$ such that if $x \in U' \cap D$ and $\mathcal{C}' = \{V_G \cap U: f(x) \in G\}$, then there is a nonempty open $W \subseteq U'$ such that $W \subseteq \bigcap \mathcal{C}'$. Thus

$$\begin{aligned} W \cap D \subseteq \bigcap \mathcal{C}' \cap D &= \bigcap\{V_G \cap U \cap D: f(x) \in G\} \\ &= U \cap \bigcap\{V_G \cap D: f(x) \in G\} \\ &= U \cap \bigcap\{f^{-1}(G) \cap D: f(x) \in G\} \\ &= U \cap D \cap \bigcap\{f^{-1}(G): f(x) \in G\} \\ &= U \cap D \cap f^{-1}(\{f(x)\}). \end{aligned}$$

Now since $x \in U' \subseteq U$, $x \in E_\alpha$ for some $\alpha < \kappa$: thus $f^{-1}(\{f(x)\}) = E_\alpha$ which is nowhere dense. We have $W \cap D \subseteq E_\alpha$ which is a contradiction since $W \cap D$ is not nowhere dense.

We now prove the sufficiency.

Claim #1. It suffices to prove: "if U is an open subspace of X with no isolated points and there exists a countable collection of open subsets of U which is not oblivious, then U is Blumberg".

PROOF. We shall assume that X is a Baire LOTS such that no open subset $U \subseteq X$ possesses both properties (i) and (ii), and show that X is Blumberg, using the hypothesis in Claim #1.

We let $f: X \to \mathbf{R}$ and show that there exists a dense subset $D \subseteq X$ such that $f|D$ is continuous. Proceed by induction, defining open subsets U_α and V_α of X and points y_α of $\mathbf{R}$. At stage 0, let $U_0 = X$.

At stage $\alpha + 1$, suppose U_α is a nonempty open subset of X and U_α is not the union of $\leqslant 2^{\aleph_0}$ nowhere dense sets. Thus, there exists $\underline{y \in \mathbf{R}}$ such that $U_\alpha \cap f^{-1}(y)$ is not nowhere dense. Let $y_\alpha = y$. Let $V_\alpha = \operatorname{int}(f^{-1}(y_\alpha)) \cap U_\alpha$. Let $U_{\alpha+1} = U_\alpha - \overline{V}_\alpha$.

At stage α if $\lim(\alpha)$ (i.e. α is a limit ordinal), suppose that for all $\beta < \alpha$, U_β is an open subset of X. Let $U_\alpha = \operatorname{int}(\bigcap\{U_\beta: \beta < \alpha\})$.

This induction can stop in only one of two ways. Let case #1 be that for some ordinal α, $\operatorname{int}(\bigcap\{U_\beta: \beta < \alpha\}) = \varnothing$. Let case #2 be that for some ordinal α, U_α is the union of $\leqslant 2^{\aleph_0}$ nowhere dense sets.

In case #1 we have $\bigcup\{X - U_\beta: \beta < \alpha\}$ dense in X. This means $\bigcup\{U_\beta - U_{\beta+1}: \beta < \alpha\}$ is dense in X, hence $\bigcup\{\overline{V}_\beta: \beta < \alpha\}$ is dense in X. So we have $\bigcup\{V_\beta: \beta < \alpha\}$ dense in X. Note that if we let $D_\beta = V_\beta \cap f^{-1}(y_\beta)$ then D_β is dense in V_β, hence if we let $D = \bigcup\{D_\beta: \beta < \alpha\}$, D is dense in X. Also note that the sets D_β are pairwise disjoint and open in D. Hence $f|D$ is continuous since f is constant on each D_β.

In case #2, since U_α cannot possess both property (i) and property (ii), there exists a countable collection of open subsets of U_α which is not oblivious. U has no isolated points since it is the union of nowhere dense sets. Hence we may, but the hypothesis of the claim, assume that U_α is Blumberg. That is, there exists a dense set $D_1 \subseteq U_\alpha$ such that $f|D_1$ is continuous. By an argument similar to case #1 we can conclude that there is a dense $D_2 \subseteq X - \overline{U}_\alpha$ such that $f|D_2$ is continuous. Thus $D_1 \cup D_2$ is dense in X and $f|(D_1 \cup D_2)$ is continuous.

Claim #2. Suppose U is a Baire LOTS with no isolated points and there exists a countable collection $\mathcal{C}$ of open subsets of U such that for all open $V \subseteq U$ there exist $z \in V$ and $\mathcal{C}' \subseteq \mathcal{C}$ such that $z \in \bigcap \mathcal{C}'$ and $\bigcap \mathcal{C}'$ contains no nonempty open set. Then U is Blumberg.

PROOF. Let U and $\mathcal{C}$ be as above. Let $\mathcal{C} = \{C_n : n < \omega\}$. Each C_n is the disjoint union of open intervals; $C_n = \bigcup\{I_{n,\beta} : \beta < \lambda_n\}$. Let $\mathcal{C}^* = \{I_{n,\beta} : \beta < \lambda_n;\ n < \omega\}$. Let $\mathcal{C}_f^* = \{$nonempty finite intersections of elements of $\mathcal{C}^*\}$.

We now prove that $\mathcal{C}_f^*$ is a π-basis for U. Let (x, y) be a nonempty open interval in U. It suffices to show that there exists $I \in \mathcal{C}_f^*$ such that $I \subseteq (x,y)$.

