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DECOMPOSITIONS OF LINEAR MAPS[1]

BY

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ABSTRACT. In the first part we show that the decomposition of a bounded selfadjoint linear map from a C^*-algebra into a given von Neumann algebra as a difference of two bounded positive linear maps is always possible if and only if that range algebra is a "strictly finite" von Neumann algebra of type I. In the second part we define a "polar decomposition" for some bounded linear maps and show that polar decomposition is possible if and only if the map satisfies a certain "norm condition". We combine the concepts of polar and positive decompositions to show that polar decomposition for a selfadjoint map is equivalent to a strict Hahn-Jordan decomposition (see Theorems 2.2.4 and 2.2.8).

0. **Introduction.** In this paper we study bounded linear maps between C^*-algebras. We are particularly concerned with decompositions of such maps. These decompositions are analogues of Hahn-Jordan and polar decompositions for linear functionals on C^*-algebras. The study of positive decomposition of bounded linear functionals of partially ordered normed linear spaces may be traced back to M. Kreĭn [12] and J. Grosberg [6] around 1939. Later Z. Takeda [18] worked out the same problem on the C^*-algebra setting. Recently some independent efforts were made to study the positive decomposition for bounded linear maps between two partially ordered normed linear spaces [20].

The major result in Chapter 1 is that the decomposition of any selfadjoint linear map as a difference of positive maps into a given von Neumann algebra is always possible if and only if that algebra is a "strictly finite" von Neumann algebra of type I (see Theorem 1.4.6). In §1.1 we state and establish some basic properties needed for the rest of the paper. In §1.2 we prove the sufficient part of Theorem 1.4.6 in a special case when the range algebra is an abelian von Neumann algebra (see Lemma 1.2.1). In §1.3 we construct examples to show that "positive decomposition" is not always possible under varying conditions of restrictiveness.

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In Chapter 2 we prove that the polar decomposition is possible if and only if the map satisfies a certain "norm condition" (see Theorem 2.1.4). We give examples of linear maps, some of which satisfy that condition and some which do not. In §2.2 we combine the concepts of polar and positive decompositions to show that polar decomposition for a selfadjoint map is equivalent to a strict Hahn-Jordan decomposition (see Theorems 2.2.4 and 2.2.8).

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I. Positive Decomposition of Selfadjoint Bounded Linear Maps

1. **Preliminaries**. Throughout this chapter the notation is as follows:

$\mathcal{A}$: C^*-algebra;

R: von Neumann algebra;

$B(\mathcal{A}_1, \mathcal{A}_2)$: the Banach space of all bounded linear maps from $\mathcal{A}_1$ into $\mathcal{A}_2$;

$\Phi, \Psi, \Lambda, \ldots$: the elements in $B(\mathcal{A}_1, \mathcal{A}_2)$;

$\mathcal{A}^*$: the dual space of $\mathcal{A}$;

$\varphi, \eta, \omega, \ldots$: the elements of $\mathcal{A}^*$;

R_* : the predual space of R;

$C(X)$: the C^*-algebra of all complex continuous functions on the compact Hausdorff space X;

l^∞ : the C^*-algebra of all bounded sequences;

M_n: the C^*-algebra of all $n \times n$ complex matrices;

$M_n \otimes \mathcal{A}$: the tensor product of M_n and $\mathcal{A}$.

Unless noted otherwise, all algebras have an identity element. As to the general theory of von Neumann algebras, we refer to [3].

1.1.1. Definitions. A linear map Φ from $\mathcal{A}_1$ into $\mathcal{A}_2$ is called *positive* if $\Phi(A)$ is positive in $\mathcal{A}_2$ for all positive A in $\mathcal{A}_1$. We write $\Phi \geqslant 0$. For given Φ in $B(\mathcal{A}_1, \mathcal{A}_2)$ we define Φ^* as a map from $\mathcal{A}_1$ into $\mathcal{A}_2$ by $\Phi^*(A) = \Phi(A^*)^*$. If $\Phi = \Phi^*$, then Φ is called *selfadjoint*.

It is evident that $\|\Phi^*\| = \|\Phi\|$, and if Φ is positive then it is selfadjoint. The "*" operation is continuous in the norm topology on $B(\mathcal{A}_1, \mathcal{A}_2)$; hence the subset of all selfadjoint elements in $B(\mathcal{A}_1, \mathcal{A}_2)$ is a closed subspace denoted by $B_{\text{s.a.}}(\mathcal{A}_1, \mathcal{A}_2)$. When $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, $B_{\text{s.a.}}(\mathcal{A})$ is a (real) closed subalgebra of $B(\mathcal{A})$. The subset of all positive elements in $B(\mathcal{A}_1, \mathcal{A}_2)$ is a closed positive cone in $B_{\text{s.a.}}(\mathcal{A}_1, \mathcal{A}_2)$ denoted by $P(\mathcal{A}_1, \mathcal{A}_2)$.

$M_n \otimes \mathcal{A}$ can be regarded as the algebra of all $n \times n$ matrices with their entries in $\mathcal{A}$. For each $\Phi\colon \mathcal{A}_0 \to \mathcal{A}_1$ induces $\Phi_n = \mathrm{id}_n \otimes \Phi$ from $M_n \otimes \mathcal{A}_0$ into $M_n \otimes \mathcal{A}_1$, where id_n is the identity map of M_n, and for T in $M_n \otimes \mathcal{A}$, with $T = (T_{ij})$, $\Phi_n(T) = (\Phi(T_{ij}))$.

1.1.2. DEFINITION. A linear map Φ from $\mathcal{A}_0$ into $\mathcal{A}_1$ is called *completely positive* if Φ_n is positive from $M_n \otimes \mathcal{A}_0$ into $M_n \otimes \mathcal{A}_1$, for all $n = 1, 2, 3, \ldots.$

Stinespring has characterized completely positive linear maps from $\mathcal{A}_0$ into $B(\mathcal{H})$. We state some theorems below. For their proofs, we refer to [17] and [1].

1.1.3. THEOREM (STINESPRING). *Let Φ be a completely positive linear map from $\mathcal{A}$ into $B(\mathcal{H})$. Then there exists a *-homomorphism Ψ from $\mathcal{A}$ into $B(\mathcal{K})$ and a bounded linear transformation V from $\mathcal{H}$ into $\mathcal{K}$ such that for A in $\mathcal{A}$,*

$$\Phi(A) = V^* \Psi(A) V,$$

where $\mathcal{K}$ is the completion of pre-Hilbert space $\mathcal{H} \odot \mathcal{A}$ (the algebraic tensor product) under the inner product norm

$$\left(\sum x_i \otimes A_i, \sum y_i \otimes B_i\right)_\Phi = \sum_{i,j} (\Phi(B_j^* A_i) x_i, y_j).$$

1.1.4. THEOREM. *When either $\mathcal{A}_0$ or $\mathcal{A}_1$ is commutative, then any positive linear map from $\mathcal{A}_0$ into $\mathcal{A}_1$ is completely positive.*