Since U contains no isolated points there exist points p, q, a, b, and c in U such that $x < a < p < b < q < c < y$. By hypothesis we can let $z \in (p,q)$ such that there exists $\{I_m : m < \omega\} \subseteq \mathcal{C}_f^*$ such that $z \in \bigcap\{I_m : m < \omega\}$ and $\bigcap\{I_m : m < \omega\}$ contains no nonempty open set. Without loss of generality we can assume that for each $m < \omega$, $I_{m+1} \subseteq I_m$.

We now show that for some $m < \omega$, $I_m \subseteq (x,y)$. Suppose that for all $m < \omega$, $I_m \cap (U - (x,y)) \neq \varnothing$. Since $z \in \bigcap\{I_m : m < \omega\}$, for each $m < \omega$, $I_m \cap (x,y) \neq \varnothing$. Thus, since $\{I_m : m < \omega\}$ is a nested sequence of intervals, either $(\forall m < \omega)(y \in I_m)$ or $(\forall m < \omega)(x \in I_m)$. Without loss of generality assume $(\forall m < \omega)(y \in I_m)$. Therefore, since each I_m is an interval, we have $[z,y] \subseteq \bigcap\{I_m : m < \omega\}$. Since $z \in (p,q)$, $\varnothing \neq (q,y) \subseteq [z,y] \subseteq \bigcap\{I_m : m < \omega\}$, which contradicts our choice of $\{I_m : m < \omega\}$.

Now, for each finite subset $F \subseteq \omega$, let $\mathcal{B}_F = \{I : I$ is the intersection of exactly one open interval from each C_n such that $n \in F\}$. Since each C_n consists of parwise disjoint intervals, the $\mathcal{B}_F$'s are pairwise disjoint, and $\mathcal{C}_f^* = \bigcup\{\mathcal{B}_F : F$ is a finite subset of $\omega\}$. Hence $\mathcal{C}_f^*$ is a σ-disjoint π-basis for U and hence U satisfies the criterion in **[W1]** for being a Blumberg space (i.e. U is Baire and has a σ-disjoint π-basis).

By combining Claim #1 and Claim #2 we have that X is Blumberg.

A compact LOTS will be constructed which, assuming $2^{\aleph_0} \geqslant \aleph_2$, will satisfy (i) and (ii) of Theorem 2. To begin, the classical construction of κ-Aronszajn trees (see **[Je]**) is modified. Recall that a *tree*, $\langle T, <_T\rangle$, is a well-founded partial order such that for each $x \in T$, $\hat{x} = \{t \in T : t <_T x\}$ is well-ordered. If α is an ordinal, $T_\alpha = \{x \in T : x$ is order-isomorphic to $\alpha\}$. If α is an ordinal and κ is a cardinal then T is a (α, κ)-tree iff $T = \bigcup\{T_\beta: \beta \leqslant \alpha\}$ and for all $\beta < \alpha$, $|T_\beta| < \kappa$.

LEMMA 4. *There exists an $(\omega_2, (2^{\aleph_0})^+)$-tree T, such that if b is a maximal chain in T and if β is the least ordinal such that $b \cap T_\beta = \varnothing$, then the cofinality of β is ω_1.*

PROOF. Recall that an η_1-*set* is a linearly ordered set $\langle X, \prec\rangle$ such that if A and B are countable subsets of X and $(\forall a \in A)(\forall b \in B)(a \prec b)$ then there exists $x \in X$ such that $(\forall a \in A)(\forall b \in B)(a \prec x \prec b)$. Let $\langle Q, \prec\rangle$ be the canonical η_1-set constructed in **[GJ]** and $\langle R, \prec\rangle$ be its Dedekind completion with endpoints 0_R and 1_R. The same argument as for Lemma B in **[Ju**, p. 122**]** shows that neither ω_2 nor ω_2^* can be order-embedded in R and hence in Q. As is shown in **[GJ]**, $|Q| = 2^{\aleph_0}$. Let $C = \{x \in R: x$ is the supremum of a countable strictly increasing sequence from $Q\}$. Let $S = \{0_R\} \cup Q \cup C$. $|S| = 2^{\aleph_0}$.

The tree T will be constructed from well-ordered sequences of elements of S which have endpoints. The order relation will be set inclusion, written $<_T$. If σ is a sequence with endpoint q, let $\sigma^* = q$. Let $\sigma|\alpha + 1$ be the sequence consisting of the first $\alpha + 1$ elements of σ.

The basic idea behind the construction of T is that we construct the tree level by level, extending all subbranches at levels indexed by successor ordinals or limit ordinals of countable cofinality. However, at levels indexed by limit ordinals of cofinality ω_1, we only extend enough subbranches to keep the tree "growing"; thus we keep the cardinality of each level at $2^{\aleph_0}$.

We will proceed by induction on the levels of T using the inductive hypothesis IH(β): suppose for every $\gamma < \beta$ and for every $\sigma \in T_\gamma$,

(i) if cf $(\beta) = \omega$ and $\{\gamma_n: 0 \leqslant n < \omega\}$ is an increasing sequence of ordinals cofinal with β such that $\gamma_0 = \gamma$, and for each $n < \omega$ there exists $\sigma_n \in T_{\gamma_n}$ such that $\sigma_{n+1}|\gamma_n + 1 = \sigma_n$ with $\sigma_0 = \sigma$, then there exists $\tau \in T$ such that for all $n < \omega$, $\tau|\gamma_n + 1 = \sigma_n$ and $\tau^* = \sup\{\sigma_n^*: n < \omega\} \in C$.

(ii) if cf $(\beta) \neq \omega$ and $q \in Q$ and $\sigma^* \prec q$, then there exists $\tau \in T$ such that $\tau^* = q$ and $\tau|\gamma + 1 = \sigma$.

Let $T_0 = \{0_R\}$.

In order to construct $T_{\alpha+1}$ assume IH(β) for all $\beta \leqslant \alpha$. Let $T_{\alpha+1} = \{\tau: \tau$ has length $\alpha + 2$ and $(\exists\sigma \in T_\alpha)(\tau|\alpha + 1 = \sigma$ and $\tau^* \in Q$ and $\sigma^* \prec \tau^*)\}$.

Claim. IH $(\alpha + 1)$.

PROOF. Let $\gamma < \alpha + 1$ and $\sigma \in T_\gamma$. Let $q \in Q$ such that $\sigma^* \prec q$.

If $\gamma = \alpha$, then IH $(\alpha + 1)$ follows immediately from the construction of $T_{\alpha+1}$.

If $\gamma < \alpha$ and cf $(\alpha) \neq \omega$, let $p \in Q$ such that $\sigma^* \prec p \prec q$. By IH(σ) there exists $\rho \in T_\alpha$ such that $\rho^* = p$ and $\rho|\gamma + 1 = \sigma$. By the construction of $T_{\alpha+1}$

there exists $\tau \in T_{\alpha+1}$ such that $\tau^* = q$ and $\tau|\alpha + 1 = \rho$. Hence $\tau|\alpha + 1 = \sigma$ and we have IH $(\alpha + 1)$ in this case.

If $\gamma < \alpha$ and cf $(\alpha) = \omega$, pick $\{\gamma_n: n < \omega\}$ as a strictly increasing sequence of ordinals such that $\gamma_0 = \gamma$ and for each $n \geqslant 1$, γ_n is a successor ordinal and $\sup\{\gamma_n: n < \omega\} = \alpha$. Pick $\{p_n: 1 \leqslant n < \omega\} \subseteq Q$ such that, for all $n < \omega$, $p_{n+1} \prec p_{n+2} \prec q$. Let $p_0 = \sigma^*$. Let $\sigma_0 = \sigma$. Since we have IH(β) for all $\beta < \alpha$, for each $n \geqslant 1$ pick $\sigma_n \in T_{\gamma_n}$ such that $\sigma_n^* = p_n$ and $\sigma_n|\gamma_{n-1} + 1 = \sigma_{n-1}$. By IH($\alpha$), there exists $\rho \in T_\alpha$ such that for all $n < \omega$, $\rho|\gamma_n + 1 = \sigma_n$ and $\rho^* = \{\sup p_n: n < \omega\} \in C$. Since Q is an η_1-set, $\rho^* \prec q$. From the construction of $T_{\alpha+1}$ there exists $\tau \in T_{\alpha+1}$ such that $\tau^* = q$ and $\tau|\alpha + 1 = \rho$. Thus $\tau|\gamma + 1 = \rho|\gamma + 1 = \rho|\gamma_0 + 1 = \sigma_0 = \sigma$, and IH $(\alpha + 1)$ also holds in this case.

In order to construct T_λ where cf $(\lambda) = \omega$ assume IH(β) for all $\beta < \lambda$. Let b be a branch (i.e. a maximal chain) cofinal in $\bigcup\{T_\alpha: \alpha < \lambda\}$ and let $\{\alpha_n: n < \omega\}$ be a sequence of ordinals cofinal with λ. Let $p = \sup\{\sigma^*: \sigma \in b \cap T_{\alpha_n}; n < \omega\}$. Thus $p \in C$ and for all $\sigma \in b$, $\sigma^* \prec p$. Hence each branch cofinal in $\bigcup\{T_\alpha: \alpha < \lambda\}$ can be extended in a unique way. Let

$$\begin{aligned} T_\lambda = \{\tau\colon\ & \tau \text{ has length } \lambda + 1 \text{ and there exists a cofinal branch} \\ & b \in \textstyle\bigcup\{T_\alpha: \alpha < \lambda\} \text{ such that for all } \alpha < \lambda, \sigma \in b \cap T_\alpha \\ & \text{implies } \tau|\alpha + 1 = \sigma \text{ and } \tau^* = \sup\{\sigma^*: \sigma \in b \cap T_\alpha; \alpha < \lambda\}\}. \end{aligned}$$

Clearly this satisfies IH(λ).

In order to construct T_δ for cf $(\delta) = \omega_1$ assume IH(β) for all $\beta < \delta$. We first describe the construction of the sequence $\tau(\alpha, \sigma, q)$. Let $\alpha < \delta$ and $\sigma \in T_\alpha$. Let $q \in Q$ such that $\sigma^* < q$. Let $\{q_{\nu+1}: \nu < \omega_1\}$ be an increasing sequence of elements of Q, greater than σ^* which converge to q. Let $\{\alpha_{\nu+1}: \nu < \omega_1\}$ be an increasing sequence of successor ordinals converging to δ.

Inductively define a chain of elements $\{\sigma_\nu: \nu < \omega_1\}$ such that $\sigma_\nu \in T_{\alpha_\nu}$ and $\sigma_\nu^* \prec q_{\nu+1}$ as follows. Let $\sigma_0 = \sigma$ and $\alpha_0 = \alpha$.