1.1.5. LEMMA. *If $\Phi \geqslant 0$ in $B(\mathcal{A}_1, \mathcal{A}_2)$ then* $\|\Phi\| = \|\Phi(1)\|$.

PROOF. It is proved by Dye and Russo [4] that

$$\|\Phi\| = \mathrm{Sup}_{U:\text{unitary in } \mathcal{A}_1} \|\Phi(U)\|.$$

Given any fixed unitary element U in $\mathcal{A}_1$, Φ can be considered from $C^*(U)$ ($=$ the C^*-algebra generated by U and U^*) into $\mathcal{A}_2$ and $C^*(U)$ is commutative. Therefore, Φ is completely positive (by 1.1.4) and $\Phi(\cdot) = V^*\Psi(\cdot)V$ where Ψ is a *-homomorphism, V is a bounded linear transformation from underlying Hilbert space $\mathcal{H}_2$ of $\mathcal{A}_2$ into $(\mathcal{H}_2 \otimes C^*(U))$-completion defined as $V(x) = x \otimes 1$ (by 1.1.3). So

$$\|V(x)\|^2 = (\Phi(1)x, x) \leqslant \|\Phi(1)\| \, \|x\|^2.$$

Hence $\|V\| \leqslant \|\Phi(1)\|^{1/2}$. Thus for $A \in C^*(U)$,

$$\|\Phi(A)\| = \|V^* \Psi(A) V\| \leqslant \|V^*\| \, \|V\| \, \|A\| = \|V\|^2 \|A\| \leqslant \|\Phi(1)\| \, \|A\|.$$

Therefore $\|\Phi(U)\| \leqslant \|\Phi(1)\|$ and $\|\Phi\| = \|\Phi(1)\|$. Q.E.D.

Let B_1, B_2 be two Banach spaces, $B_1 \odot B_2$ the algebraic tensor product of B_1 and B_2. Denote by $B_1 \otimes_\Lambda B_2$ the projective tensor product of B_1 and B_2, i.e., the completion of $B_1 \odot B_2$ with respect to the norm "Λ" defined as follows [7, p. 28]: for x in $B_1 \odot B_2$,

$$\|x\|_\Lambda = \inf\left\{ \sum_{i=1}^n \|a_i\| \, \|b_i\| \;\middle|\; x = \sum_{i=1}^n a_i \otimes b_i, \, a_i \in B_1, \, b_i \in B_2 \right\}.$$

1.1.6. THEOREM. *$B(\mathcal{Q}, R) \cong (\mathcal{Q} \otimes_\Lambda R_*)^*$. $\mathcal{Q} \otimes_\Lambda R_*$ is the projective tensor product of $\mathcal{Q}$ and the predual R_* of R. The isomorphism here is also an isometry.*

Note. This theorem is very useful and it also gives insight into the structure of $B(\mathcal{Q}, R)$. But the positive cone (the (real) subalgebra of all selfadjoint elements) in $(\mathcal{Q} \otimes_\Lambda R_*)^*$ corresponds to the *completely positive* cone, instead of positive cone, in $B_{\text{s.a.}}(\mathcal{Q}, R)$.

1.1.7. LEMMA. *Let $\mathcal{Q} \otimes_\Gamma R_*$ be the completion of $\mathcal{Q} \odot R_*$ under a given cross norm "Γ". Then there exists a linear map ι from $(\mathcal{Q} \otimes_\Gamma R_*)^*$ into $B(\mathcal{Q}, R)$ such that $f \in (\mathcal{Q} \otimes_\Gamma R_*)^* \to \iota(f)$ with $\|\iota(f)\| \leqslant \|f\|_\Gamma$.*

PROOF. Define $\iota(f)$ by $\iota(f)(x)(y) = f(x \otimes y)$ where $x \in \mathcal{Q}, y \in R_*$. In this way, $\iota(f)(x)$ appears as an element of the dual space of R_*, i.e., as an element of R. Clearly $\iota(f)$ is a linear map from $\mathcal{Q}$ into R; and

$$\begin{aligned}\|\iota(f)\| &= \sup_{\|x\| \leqslant 1; \|y\| \leqslant 1} |\iota(f)(x)(y)| \leqslant \sup_{\|x\| \leqslant 1; \|y\| \leqslant 1} \|f\|_\Gamma \|x \otimes y\|_\Gamma \\ &= \sup_{\|x\| \leqslant 1; \|y\| \leqslant 1} \|f\|_\Gamma \|x\| \, \|y\| \leqslant \|f\|_\Gamma .\end{aligned}$$

Also ι is linear and $\|\iota\| \leqslant 1$. Q.E.D.

The notation above is adapted from papers by Lance [13] and by Effros and Lance [5].

1.1.8. LEMMA. *There exists a linear map ι^{-1} (it turns out to be the inverse of ι in 1.1.7) from $B(\mathcal{Q}, R)$ onto $(\mathcal{Q} \otimes_\Lambda R_*)^*$ such that $\|\iota^{-1}(\Phi)\| \leqslant \|\Phi\|$ for $\Phi \in B(\mathcal{Q}, R)$.*

PROOF. Define ι^{-1} by

$$\iota^{-1}(\Phi)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} \Phi(x_i)(y_i)$$

for $x_i \in \mathcal{Q}, y_i \in R_*$. We observe that

$$\left|\iota^{-1}(\Phi)\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right| = \left|\sum_{i=1}^{n} \Phi(x_i)(y_i)\right| \leqslant \|\Phi\| \sum_{i=1}^{n} \|x_i\| \, \|y_i\|.$$

Hence

$$\left|\iota^{-1}(\Phi)\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right| \leqslant \|\Phi\| \left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|_\Lambda .$$

Hence $\iota^{-1}(\Phi)$ can be continuously extended to $\mathcal{Q} \otimes_\Lambda R_*$, and $\|\iota^{-1}(\Phi)\| \leqslant \|\Phi\|$. Q.E.D.

PROOF OF THE THEOREM. It is clear that ι and ι^{-1} are inverse to each other; therefore we have established an isometry between $B(\mathcal{Q}, R) \cong (\mathcal{Q} \otimes_\Lambda R_*)^*$. Q.E.D.

1.1.9. DEFINITION. A selfadjoint map Φ from $\mathcal{Q}_0$ into $\mathcal{Q}_1$ admits a *positive decomposition* if there exists a bounded positive linear map Φ^+ from $\mathcal{Q}_0$ into $\mathcal{Q}_1$ such that $\Phi^+ - \Phi \geqslant 0$. In this case, we say that Φ is *positively decomposable.*

1.1.10. REMARK. By a *nuclear linear map* $\Phi: \mathcal{Q}_1 \to \mathcal{Q}_2$ we mean $\Phi(A) = \sum_{i=1}^{\infty} \lambda_i f_i(A) U_i$, where f_i's are in $\mathcal{Q}_1^*$ (the dual space of $\mathcal{Q}_1$), U_i's are in $\mathcal{Q}_2$ with $\|U_i\| = \|f_i\| = 1$ for all $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} |\lambda_i| = \alpha < \infty$. If Φ is a selfadjoint nuclear map, $\Phi = \sum_{k=1}^{\infty} \lambda_k f_k \odot U_k$. (We write $f_k \odot U_k$ for the map $A \to f_k(A) U_k$.)