Suppose $\sigma_\nu \in T_{\alpha_\nu}$ has been defined and $\sigma_\nu^* \prec q_{\nu+1}$. By IH $(\alpha_{\nu+1})$ there exists $\sigma_{\nu+1} \in T_{\alpha_{\nu+1}}$ such that $\sigma_{\nu+1}^* = q_{\nu+1}$ and $\sigma_{\nu+1}|\alpha_\nu + 1 = \sigma_\nu$. Since $q_{\nu+1} \prec q_{\nu+2}$, $\sigma_{\nu+1}^* \prec q_{\nu+2}$. Suppose $\sigma_\nu \in T_{\alpha_\nu}$ has been defined for all $\nu < \eta$ and cf $(\eta) = \omega$. Pick $\alpha_\eta = \sup\{\alpha_\nu: \nu < \eta\}$. By IH (α_η) there exists $\sigma_\eta \in T_{\alpha_\eta}$ such that for all $\nu < \eta$, $\sigma_\eta|\alpha_\nu + 1 = \sigma_\nu$ and $\sigma_\eta^* = \sup\{\sigma_\nu^*: \nu < \eta\}$. Since Q is an η_1-set, $\sigma_\eta^* \prec q_{\eta+1}$.

Let $\tau(\alpha, \sigma, q)$ be that sequence τ such that τ has length $\delta + 1$, $\tau|\alpha_\nu + 1 = \sigma_\nu$ for all $\nu < \omega_1$, and $\tau^* = q$. Let $T_\delta = \{\tau(\alpha, \sigma, q): \alpha < \delta, \sigma \in T_\alpha$, and $q \in Q$ such that $\sigma^* \prec q\}$. Clearly this satisfies IH(δ).

Let $T = \bigcup\{T_\alpha: \alpha < \omega_2\}$. T is a *normal* tree **[De]**. That is:

(i) $|T_0| = 1$,

(ii) $(\forall\alpha < \omega_2)(\forall\sigma \in T_\alpha)(\exists\tau_1, \tau_2 \in T_{\alpha+1})(\tau_1 \neq \tau_2 \;\&\; \sigma <_T \tau_1 \;\&\; \sigma <_T \tau_2)$,

(iii) $(\forall\alpha < \beta < \omega_2)(\forall\sigma \in T_\alpha)(\exists\tau \in T_\beta)(\sigma <_T \tau)$,

(iv) $(\forall\alpha < \omega_2)(\text{if } \lim(\alpha) \text{ then } (\forall\sigma, \tau \in T_\alpha)(\hat{\sigma} = \hat{\tau} \text{ implies } \sigma = \tau))$.

By induction on $\alpha < \omega_2$ it can be shown that $(\forall\alpha < \omega_2)(|T_\alpha| \leqslant 2^{\aleph_0})$. So clearly T is an $(\omega_2, (2^{\aleph_0})^+)$-tree. No branch of T "ends" at an ordianl with cofinality ω (i.e. if b is a branch of T and β is the least ordinal such that $b \cap T_\beta = \varnothing$, then cf $(\beta) \neq \omega$). Also, if T has a cofinal branch b, of length ω_2, then $\{\sigma^*: \sigma \in b\}$ would determine an embedding of ω_2 into S, which is impossible. Hence, if b is a branch of T and β is the least ordinal such that $T_\beta \cap b = \varnothing$, then cf $(\beta) = \omega_1$, and the lemma is proved.

Using Lemma 4 and modifying the "Souslin tree yields Souslin line" construction in **[Mi]**, we have

Lemma 5. *There exists a linear order $\langle L, \prec\rangle$ such that*

(i) *neither ω_2 nor ω_2^* is order-embeddable in L,*

(ii) *if $\{a_m: m < \omega\}$ and $\{b_n: n < \omega\}$ are subsets of L such that $(\forall m < \omega)$ $(\forall n < \omega)(a_m \preccurlyeq a_{m+1} \prec b_{n+1} \preccurlyeq b_n)$, then there exists three points x, y, z of L such that $(\forall m < \omega)(\forall n < \omega)(a_m \prec x \prec y \prec z \prec b_n)$,*

(iii) *L is the union of an increasing chian of $\aleph_2$ proper subsets each of which is nowhere dense in the order topology.*

Proof. Let T be the tree constructed in Lemma 4. For each $\tau \in T$, let $S(\tau) = \{\sigma: \sigma \text{ is an immediate successor of } \tau\}$. For each $\tau \in T$, $S(\tau) = 2^{\aleph_0}$, so we can impose a linear order $<_\tau$ on $S(\tau)$ to make $S(\tau)$ order-isomorphic to the η_1-set Q in Lemma 4.

If b is a branch of T and $\alpha < \omega_2$, denote the element of $b \cap T_\alpha$, if it exists, by b^α. Define a linear order $\prec$ on the branches of T as follows. $b \prec c$ iff the following condition holds: α is the greatest ordinal such that $b \cap c \cap T_\alpha \neq \varnothing$ and $b \cap c \cap T_\alpha = \{\tau\}$ and $b_{\alpha+1} <_\tau c_{\alpha+1}$. The next claim shows that $\prec$ is well defined.

Claim #1. If $b \neq c$ are branches of T, then there exists a maximal ordinal α such that $b \cap c \cap T_\alpha \neq \varnothing$.

Proof. It suffices to show that if $\lim(\alpha)$ and for all $\beta < \alpha$, $b \cap c \cap T_\beta \neq \varnothing$, then $b \cap c \cap T_\alpha \neq \varnothing$. For all $\beta < \alpha$, $b^\beta = c^\beta$; since $b \neq c$ we must have $\varnothing \neq b \cap T_\alpha = \{b^\alpha\}$ and $\varnothing \neq c \cap T_\alpha = \{c^\alpha\}$. We have $\lim(\alpha)$ and $\hat{b}^\alpha = \hat{c}^\alpha$, hence since T is normal, $b^\alpha = c^\alpha$ and $b \cap c \cap T_\alpha = \{b^\alpha\} \neq \varnothing$.

It is now clear that $\prec$ is an antisymmetric total order on the set of branches of T. A straightforward argument shows that $\prec$ is transitive. Thus let L be the set of branches of T so that $\prec$ is a linear order on L.