$$
\begin{aligned}
\Phi = {} & \sum_{n=1}^{\infty} (\lambda_k^r f_k^r - \lambda_k^i f_k^i) \odot U_k^r - (\lambda_k^i f_k^r + \lambda_k^r f_k^i) \odot U_k^i \\
& + i \sum_{k=1}^{\infty} (\lambda_k^i f_k^r + \lambda_k^r f_k^i) \odot U_k^r + (\lambda_k^r f_k^r - \lambda_k^i f_k^i) \odot U_k^i
\end{aligned} \tag{2}
$$

where $\lambda_k^r, \lambda_k^i, f_k^r, f_k^i, U_k^r, U_k^i$ are the real and imaginary parts of λ_k, f_k, U_k respectively. Each of the two summations in (2) is selfadjoint. The selfadjointness of Φ implies the second summation in (2) is a zero map, and consequently Φ can be expressed as

$$\sum_{k=1}^{\infty} g_k \odot U_k^r - \sum_{k=1}^{\infty} b_k \odot U_k^i$$

where

$$g_k = (\lambda_k^r f_k^r - \lambda_k^i f_k^i), \qquad b_k = (\lambda_k^i f_k^r + \lambda_k^r f_k^i)$$

with $g_k^* = g_k$, $b_k^* = b_k$ and

$$\sum_{k=1}^{\infty} \|g_k\| \leqslant 2 \sum_{k=1}^{\infty} |\lambda_k|, \qquad \sum_{k=1}^{\infty} \|b_k\| \leqslant 2 \sum_{k=1}^{\infty} |\lambda_k|.$$

Then each of $g_k, b_k, (U_k^r, U_k^i)$ $(k = 1, 2, 3, \dots)$ can be decomposed as the difference of two positive functionals (elements) and Φ becomes

$$
\begin{aligned}
(\Phi =) & \sum_{k=1}^{\infty} g_k^+ \odot (U_k^r)^+ + g_k^- \odot (U_k^r)^- - b_k^+ \odot (U_k^i)^+ - b_k^- \odot (U_k^i)^- \\
& - \sum_{k=1}^{\infty} g_k^- \odot (U_k^r)^+ + g_k^+ \odot (U_k^r)^- - b_k^- \odot (U_k^i)^+ - b_k^+ \odot (U_k^i)^-
\end{aligned}
$$

By straightforward calculation we see each of the two summations in the above equation has norm less than 8α. This shows that Φ is positively decomposable.

2. **Lemmas.** In this section we establish a positive decomposition for selfadjoint linear maps with certain kinds of range algebras.

1.2.1. LEMMA. *Let Φ be a selfadjoint bounded linear map from a C^*-algebra $\mathcal{Q}$*

into an abelian von Neumann algebra $C(X)$: then Φ admits a positive decomposition, i.e., $\Phi = \Phi^+ - \Phi^-$, where $\Phi^+, \Phi^- \geqslant 0$, with $\|\Phi^+\| \leqslant \|\Phi\|$.

PROOF. Let $\mathcal{G}$ be the family of all partitions of unity $\{g_i | i \in I\}$ on X with nonredundant supports $\{V_i\}$ for $\{g_i\}$, i.e., $\bigcup_{i\in I; i\neq j} V_i \neq X$ for any fixed j. We can pick x_i in U_i in such a way that $g_i(x_j) = \delta_{ij}$ for all $i, j \in I$.

Next, for each $I \in \mathcal{G}$, we define a map π_I: $C(X) \to$ linear subspace generated by $\{g_i | i \in I\}$ by $\pi_I(f) = \sum_{i\in I} f(x_i) g_i$ and observe the following:

(i) π_I is a linear map and, for $f \geqslant 0$ in $C(X)$, $\pi_I(f) = \sum_{i\in I} f(x_i) g_i \geqslant 0$.

(ii) $\pi_I(1) = \sum_{i\in I} g_i = 1$, so $\|\pi\|_I = 1$.

(iii) $\pi_I \circ \Phi$ is a selfadjoint linear map with $\|\pi_I \circ \Phi\| \leqslant \|\Phi\|$.

Consider $\varphi_i(A) = \Phi(A)(x_i)$ for each $i \in I$. It is clear that the φ_i's are selfadjoint linear functionals of $\mathcal{A}$ with $\|\varphi_i\| \leqslant \|\Phi\|$ for all $i \in I$, i.e.,

$$|\varphi_i(A)| = |\Phi(A)(x_i)| \leqslant \|\Phi(A)\| \leqslant \|\Phi\| \, \|A\|.$$

Hence we have $\varphi_i = \varphi_i^+ - \varphi_i^-$ where φ_i^+, φ_i^- are positive linear functionals of $\mathcal{A}$ with $\|\varphi_i\| = \|\varphi_i^+\| + \|\varphi_i^-\|$.

$$(\pi_I \circ \Phi)(A) = \sum_{i\in I} \Phi(A)(x_i) g_i = \sum_{i\in I} \varphi_i^+(A) g_i - \varphi_i^-(A) g_i$$

for all $A \in \mathcal{A}$. We define Φ_I^+ of $\mathcal{A}$ into $C(X)$ by $A \to \sum_{i\in I} \varphi_i^+(A) g_i$. It is evident that Φ_I^+ is a positive linear map; thus

$$\begin{aligned} \|\Phi_I^+\| &= \|\Phi_1^+(1)\| = \left\| \sum_{i\in I} \varphi_i^+(1) g_i \right\| \\ &= \left\| \sum_{i\in I} \|\varphi_i^+\| g_i \right\| \leqslant \|\Phi\| \left\| \sum_{i\in I} g_i \right\| = \|\Phi\| \end{aligned}$$

and also that $\Phi_I^+ \geqslant \pi_I \circ \Phi$.

Now consider $B(\mathcal{A}, C(X))$, the Banach space of all bounded linear maps from $\mathcal{A}$ into $C(X)$, which is isometrically isomorphic to $[\mathcal{A} \otimes_\Lambda C_*(X)]^*$ (the dual of the projective tensor product of $\mathcal{A}$ and the predual $C_*(X)$ of $C(X)$). Compactness of X implies that for any given f in $C(X)$, x_0 in X and $\varepsilon > 0$, we can find I_0 in $\mathcal{G}$ such that x_0 lies in a support V_{i_0} of g_{i_0} but not in $\bigcup_{i\neq i_0; i\in I_0} V_i$ and $|f(x) - f(x_0)| < \varepsilon$ for all x in V_{i_0}. Thus we have

$$|\pi_{I_0}(f)(x_0) - f(x_0)| = \left| \sum_{i\in I_0} f(x_i) g_i(x_0) - f(x_0) \right| = |f(x_{i_0}) - f(x_0)| < \varepsilon,$$

where $x_{i_0} \in V_{i_0}$. Therefore we can select a net $\{\pi_I | I \in \mathcal{G}' \subset \mathcal{G}\}$ convergent to the identity map on $C(X)$ in $B(C(X), C(X))$ under the point-weak*-topology. On the other hand $\{\Phi_I^+ | I \in \mathcal{G}'\}$ is a bounded subset in $(\mathcal{A} \otimes_\Lambda C_*(X))^*$. Hence it must have a cluster point Φ^+ under the weak*-topology with $\|\Phi^+\| \leqslant \|\Phi\|$.