Claim #2. Condition (i) holds for $\langle L, \prec\rangle$.

Proof. Suppose $\{b_\gamma: \alpha < \omega_2\}$ embeds ω_2 into L. We will inductively define a branch $d \subseteq T$ such that for each $\alpha < \omega_2$, $|\{\gamma: b_\gamma \cap d \cap T_\alpha = \varnothing\}| \leqslant \aleph_1$.

Let $A_\alpha = \{\gamma: b_\gamma \cap d \cap T_\alpha = \varnothing\} = \{\gamma: d^\alpha \notin b_\gamma\}$.

Let d^0 be the element of T_0.

Suppose d^α is defined and $|A_\alpha| \leqslant \aleph_1$. We shall define $d^{\alpha+1}$ as follows. Let $B = \{\tau \in S(d^\alpha): (\exists\gamma < \omega_2)(\tau \in b_\gamma)\}$. Since $|\omega_2 - A_\alpha| = \aleph_2$, $|B| \geqslant 1$.

Suppose $|B| = \aleph_2$. Since $\{b_\gamma: \gamma < \omega_2\}$ is well-ordered by $\prec$, B is well-ordered by $<_{d^\alpha}$. Hence B gives rise to an embedding of ω_2 into $S(d^\alpha)$; but this is impossible since $S(d^\alpha)$ is order-isomorphic to Q.

We must have $|B| \leqslant \aleph_1$. Since $|A_\alpha| \leqslant \aleph_1$, $|\{\gamma: d^\alpha \in b_\gamma\}| = \aleph_2$, so we have $|\{\gamma: (\exists\tau \in S(d^\alpha))(\tau \in b_\gamma)\}| = \aleph_2$. Since $|B| \leqslant \aleph_1$, there exists $\tau \in S(d^\alpha)$ such that $|\{\gamma: \tau \in b_\gamma\}| = \aleph_2$.

We claim that τ is unique. Suppose there exist $\tau_1, \tau_2 \in S(d^\alpha)$ such that $\tau_1 <_{d^\alpha} \tau_2$ and $|\{\gamma: \tau_1 \in b_\gamma\}| = \aleph_2 = |\{\gamma: \tau_2 \in b_\gamma\}|$. If $\gamma_0 \in \{\gamma: \tau_2 \in b_\gamma\}$, then for all $\alpha \in \{\gamma: \tau_1 \in b_\gamma\}$, $b_\gamma \prec b_{\gamma_0}$. Since $|\{\gamma: \tau_1 \in b_\gamma\}| = \aleph_2$ this contradicts the fact that $\{b_\gamma: \gamma < \omega_i\}$ is a well-ordered sequence of type ω_2.

Since τ is unique, we let $d^{\alpha+1} = \tau$, and we have $|A_{\alpha+1}| \leqslant \aleph_1$ since $A_{\alpha+1} = \bigcup\{A_\alpha: \beta \leqslant \alpha\} \cup \{\gamma: d^{\alpha+1} \notin b_\gamma\}$.

Suppose $\lim(\alpha)$ and for all $\beta < \alpha$, d^β is defined and $|A_\beta| \leqslant \aleph_1$. Since $\alpha < \omega_2$, $|\bigcup\{A_\beta: \beta < \alpha\}| \leqslant \aleph_1$; hence $|\{\gamma: (\forall\beta < \alpha)(b_\gamma \cap T_\beta = \{d^\beta\})\}| = \aleph_2$. Thus, there must exist $\tau \in T_\alpha$ such that $\hat{\tau} = \{d^\beta: \beta < \alpha\}$. Since T is normal, τ is unique; hence we can let $d^\alpha = \tau$ and then

$$|A_\alpha| = |\bigcup\{A_\beta: \beta < \alpha\}| \leqslant \aleph_1.$$

We have defined an ω_2-branch d of T, which contradicts Lemma 4, so ω_2 is not embeddable in $\langle L, \prec\rangle$. The case for ω_2^* is similar.

Claim #3. Condition (ii) holds for $\langle L, \prec\rangle$.

PROOF. Let $A = \{a_m: m < \omega\}$ be a nondecreasing subset of L; let $B = \{b_n: n < \omega\}$ be a nonincreasing subset of L such that $(\forall m < \omega)(\forall n < \omega)(a_m \prec b_n)$. For each $n < \omega$, let α_n be the greatest ordinal such that $a_n \cap b_n \cap T_{\alpha_n} \neq \varnothing$ and let $\{\tau_n\} = a_n \cap b_n \cap T_{\alpha_n}$.

We now show that for all $n < \omega$, $\tau_n \leqslant_T \tau_{n+1}$. We first show that $\tau_n \in a_{n+1}$. Suppose $\tau_n \notin a_{n+1}$. Let β be the greatest ordinal such that $a_n \cap a_{n+1} \cap T_\beta \neq \varnothing$. Since $\tau_n \in a_n \cap T_{\alpha_n}$, we must have $\beta < \alpha_n$. Let $a_n \cap a_{n+1} \cap T = \{\sigma\}$. Since $a_n \preccurlyeq a_{n+1}$, $a_n^{\beta+1} \leqslant_\sigma a_{n+1}^{\beta+1}$. Since $\beta < a_n$, $b_n^{\beta+1} = a_n^{\beta+1}$ so that $d_n^{\beta+1} \leqslant_\sigma a_n^{\beta+1}$. Therefore $b_n \preccurlyeq a_{n+1}$ which is a contradiction. Similarly, $\tau_n \in b_{n+1}$, so that $\tau_n \in a_{n+1} \cap b_{n+1}$. Hence, by the definition of τ_{n+1}, $\tau_n \leqslant_T \tau_{n+1}$.

Let $\alpha = \sup\{\alpha_n: n < \omega\}$. Suppose for some $n < \omega, \alpha = \alpha_n$. Since $(\forall n < \omega)(\tau_n \leqslant_T \tau_{n+1})$, we have $(\forall n < \omega)(\alpha_n \leqslant \alpha_{n+1})$. Thus, in this case, we have $(\forall j \geqslant n)(\alpha_j = \alpha_n$ and $\tau_j = \tau_n)$. Let $A' = \{a_j^{\alpha_n+1}: j \geqslant n\}$ and $B' = \{b_j^{\alpha_n+1}: j \geqslant n\}$. We have $A' \cup B' \subseteq S(\tau_n)$, and for all $\rho \in A'$ and $\sigma \in B'$, $\rho <_{\tau_n} \sigma$. Since $S(\tau_n)$ is a η_1-set, there exists $\pi_1, \pi_2, \pi_3 \in S(\tau_n)$ such that for all $j \geqslant n$, $a_j^{\alpha_n+1} <_{\tau_n} \pi_1 <_{\tau_n} \pi_2 <_{\tau_n} \pi_3 <_{\tau_n} b_j^{\alpha_n+1}$. So by picking branches x, y, z of

T through π_1, π_2, π_3 respectively we have that for all $m, n < \omega$, $a_m \prec x \prec y \prec z \prec b_n$.

Now suppose $\alpha \notin \{\alpha_n: n < \omega\}$. Since $\mathrm{cf}(\alpha) = \omega$, there exists $\tau \in T_\alpha$ such that for all $n < \omega$, $\tau_n <_T \tau$. Let $A'' = \{\rho \in S(\tau): (\exists n < \omega)(\rho \in a_n)\}$. Let $B'' = \{\sigma \in S(\tau): (\exists n < \omega)(\sigma \in b_n)\}$. A'' and B'' may be empty, but in any case for all $\rho \in A''$ and $\sigma \in B''$, $\rho <_\tau \sigma$. Hence we can pick elements π_1, π_2, π_3 of $S(\tau)$ such that for all $\rho \in A''$ and $\sigma \in B''$ we have $\rho <_\tau \pi_1 <_\tau \pi_2 <_\tau \pi_3 <_\tau \sigma$. By picking branches x, y, z of T through π_1, π_2, π_3 respectively, we have that for all $m, n < \omega$, $a_m \prec x \prec y \prec z \prec b_n$.

Claim #4. Condition (iii) holds in $\langle L, \prec\rangle$.

PROOF. For $\tau \in T$, let $U_\tau = \{\text{branches } b \in L: \tau \in b\}$. We will show that each U_τ is open in the order-induced topology on L. Suppose $b \in U_\tau$, and $\tau \in T_\alpha$. Since T is normal, $\varnothing \neq b \cap S(\tau) = \{b^{\alpha+1}\}$. Pick $\rho, \sigma \in S(\tau)$ such that $\rho <_\tau b^{\alpha+1} <_\tau \sigma$; pick branches a, c through ρ, σ respectively. $a \prec b \prec c$ and $(a,c) \subseteq U_\tau$ so U_τ is open.

Let $\mathcal{U} = \{U_\tau: \tau \in T\}$. We will show that $\mathcal{U}$ is a basis for the order topology on L. Let $b \in (a,c) \subseteq L$. Pick $\tau \in T$ such that $\tau \in b$, $\tau \in a$, and $\tau \in c$. Thus $b \in U \subseteq (a,c)$.

For each $\alpha < \omega_2$, let G_α denote the open set $\bigcup\{U_\tau: \tau \in T_\alpha\}$. We will show that G_α is a dense subset of L. Let U_τ be a basic open subset of L. If $\tau \in T_\alpha$ for some $\beta \geqslant \alpha$, then $U_\tau \subseteq G$. If $\tau \in T_\beta$ for some $\beta < \alpha$, then since T is normal there exists $\sigma \in T_\alpha$ such that $\tau <_T \sigma$. Hence $U_\sigma \subseteq U_\tau$, so $U_\tau \cap G_\alpha \neq \varnothing$. Thus G_α is dense in L.

For each $\alpha < \omega_2$, let $E_\alpha = \{b \in L: b \cap T_\alpha = \varnothing\}$. If $\alpha < \beta$, then $E_\alpha \subseteq E_\beta$. Since T has no branches of length ω_2, $L = \bigcup\{E_\alpha: \alpha < \omega_2\}$. Each E_α is nowhere dense since G_α is an open dense subset of L and $E_\alpha = L - G_\alpha$.

THEOREM 3. *There exists a compact LOTS with the following properties*:

(a) *it is the union of* $\aleph_2$ *nowhere dense subsets*,

(b) *any countable collection of open subsets of it is oblivious.*

PROOF. Let $\tilde{L}$ be the Dedekind completion of the line L in Lemma 5. L is a compact LOTS.

In order to show that $\tilde{L}$ has property (a), note that as in Lemma 5, $L = \bigcup\{E_\alpha: \alpha < \omega_2\}$. Since each E_α is nowhere dense in L it is nowhere dense in $\tilde{L}$, hence each $\overline{E}_\alpha$ is nowhere dense in $\tilde{L}$. Also if $\alpha < \beta$, then $\overline{E}_\alpha \subseteq \overline{E}_\beta$. Each $x \in \tilde{L}$ is the supremum (or infimum) of $\aleph_1$ elements of L, since otherwise ω_2 (or ω_2^*) would be embeddable in L, contradicting Lemma 5. Thus, for each $x \in \tilde{L}$, there exists $\alpha < \omega_2$ such that $x \in \overline{E}_\alpha$. Therefore $\tilde{L} = \bigcup\{\overline{E}_\alpha: \alpha < \omega_2\}$, the union of $\aleph_2$ nowhere dense sets.