Finally we show that $\Phi^+ \geqslant 0$ and $\Phi^+ - \Phi \geqslant 0$. For any $A \geqslant 0$ in $\mathcal{A}$, x_0

$\in X$ and $\varepsilon > 0$ we can find I_0 in $\mathcal{G}'$ such that $|\Phi^+(A)(x_0) - \Phi^+_{I_0}(A)(x_0)| < \varepsilon$. This inequality and $\Phi^+_{I_0}(A)(x_0) \geqslant 0$ imply that $\Phi^+(A)(x) \geqslant 0$. Therefore $\Phi^+ \geqslant 0$. In addition, $\Phi^+_I(A)(x) - \pi_I \circ \Phi(A)(x) \geqslant 0$ for all $A \geqslant 0$ in $\mathcal{Q}$, $x \in X$ and $I \in \mathcal{G}'$. This implies that $\Phi^+(A)(x) - \Phi(A)(x) \geqslant 0$ for all $A \geqslant 0$ in $\mathcal{Q}$, x in X.

1.2.2. LEMMA. *Let Φ be a bounded selfadjoint linear map from a C^*-algebra $\mathcal{Q}$ into $M_n \otimes C(X)$ where $C(X)$ is an abelian von Neumann algebra; then Φ admits a positive decomposition.*

PROOF. Each A in $M_n \otimes C(X)$ is an $n \times n$ matrix (α_{ij}) with entries $\alpha_{ij} \in C(X)$, $i, j = 1, \ldots, n$. Let $\{e_1, \ldots, e_n\}$ be the canonical orthonormal basis for $\mathcal{C}^n$ and ω_{ij} the linear functional on M_n defined by $\omega_{ij}(T) = (Te_j, e_i) = t_{ij}$ when $T = (t_{ij})$ for $i, j = 1, \ldots, n$. Each ω_{kl} induces a map Φ_{kl} from $M_n \otimes C(X)$ into $C(X)$ by

$$(A \in M_n \otimes C(X)) \quad \Phi_{kl}(A) = \Phi_{kl}\left(\sum_{i,j=1}^{n} e_{ij} \otimes \alpha_{ij}\right)$$

$$= \sum_{i,j=1}^{n} \omega_{kl}(e_{ij})\alpha_{ij} = \alpha_{kl}$$

where $A = (\alpha_{ij})$ and $\{e_{ij} | i,j = 1, 2, \ldots, n\}$ are the usual matrix units of M_n. Note that Φ_{kl} is a linear map from $M_n \otimes C(X)$ into $C(X)$ for all k, $l = 1, \ldots, n$ with $\|\Phi_{kl}\| \leqslant \|\omega_{kl}\| \leqslant 1$. For each x_0 in X induces a pure state of $C(X)$ by $\varphi_{x_0}(f) = f(x_0)$ and

$$|\Phi_{kl}(A)(x_0)| = |(\omega_{kl} \otimes \varphi_{x_0})(A)| \leqslant \|\omega_{kl} \otimes \varphi_{x_0}\| \, \|A\|$$

$$= \|\omega_{kl}\| \, \|\varphi_{x_0}\| \, \|A\| = \|A\|.$$

Set

$$A_{kl} = \begin{cases} (1/2)(e_{kl} + e_{lk}), & k < l, \\ e_{kk}, & k = l, \\ (1/2i)(e_{lk} - e_{kl}), & k > l, \end{cases}$$

where e_{lk}'s are the usual matrix units for M_n and

$$\Psi_{kl}(A) = \begin{cases} (\Phi_{kl} + \Phi_{lk})(A), & k < l, \\ \Phi_{kk}(A), & k = l, \\ (1/i)(\Phi_{kl} - \Phi_{lk})(A), & k > l. \end{cases}$$

Since $\Phi_{kl}(A) = \overline{\Phi_{lk}(A^*)}$, Ψ_{kl} is selfadjoint and $\|\Psi_{kl}\| \leqslant 2\|\Phi_{kl}\|$. Thus

$$\Phi(A) = \sum_{k,l=1}^{n} A_{kl} \otimes \Psi_{kl}(\Phi(A)).$$

By Lemma 1.2.1, $\Psi_{kl}(\Phi(A)) = \Psi_{kl}^+(A) - \Psi_{kl}^-(A)$ where Ψ_{kl}^+, Ψ_{kl}^- are positive linear maps with $\|\Psi_{kl}^{\pm}\| \leqslant \|\Psi_{kl} \circ \Phi\|$. With A in $\mathfrak{A}$, let $\Phi^+(A)$ be

$$\sum_{kl=1}^{n} \left[A_{kl}^+ \otimes \Psi_{kl}^+(A) + A_{kl}^- \otimes \Psi_{kl}^-(A)\right]$$

where A_{kl}^+ (A_{kl}^-) is the positive (negative) part of A_{kl} in M_n. It is easy to see that Φ^+ is positive, $\Phi^+ \geqslant \Phi$ and

$$\|\Phi^+\| = \|\Phi^+(I)\| \leqslant \sum_{k,l=1}^{n} 2\|\Phi\|(\|A_{kl}^+\| + \|A_{kl}^-\|) \leqslant 4n^2\|\Phi\|. \qquad \text{Q.E.D.}$$

1.2.3. Corollary. *Every selfadjoint linear map from* $\mathfrak{A}$ *into* M_n $(\cong M_n \otimes C)$ *admits a positive decomposition.*

1.2.4. Definition. A finite type I von Neumann algebra R is called *strictly* finite if $R = \sum_{i\in I} \oplus R_i$ where R_i is of type I_{n_i} and $\operatorname{Sup}_{i\in I} n_i < +\infty$.

1.2.5. Lemma. *If R is a strictly finite von Neumann algebra of type* I, *then any selfadjoint linear map Φ from $\mathfrak{A}$ into R admits a positive decomposition.*

Proof. Let $R = \sum_{i\in I} \oplus R_i$ with R_i of type I_{n_i} and $\operatorname{Sup}_{i\in I} n_i = k < +\infty$ and π_i be a projection map of R onto R_i ($i \in I$). We may assume $\|\Phi\| = 1$. Since $R_i = M_{n_i} \otimes C(X_i)$, by 1.2.2 $\pi_i \circ \Phi$, being a selfadjoint linear map, is majorized by a positive linear map $\Phi_i^+ : \mathfrak{A} \to R_i$ with $\|\Phi_i^+\| \leqslant 4n_i^2 \leqslant 4k^2 \leqslant +\infty$. Thus $\sum_{i\in I} \oplus\Phi_i^+$, denoted by Φ^+, is a positive linear map from $\mathfrak{A}$ into R and $\Phi^+ > \Phi$ with $\|\Phi^+\| \leqslant 4k^2 < +\infty$. Q.E.D.

In the following we will develop a somewhat more general version of Lemma 1.2.2 by replacing the range algebra, a commutative von Neumann algebra, with a commutative AW^*-algebra, i.e., $C(X)$, the C^*-algebra of all continuous functions on a *Stonean* space X.

1.2.6. Definition. A compact Hausdorff space X is called *Stonean* if for every open set U in X the closure $\overline{U}$ of U is open.

We state basic facts about Stonean spaces needed in this section without proofs (they can be found in [2, pp. 66–77]). First of all, the most frequently used example of a Stonean space is the Stone-Čech compactification of a discrete set D, denoted by $\beta(D)$. Let X be a compact Hausdorff space and X_d be X with the discrete topology. We can always embed $C(X)$ into $C(\beta(X_d))$ as a subalgebra with the same unit.

1.2.7. Definition. Let X be a compact Hausdorff space. $C(X)$ is called *injective* if $C(X)$ is an injective object in the category consisting of "objects" such as $C(X)$'s, where X is compact Hausdorff and "morphisms" are *-homomorphisms between $C(X)$'s.

1.2.8. Theorem. *Let X be compact Hausdorff space. Then $C(X)$ is injective if and only if X is Stonean.*

PROOF. The proof can be found in [2].

1.2.9. COROLLARY. *Let X be Stonean. Then there exists a projection map from $C(\beta(X_d))$ onto $C(X)$ of norm* 1.

PROOF. Since $C(X)$ can be embedded into $C(\beta(X_d))$ as a subalgebra with the same unit, the identity map id can be lifted to $C(\beta(X_d))$. It is illustrated below:

$$\begin{tikzcd}
C(X) & C(\beta(X_d)) \arrow[l, dashed, "\Lambda"'] \\
 & C(X) \arrow[u, "\iota\ \text{embedding}"'] \arrow[ul, "\mathrm{id}"]
\end{tikzcd}$$

Since Λ is a*-homomorphism and preserves the unit element, Λ is of norm 1. Q.E.D.

1.2.10. COROLLARY. *Let Φ be a selfadjoint linear map from $\mathcal{A}$ into $C(X)$, where X is Stonean. Then Φ admits a positive decomposition.*

PROOF. We may embed $C(X)$ into $l^\infty(X_d)$, the C^*-algebra of all bounded functions on the set X with discrete topology. Again, every element in $l^\infty(X_d)$ has a unique extension to an element in $C(\beta(X_d))$. Thus we have an embedding $\iota_2 \colon l^\infty(X_d) \to C(\beta(X_d))$. By Lemma 1.2.1, $\Phi = \Phi_1 - \Phi_2$, where Φ_1, Φ_2 are bounded positive linear maps from $\mathcal{A}$ into $l^\infty(X_d)$. By Corollary 1.2.9 there exists a projection map Λ of norm 1 from $C(\beta(X_d))$ onto $C(X)$.

$$\begin{tikzcd}
C(X) & C(\beta(X_d)) \arrow[l, dashed, "\Lambda"'] \\
 & l^\infty(X_d) \arrow[u, "\iota_2"'] \\
 & C(X) \arrow[u, "\iota_1"'] \arrow[uul, "\mathrm{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C(X) & C(\beta(X_d)) \arrow[l, dashed, "\Lambda"'] \\
 & \mathcal{A} \arrow[u, "\Phi_i"'] \arrow[ul, "\Lambda \circ \Phi_i"]
\end{tikzcd}$$

$i = 1, 2$

Hence $\Phi = \Lambda \circ \Phi = \Lambda \circ \Phi_1 - \Lambda \circ \Phi_2$, where $\Lambda \circ \Phi_1, \Lambda \circ \Phi_2$ are two positive linear maps from $\mathcal{A}$ into $C(X)$ with $\|\Lambda \circ \Phi_i\| \leqslant \|\Phi_i\|$, $i = 1, 2$. Q.E.D.

3. **Counterexamples.** In this section we shall exhibit several examples in which positive decompositions for certain selfadjoint linear maps with range algebra other than strictly finite von Neumann algebra of type I fail to exist.

1.3.1. EXAMPLE I. This is derived from O. Lanford's example. It shows that if $\mathcal{H}$ is a separable infinite dimensional Hilbert space then there exists a selfadjoint linear map from $B(\mathcal{H})$ into itself which does not admit any positive decompositions.

1.3.2. LEMMA. *For integer $n \geqslant 1$ there exist $A_1, \ldots, A_n \in M_{2^n}$ such that*
(1) $A_i = A_i^*$.
(2) $A_iA_j + A_jA_i = 2\delta_{ij}I_{2^n}$.
(3) $\operatorname{tr}(A_i) = 0$; tr *is the trace function on* M_{2^n}.
(4) *If* $A \in M_{2^n}$ *and* $A = \sum \alpha_i A_i$, *then* $\|A\| \leqslant 2^{1/2}(\sum |\alpha_i|)^{1/2}$.

Note. These A_i's actually generate a *Clifford algebra* [14].
PROOF. Let

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see $H^* = H$, $J^* = J$, $H^2 = I = J^2$, $HJ + JH = 0$. Let

$$\begin{aligned} A_1 &= H \otimes \underbrace{I \otimes \cdots \otimes I}_{n-1}, \\ A_2 &= J \otimes H \otimes \underbrace{I \otimes \cdots \otimes I}_{n-2}, \\ &\vdots \\ A_k &= \underbrace{J \otimes \cdots \otimes J}_{k-1} \otimes H \otimes \underbrace{I \otimes \cdots \otimes I}_{n-k}, \qquad 3 \leqslant k \leqslant n. \end{aligned}$$

Hence $A_i^* = A_i$, $A_i^2 = I$, and $\operatorname{tr}(A_i) = 0$. If $i \neq j$, $A_iA_j + A_jA_i = 0$. If $A = \sum_i \alpha_i A_i$,

$$\begin{aligned} A^*A + AA^* &= \sum_{i,j} \bar{\alpha}_j \alpha_i A_j A_i + \sum_{i,j} \alpha_i \bar{\alpha}_j A_i A_j \\ &= \sum_{i,j} \alpha_i \bar{\alpha}_j [A_iA_j + A_jA_i] = \sum_{i,j} \alpha_i \bar{\alpha}_j (2\delta_{ij}) I. \end{aligned}$$

Thus

$$\|A\|^2 \leqslant \|A^*A + AA^*\| = 2 \sum |\alpha_i|^2 \quad \text{and} \quad \|A\| \leqslant 2^{1/2}(\sum |\alpha_i|^2)^{1/2}. \quad \text{Q. E.D.}$$

1.3.3. Let $\varphi_1, \ldots, \varphi_n$ be positive linear normal functionals on $B(\mathcal{H})$ with orthogonal supports and $\|\varphi_i\| = n^{-1/2}$. We define $\Phi_n\colon B(\mathcal{H}) \to M_{2^n}$ by

$$\Phi_n(A) = \sum_{i=1}^n \varphi_i(A) A_i, \qquad A \in B(\mathcal{H}).$$

Then

$$\|\Phi_n(A)\| \leqslant 2^{1/2}\left(\sum_{i=1}^n |\varphi_i(A)|^2\right)^{1/2} \leqslant 2^{1/2}\left(\sum_{i=1}^n n^{-1}\|A\|^2\right)^{1/2} = 2^{1/2}\|A\|.$$

So Φ_n is selfadjoint with $\|\Phi_n\| \leqslant \sqrt{2}$. Now we construct a counterexample by setting

$$\Phi_0(A) = \sum_{n=1}^{\infty} \oplus \Phi_n(A) \qquad (A \in B(\mathcal{H})).$$

Then $\|\Phi_0\| = \operatorname{Sup}_n \|\Phi_n\| \leqslant \sqrt{2}$. Next we show that there is no positive element Φ in $B(B(\mathcal{H}))$ such that $\Phi - \Phi_0 \geqslant 0$.