In order to show that $\tilde{L}$ has property (b), let $\mathcal{C}$ be a countable collection of open subsets of $\tilde{L}$ and show $\mathcal{C}$ is oblivious. Suppose $x \in \tilde{L}$ and $\{C_n: n < \omega\}$ is a subcollection of $\mathcal{C}$ such that $x \in \bigcap\{C_n: n < \omega\}$. Since L is dense in $\tilde{L}$, for all $n < \omega$ there exist a_n, $b_n \in L$ such that $x \in (a_n, b_n) \subseteq \bigcap\{C_j : j \leqslant n\}$, and $a_n \preccurlyeq a_{n+1}$, and $b_{n+1} \preccurlyeq b_n$. Thus for all $m, n < \omega$, $a_m \preccurlyeq a_{m+1} \prec b_{n+1} \preccurlyeq b_n$. By Lemma 5 there exist $w, y, z \in L$ such that for all $m, n < \omega$, $a_m \prec w \prec y \prec z \prec b_n$. Thus $\varnothing \neq (w, z) \subseteq \bigcap\{(a_n, b_n): n < \omega\} \subseteq \bigcap\{C_n : n < \omega\}$. Therefore $\mathcal{C}$ is oblivious and the theorem is proved.

If $2^{\aleph_0} \geqslant \aleph_2$, the compact LOTS in Theorem 3 satisfies conditions (i) and (ii) of Theorem 2. Hence the following corollary.

COROLLARY. *Assume $2^{\aleph_0} \geqslant \aleph_2$. The space $\tilde{L}$ in Theorem 3 is a compact LOTS which is not Blumberg.*

THEOREM 4. *There exists a compact Hausdorff space which is not Blumberg.*

PROOF. Let X be the disjoint union of $\mathrm{St}(\mathfrak{B}/\mathfrak{I})$ and $\tilde{L}$. If X is Blumberg, then clearly both $\mathrm{St}(\mathfrak{B}/\mathfrak{I})$ and $\tilde{L}$ must be Blumberg. However, by the corollaries to Theorem 1 and Theorem 3, regardless of the value of $2^{\aleph_0}$, $\mathrm{St}(\mathfrak{B}/\mathfrak{I})$ and $\tilde{L}$ cannot *both* be Blumberg.

We shall now relate some miscellaneous results regarding Blumberg spaces. Theorem 2 gives necessary and sufficient conditions for a Baire LOTS to be Blumberg. By examining the proof, we notice that the following were proved.

THEOREM 5. *If X is a space such that*
(i) *X is the union of $\leqslant 2^{\aleph_0}$ nowhere dense subsets, and*
(ii) *every countable collection of open subsets of X is oblivious*
then X is not Blumberg.

PROPOSITION 3. *If X is a space such that no open subset of X is the union of $\leqslant 2^{\aleph_0}$ nowhere dense subsets, then X is Blumberg.*

Proposition 3 appears in **[W1]** and it, together with Theorem 5, enables us to tell whether some spaces are Blumberg. For example, consider the compact LOTS $\tilde{L}$ constructed previously. It is straightforward to show that nonempty G_δ's of $\tilde{L}$ have nonempty interiors, and hence that $\tilde{L}$ is not the union of $\leqslant \aleph_1$ nowhere dense subsets. Thus, by Proposition 3 and Theorem 3, we conclude $\tilde{L}$ is Blumberg iff $2^{\aleph_0} = \aleph_1$. The following theorem was proved by White in **[W2]**, using the additional assumption: $2^{\aleph_0} = \aleph_1$.

THEOREM 6. *A Souslin space (i.e. a nonseparable LOTS with cellularity $\aleph_0$) is not Blumberg.*

PROOF. As shown in **[Ju]**, a LOTS with cellularity $\aleph_0$ has cardinality $\leqslant 2^{\aleph_0}$. We can assume without loss of generality that the Souslin space, X, has no isolated points. Thus by Theorem 5, it suffices to show that if $\mathcal{C}$ is a countable collection of open subsets of X, then $\mathcal{C}$ is oblivious. To this end, let $\mathcal{C} = \{C_n : n < \omega\}$ where each C_n is open. Since $c(X) = \aleph_0$, each

$$C_n = \bigcup\{I_k^n : k < \omega\}$$

where each I_k^n is an open interval. Let $\mathcal{I} = \{I_k^n: k < \omega, n < \omega\}$. Let $\mathcal{G} = \{$finite intersections of elements of $\mathcal{I}\}$. Thus $|\mathcal{G}| = \aleph_0$.

Since X has uncountable π-weight, there exist elements a, b, c, d of X such that $a < b < c < d$ and $(b, c) \neq \varnothing$ and no element of $\mathcal{G}$ is included in (a, d).

Let $x \in (b, c)$. Since elements of $\mathcal{G}$ are intervals, if $G \in \mathcal{G}$ and $x \in G$, then either $a \in G$ or $d \in G$. Since $\mathcal{G}$ is closed under finite intersections, either $(\forall G \in \mathcal{G})(x \in G$ implies $a \in G)$ or $(\forall G \in \mathcal{G})(x \in G$ implies $d \in G)$. Hence, either $a \in \bigcap\{G \in \mathcal{G}: x \in G\}$ and $(a, x) \subseteq \bigcap\{G \in \mathcal{G}: x \in G\}$, or $d \in \bigcap\{G \in \mathcal{G}: x \in G\}$ and $(x, d) \subseteq \bigcap\{G \in \mathcal{G}: x \in G\}$.