Suppose Φ is a positive linear map from $B(\mathcal{H})$ into itself such that $\Phi - \Phi_0 \geqslant 0$. We can find a family of orthogonal projections $\{P_n: n = 1, 2, 3, \ldots\}$ in $B(\mathcal{H})$ whose range spaces are $\{\mathcal{H}_{2^n}: n = 1, 2, 3, \ldots\}$ respectively with dimension 2^n. We may suppose that M_{2^n} operates on $\mathcal{H}_{2^n}$. Let

$$\Phi'(A) = \sum_{i=1}^{\infty} \oplus P_n \Phi(A) P_n, \qquad A \in B(\mathcal{H}).$$

Since $\Phi_0(\cdot) = \sum_{n=1}^{\infty} \oplus \Phi_n(\cdot)$ and $P_n \Phi'(\cdot) P_n \geqslant P_n \Phi_0(\cdot) P_n = \Phi_n(\cdot)$, $\Phi' \geqslant \Phi_0$. We may assume the range of Φ is in $\sum_{n=1}^{\infty} \oplus M_{2^n}$.

For a fixed n, let E_i be the support of φ_i, $i = 1, \ldots, n$; then $P_n \Phi(\cdot) P_n$ (denoted by $\Psi_n(\cdot)$) $\geqslant \Phi_n(\cdot)$ and $\Psi_n(E_i) \geqslant \Phi_n(E_i) = \|\varphi_i\| A_i$ for all $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ we can choose an orthonormal basis for $\mathcal{H}_{2^n}$ as the set of unit eigenvectors of A_i such that the diagonal of A_i has half of its entries $+1$, and half of them -1. (Note that if T and S are $n \times n$ matrices and $T \geqslant 0$ then each diagonal entry of T is real and nonnegative, and that if $T \geqslant S$ each diagonal entry of T is not less than the corresponding diagonal entry of S.) But $\Psi_n(E_i) \geqslant \|\varphi_i\| A_i$ and the positivity of $\Psi_n(E_i)$ implies that $\operatorname{tr}(\Psi_n(E_i)) \geqslant 2^{n-1}\|\varphi_i\| + 2^{n-1} \cdot 0$, where $\|\varphi_i\|$ terms arise from $+1$ eigenvalues of A_i, and 0 terms arise from -1 eigenvalues of A_i. Hence

$$\begin{aligned} \operatorname{tr}(\Psi_n(1)) &\geqslant \operatorname{tr}\left(\Psi_n\left(\sum_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} \operatorname{tr}(\Psi_n(E_i)) \\ &\geqslant \sum_{i=1}^{n} 2^{n-1}\|\varphi_i\| = \sum_{i=1}^{n} 2^{n-1} \cdot n^{-1/2} = 2^{n-1} n^{1/2}. \end{aligned}$$

So

$$\|\Psi_n(1)\| \geqslant (1/2^n) \operatorname{tr}(\Psi_n(1)) \geqslant (2^{n-1}/2^n) n^{1/2} = (1/2) n^{1/2}.$$

But

$$\|\Phi\| = \operatorname*{Sup}_n \|\Psi_n\| = \operatorname*{Sup}_n \|\Psi_n(1)\| \geqslant \operatorname*{Sup}_n \tfrac{1}{2} n^{1/2} = \infty.$$

Therefore Φ is unbounded.

1.3.4. Example II. In this second example we exhibit a selfadjoint linear map with its range algebra an abelian C^*-algebra, but not a von Neumann algebra, which fails to admit a positive decomposition. This shows that in order to have all selfadjoint maps from $\mathfrak{A}$ into $C(X)$ admitting a positive decomposition, X cannot be any arbitrary Hausdorff compact space, and it seems that X being Stonean is necessary. This example is a modification of one which is due to Samuel Kaplan and Ulrich Krengel [11].

(1) Let X_0 be the one-point compactification of the set of natural numbers and $C(X_0)$ be the Banach space of all real continuous functions on X_0. Define Φ: $C(X_0) \to C(X_0)$ as follows: for each $f \in C(X_0)$

$$(\Phi f)(n) = f(n) - f(n+1),$$
$$(\Phi f)(\omega) = 0 \qquad (\omega \text{ is the point at infinity}).$$

For $f \geqslant 0$ in $C(X_0)$ and fixed n_0, we consider $f_{n_0}(n) = f(n)$ for all n in X_0 except $f_{n_0}(n_0+1) = 0$. Hence $f_{n_0} \leqslant f$. For any positive linear map Ψ of $C(X_0)$ into itself if $\Psi \geqslant \Phi$ we have

$$\begin{aligned}\Psi(f)(n_0) \geqslant \Psi(f_{n_0})(n_0) &\geqslant \Phi(f_{n_0})(n_0) \\ &= f_{n_0}(n_0) - f_{n_0}(n_0+1) = f(n_0).\end{aligned}$$

So we have $\Psi(f)(n) \geqslant f(n)$ for all n.

(2) Let $\{X_n\}$ $(n = 1, 2, \dots)$ be a sequence of copies of X_0,

$$X_n = \{x_{n1}, x_{n2}, \dots, x_{nm}, \dots, x_{n\omega}\} \qquad (n = 1, 2, \dots),$$

$\sum_n X_n$ be their topological sum and let X be the 1-point compactification of $\sum_n X_n$. We may picture X as follows:

$$X = \begin{array}{ccccc} & & x_\omega & & \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} & x_{n\omega} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{21} & x_{22} & \cdots & x_{2m} & x_{2\omega} \\ x_{11} & x_{12} & \cdots & x_{1m} & x_{1\omega} \end{array}$$

Let $\{Y_n\}$ $(n = 1, 2, \dots)$ be a sequence of copies of N,

$$Y_n = \{y_{n1}, \dots, y_{nm}, \dots\} \quad (n = 1, 2, \dots),$$

$\sum_n Y_n$, be their topological sum and Y be the 1-point compactification of $\sum_n Y_n$. Define Φ: $C(X) \to C(Y)$ as follows: for each $f \in C(X)$,

$$(\Phi f)(y_{nm}) = f(x_{n,m}) - f(x_{n,m+1}),$$
$$(\Phi f)(y_\omega) = 0.$$

To see that Φf is continuous of Y, that is, that it is continuous at y_ω, let a positive ε be given. Then there is an n_0 such that, for $n > n_0$, $|f(x_{n,m}) - f(x_\omega)|$

$< \varepsilon/2$ for all m. Thus $|f(x_{n,m}) - f(x_{n,m+1})| < \varepsilon$ for all m and $n \geqslant n_0$. Also we may choose m_0 such that, for $n = 1, \ldots, n_0$ and $m > m_0$, $|f(x_{n,m}) - f(x_{n,\omega})| < \varepsilon/2$, whence $|f(x_{n,m}) - f(x_{n,m+1})| < \varepsilon$. Thus, for all $y_{n,m}$ outside the finite set $\{y_{n,m} | n = 1, \ldots, n_0, m = 1, \ldots, m_0\}$, $|(\Phi f)(y_{n,m})| < \varepsilon$.

Finally suppose there is $\Psi \geqslant 0$ and $\Psi - \Phi \geqslant 0$. Let e_n be the characteristic function of the subset X_n of X. From (1) we have $\Psi(e_n)(y_\omega) \geqslant 1$. Therefore

$$\Psi(1)(y_\omega) = \Psi\left(\sum_{n=1}^{\infty} e_n\right)(y_\omega) \geqslant \left(\sum_{n=1}^{k} \Psi(e_n)\right)(y_\omega)$$
$$= \sum_{n=1}^{k} \Psi(e_n)(y_\omega) \geqslant k$$

for any positive ingeger k. This shows the unboundedness of Ψ.

4. Main theorem.

1.4.1. Definition. A von Neumann algebra is called *positively decomposable as a range* if every selfadjoint element in $B(\mathcal{A}, R)$ admits a positive decomposition.

In §2 we proved that all strictly finite type I von Neumann algebras are positively decomposable as a range. We show in this section that these are the only positively decomposable von Neumann algebras as a range. We state a theorem proved by W. Arveson [1] without giving the proof.

1.4.2. Theorem (Arveson). *Let $\mathcal{H}$ be a separable Hilbert space and $0 \to \mathcal{A}_0 \xrightarrow{\Phi} \mathcal{A}_1$ be an exact sequence, where Φ is completely positive. Then any completely positive linear map Ψ_0 from $\mathcal{A}_0$ into $B(\mathcal{H})$ can be lifted to a completely positive map Ψ_1 of $\mathcal{A}_1$, i.e., the following diagram commutes*:

$$\begin{tikzcd}
 & B(\mathcal{H}) & \\
0 \arrow[r] & \mathcal{A}_0 \arrow[u, "\Phi_0"] \arrow[r, "\Phi"] & \mathcal{A}_1 \arrow[ul, dashed, "\Psi_1"']
\end{tikzcd}$$

where all maps in the above diagram are completely positive.

Note. A C^*-algebra $\mathcal{A}$ satisfying the condition that for any given exact sequence $0 \to \mathcal{A}_0 \xrightarrow{\Phi} \mathcal{A}_1$, where Φ is completely positive, and a completely positive linear map Ψ_0 from $\mathcal{A}_0$ into $\mathcal{A}$, Ψ_0 can always be lifted to a completely positive linear map Ψ_1 on $\mathcal{A}_1$ such that $\Psi_0 = \Psi_1 \circ \Phi$, is called *injective*.

1.4.3. Definition. If E and F are projections in a von Neumann algebra R, we write $E \sim F$ when there is an element U in R such that $U^* U = E$, $UU^* = F$. We write $E \prec F$ $(E \preccurlyeq F)$ when there is a projection F_0 in R such that $E \sim F_0$ and $F_0 < F$ $(F_0 \leqslant F)$.

Infinite case. Let R be an infinite von Neumann algebra. In this case there is a proper subprojection E_0 of I such that $I \sim E_0$, i.e., there is a partial isometry U in R such that $U^* U = 1$, $UU = E_0$, and $I - E_0 = E_1 \neq 0$.

Consider $E_2 = (UE_1)(UE_1)^*$ which is equivalent to $(UE_1)^*(UE_1) = E_1 U^* UE = E_1$ and E_1, E_2 are orthogonal since $E_2 < E_0$ and E_1, E_0 are orthogonal. Let

$$
\begin{aligned}
&E_3 \text{ be } (UE_2)(UE_2)^* \sim (UE_2)^*(UE_2) = E_2, \\
&\vdots \\
&E_k \text{ be } (UE_{k-1})(UE_{k-1})^* \sim (UE_{k-1})^*(UE_{k-1}) = E_{k-1}, \\
&\vdots
\end{aligned}
$$

Observe that $E_1 \sim E_2 \sim E_3 \sim \cdots \sim E_k \sim \cdots$ and $\{E_i | i = 1, \ldots, k, \ldots\}$ are mutually orthogonal.

Let R be the strong-operator closed *-algebra generated by $\{E_1, E_2, \ldots, E_k, \ldots, UE_1\}$ by R_1. Observe that R_1 is *-isomorphic to $B(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space having E_i as a 1-dimensional projection. We show that R is not positively decomposable as a range. We may assume that $B(\mathcal{H}) \subseteq R$. Suppose R is positively decomposable as a range. Suppose that Φ is a selfadjoint map from $\mathcal{Q}$ into $B(\mathcal{H})$ that is not positively decomposable (see 1.3.1, Example I). Then Φ is a map from $\mathcal{Q}$ into R and $\Phi = \Phi^+ - \Phi^-$ where Φ^+, Φ^- are positive bounded linear maps from $\mathcal{Q}$ into R. By Theorem 1.4.2 there is a completely positive linear map Ψ from R onto $B(\mathcal{H})$ extending the identity embedding of $B(\mathcal{H})$ into R. Hence

$$\Phi = \Psi \circ \Phi = \Psi(\Phi^+ - \Phi^-) = \Psi \circ \Phi^+ - \Psi \circ \Phi^-,$$

where $\Psi \circ \Phi^+$ and $\Psi \circ \Phi^-$ are bounded positive linear maps from $\mathcal{Q}$ into $B(\mathcal{H})$. This contradicts the choice of Φ. Therefore R is not positively decomposable as a range.

Type II_1 *case.*

1.4.4. LEMMA. *$\sum_{n=1}^{\infty} \oplus M_{2^n}$ can be embedded into any given type II_1 von Neumann algebra R.*

PROOF. Each type II_1 von Neumann algebra contains a hyperfinite factor. We can decompose the identity projection I as sum of countably many orthogoanl projections $\{P_n | n = 1, 2, \ldots\}$ with each P_n the sum of 2^n orthogonal equivalent subprojections $\{E_{n_i} | i = 1, \ldots, 2^n\}$. The *-subalgebra generated by $\{E_{n_i} | i = 1, \ldots, 2^n\}$ and those partial isometries among them is *-isomorphic to M_{2^n}. Hence we have an ultraweakly closed *-subalgebra *-isomorphic to $\sum_{n=1}^{\infty} \oplus M_{2^n}$. Q.E.D.

It is not difficult to see that $\sum_{n=1} \oplus M_{2^n}$ is injective. However, this will follow from the following more general lemma.

1.4.5. Lemma. *If* $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ *is a family of* C^**-algebras, then* $\sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\Lambda$ *is injective if and only if each* $\mathcal{A}_\Lambda$ *is injective.*

Proof. Assume that $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ are all injective. Given

$$\begin{tikzcd} & \sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\lambda & \\ 0 \arrow[r] & \mathcal{A}_0 \arrow[u, "\Psi_0"] \arrow[r, "\Phi"] & \mathcal{A}_1 \end{tikzcd}$$

(all maps are completely positive linear maps). Let P_λ be the projection of $\sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\lambda$ on $\mathcal{A}_\lambda$. Then, for each λ,

$$\begin{tikzcd} & \mathcal{A}_\lambda & \\ & \sum \oplus \mathcal{A}_\lambda \arrow[u, "P_\lambda"] & \\ 0 \arrow[r] & \mathcal{A}_0 \arrow[u, "\Psi_0"] \arrow[r, "\Phi"] & \mathcal{A}_1 \arrow[uul, dashed, "\Psi_\lambda"'] \end{tikzcd}$$

$P_\lambda \circ \Psi_0$ can be lifted to Ψ_λ on $\mathcal{A}_1$, and $\sum_{\lambda\in\Lambda} \oplus \Psi_\lambda$ is a lifting of Ψ on $\mathcal{A}_1$. Conversely assume that $\sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\lambda$ is injective. Given a completely positive map Ψ_0 of $\mathcal{A}_0$ into $\mathcal{A}_\lambda$.

$$\begin{tikzcd} & \mathcal{A}_\lambda & \\ 0 \arrow[r] & \mathcal{A}_0 \arrow[u, "\Psi_0"] \arrow[r, "\Phi"] & \mathcal{A}_1 \end{tikzcd} \qquad \begin{tikzcd} & \sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\lambda & \\ & \mathcal{A}_\lambda \arrow[u, phantom, "\cup"] & \\ 0 \arrow[r] & \mathcal{A}_0 \arrow[u, "\Psi_0"] \arrow[r, "\Phi"] & \mathcal{A}_1 \arrow[uul, dashed, "\Psi_1"'] \end{tikzcd}$$

Ψ_0 can be regarded as a map from $\mathcal{A}_0$ into $\sum_{\lambda\in\Lambda} \oplus \mathcal{A}_\lambda$. As such, it can be lifted to Ψ_1 on $\mathcal{A}_1$. Hence $P_\lambda \circ \Psi_1$ lifts Ψ_0 (with range in $\mathcal{A}_\lambda$). Q.E.D.