Thus, if $\mathcal{C}' \subseteq \mathcal{C}$ and x is an element of the nonempty open set (b, c) such that $x \in \bigcap\mathcal{C}'$, then either $\varnothing \neq (a, x) \subseteq \bigcap\{G \in \mathcal{G}: x \in G\} \subseteq \bigcap\mathcal{C}$ or $\varnothing \neq (x, d) \subseteq \bigcap\{G \in \mathcal{G}: x \in G\} \subseteq \bigcap\mathcal{C}'$. Therefore $\mathcal{C}$ is oblivious and X is not Blumberg.

The next theorem was proved by Levy in **[L2]**, using the stronger assumption: $2^{\aleph_0} = 2^{\aleph_1}$.

THEOREM 7. *Assume* $\mathrm{cf}(2^{\aleph_0}) > \omega_1$. *The Dedekind completion* $\tilde{Q}$ *of the canonical* η_1*-set* Q *in* **[GJ]** *is not Blumberg.*

PROOF. Since the η_1-set Q is dense in $\tilde{Q}$, a straightforward argument shows that every countable collection of open subsets of $\tilde{Q}$ is oblivious. Thus, by Theorem 5, it only remains to show that $\tilde{Q}$ is the union of $\leqslant 2^{\aleph_0}$ nowhere dense subsets.

As is shown in **[Ju]**, neither ω_2 nor ω_2^* is embeddable in Q; hence every element of $\tilde{Q} - Q$ is the supremum or infimum of at most $\aleph_1$ elements of Q. Since $|Q| = 2^{\aleph_0}$, we will enumerate Q as $\{q_\alpha: \alpha < 2^{\aleph_0}\}$. For each $\beta < 2^{\aleph_0}$, let $E_\beta = \overline{\{q_\alpha: \alpha < \beta\}}$. Thus $\tilde{Q} = \overline{Q} = \bigcup\{E_\beta: \beta < 2^{\aleph_0}\}$ since $\mathrm{cf}(2^{\aleph_0}) > \omega_1$.

We now show that each E_β is nowhere dense. Suppose E_β is dense in some open interval $(a, b) \subseteq \tilde{Q}$. Therefore $\{q_\alpha: \alpha < \beta\}$ is dense in (a,b); hence $\{q_\alpha: \alpha < \beta\}$ is dense in $Q \cap (a, b)$. Therefore $\{q_\alpha: \alpha < \beta\}$ is an η_1-set. However, as is shown in **[GJ]**, every η_1-set has cardinality at least $2^{\aleph_0}$, which gives a contradiction since $\beta < 2^{\aleph_0}$.

THEOREM 8. *If* $2^{\aleph_0} = \aleph_1$, *then* $\beta N - N$ *is Blumberg. If* $2^{\aleph_0} > \aleph_1$, *then it is consistent with the usual axioms of set theory that* $\beta N - N$ *be Blumberg. If* $2^{\aleph_0} > \aleph_1$, *it is also consistent with the usual axioms of set theory that* $\beta N - N$ *be not Blumberg.*

PROOF. The first statement was proved in **[W1]**, using Proposition 3. The second statement is also proved using Proposition 3 and noting that Martin's Axiom implies that no open subset of $\beta N - N$ is the union of $\leqslant 2^{\aleph_0}$ nowhere dense subsets; see e.g. **[Ta]**.

The third statement is proved using Theorem 5. We note that since nonempty G_δ's of $\beta N - N$ have nonempty interiors, every countable collection of open subsets of $\beta N - N$ is oblivious. There are models of set theory constructed in [He], in which $2^{\aleph_0} > \aleph_1$ and $\beta N - N$ is the union of $\leqslant 2^{\aleph_0}$ nowhere dense subsets.

Although Theorem 2 characterizes LOTS which are Blumberg spaces, it is not a characterization of all Blumberg spaces. In [W1] it is shown, using $2^{\aleph_0} = \aleph_1$, that the density topology on the real line is a Baire space which is not Blumberg. However it is the union of $2^{\aleph_0}$ nowhere dense subsets, namely the singleton sets. As well, it has a countable collection of open sets which is not oblivious, since the density topology is finer than the usual topology on the real line.

THEOREM 9. *The Blumberg property is not preserved under perfect maps.*

PROOF. Let X be the compact T_2 non-Blumberg space constructed previously. Let $d(X) = \kappa$. Let Y be the discrete space of cardinality κ and βY its Stone-Čech compactification. Since Y is discrete, βY is clearly Blumberg, so it just remains to show that there exists a perfect map from Y onto X.

Let D be a dense subset of X of cardinality κ. Let f be a function mapping Y onto D. Since Y is discrete f is continuous and thus has a continuous extension F to βY. It is now straightforward to show that $F: \beta Y \to X$ is a perfect map.

Several open problems immediately suggest themselves:

Question 1. Is it consistent with the usual axioms of set theory that $\mathrm{St}(\mathfrak{B}/\mathfrak{I})$ be Blumberg?

Question 2. Are the uncountable products of the closed unit interval Blumberg? In particular is $[0, 1]^{\aleph_1}$ Blumberg?

Question 3. If D is the two-point discrete space and $\kappa > \omega$, is D^κ Blumberg?

Question 4. Is the Blumberg property preserved under continuous open surjections?

Question 5. If $X \times Y$ is a Blumberg space, must both X and Y be Blumberg?

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