Now we show any type II_1 von Neumann algebra R is not positively decomposable as a range. Suppose the contrary, and let Φ be a selfadjoint linear map from $\mathcal{A}$ into $\sum_{n=1} \oplus M_{2^n}$ that is not positively decomposable. As Φ maps into R, $\Phi = \Phi^+ - \Phi^-$ where Φ^+, Φ^- are two positive bounded linear maps from $\mathcal{A}$ into R. Let Ψ be a completely positive linear map from R onto $\sum_{n=1} \oplus M_{2^n}$ extending the identity embedding of $\sum_{n=1} \oplus M_{2^n}$ into R (by 1.4.3 and 1.4.4). Then $\Phi = \Psi \circ \Phi = \Psi(\Phi^+ - \Phi^-) = \Psi \circ \Phi^+ - \Psi \circ \Phi^-$ where $\Psi \circ \Phi^+$, $\Psi \circ \Phi^-$ are bounded positive linear maps from $\mathcal{A}$ into $\sum_{n=1} \oplus M_{2^n}$. This contradicts the choice of Φ. Therefore R is not positively decomposable as a range.

1.4.6. THEOREM. *A von Neumann algebra R is positively decomposable as a range if and only if R is strictly finite of type* I.

PROOF. Because of Lemma 1.2.5 and two case studies in this section it is sufficient to show the following to complete this proof:

If R is of finite type I, but not strictly finite, we can embed $\sum_{n=1}^{\infty} M_{2^n}$ into R. But this follows from the definition of nonstrictly finite type I of R. Then we can have a selfadjoint linear map Φ from $\mathcal{Q}$ into $\sum_{n=1}^{\infty} M_{2^n}$ that is not positively decomposable, and subsequently that is not positively decomposable as a map with range R. Q.E.D.

1.4.7. CONCLUDING REMARK. As to the question "What are those selfadjoint positively decomposable linear maps from $\mathcal{Q}$ into an infinite or continuous type of von Neumann algebra?", it is still not completely answered. Consider R to be a hyperfinite factor containing an increasing sequence of type I_{n_p} factors M_p with $n_p < +\infty$ $(p = 1,2,3,\dots)$ containing the identity of R such that the strong-operator closure of $\bigcup_{p=1}^{\infty} M_p$ is R. Let Ψ_p be a completely positive linear map from R onto M_p extending the identity embedding of M_p into R. For any given selfadjoint linear map Φ from $\mathcal{Q}$ into R it is clear that $\Psi_p \circ \Phi$ is positively decomposable, i.e., $\Psi_p \circ \Phi = \Phi_p^+ - \Phi_p^-$. But $\|\Phi_p^+\|$ may increase to infinity as p gets large. If $\{\|\Phi_p^+\|: p = 1,2,\dots\}$ is bounded, there is a cluster point Φ_+ of $\{\Phi_p^+: p = 1,2,\dots\}$, under the point-weak*-topology on $B(\mathcal{Q}, R)$. It yields the desired positive decomposition for Φ, i.e., $\Phi^+ \geqslant \Phi$ with

$$\|\Phi^+\| \leqslant \operatorname*{Sup}_{p=1,2,\dots} \|\Phi_p^+\|.$$

As an example of this, the nuclear selfadjoint linear map can be viewed as one linear map with a uniform boundedness condition on $\{\|\Phi_n^+\|: n = 1,2,\dots\}$.

Although we know that the subspace of the nuclear selfadjoint linear maps is dense in $B_{\mathrm{s.a.}}(\mathcal{Q}, R)$ under point-weak* topology [16], we suspect that the set of selfadjoint linear maps with a uniform bounded condition is far from encompassing all selfadjoint positively decomposable maps. Studying the closure of all selfadjoint nuclear maps under some topology finer than point-weak* topology may shed light on what constitutes the family of all positively decomposable selfadjoint maps.

II. POLAR DECOMPOSITION

1. A theorem of polar decomposition. In this chapter, the notation of Chapter I applies.

2.1.1. LEMMA. *Let Φ be a bounded linear map from $\mathcal{Q}_0$ into $\mathcal{Q}_1$. Then Φ is positive if and only if $\varphi \circ \Phi(1) = \mathrm{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)|$ for all pure states φ of $\mathcal{Q}_1$.*

PROOF. Suppose that $\varphi \circ \Phi(1) = \mathrm{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)|$ for all pure states φ of $\mathfrak{A}_1$. This implies that $\|\varphi \circ \Phi\| = |\varphi \circ \Phi(1)|$ and $\varphi \circ \Phi$ is a positive linear functional on $\mathfrak{A}_0$. Since this is true for all pure states φ of $\mathfrak{A}_1$, Φ is positive.

Conversely, suppose that Φ is positive; then $\varphi \circ \Phi$ is a positive linear functional of $\mathfrak{A}_0$ for all pure states φ of $\mathfrak{A}_1$. Hence $\varphi \circ \Phi(1) = \|\phi \circ \Phi\|$ for all φ, i.e., $\varphi \circ \Phi(1) = \mathrm{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)|$ for all pure states φ. Q.E.D.

Note. In the above lemma when $\mathfrak{A}_1$ is one dimensional, we have the known necessary and sufficient condition for Φ to be positive. That is, "$\Phi(1) = \|\Phi\|$".

2.1.2. DEFINITION. Let Φ be a positive linear map from $\mathfrak{A}_0$ into $\mathfrak{A}_1$. The *left kernel* of Φ is the set $= \{A \in \mathfrak{A}_0 | \Phi(A^*A) = 0\}$ denoted by $\ker_\Phi$.

Note that A is in $\ker_\Phi$ if and only if A is in the left kernel of $\varphi \circ \Phi$ for each state φ of $\mathfrak{A}_1$. Thus $\ker_\Phi$ is a norm-closed left ideal in $\mathfrak{A}_0$, the intersection of left kernels of positive linear functionals $\varphi \circ \Phi$ on $\mathfrak{A}_0$ where φ runs through states of $\mathfrak{A}_1$.

When Φ is a positive linear map from a von Neumann algebra into a C^*-algebra $\mathfrak{A}$, continuous in the ultraweak topology in R and norm topology in $\mathfrak{A}$, $\ker_\Phi$ is a ultraweakly closed left ideal in R. There is a unique projection P' in R such that $RP' = \ker_\Phi$. We call $(P =)\ 1 - P'$ the *support* of Φ. We have $\Phi(AP) = \Phi(A)$ for all A in R. Φ is said to be *faithful* if $\ker_\Phi = 0$.

If Φ is a map of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, we write $\Phi . U$ for the map $A \to \Phi(UA)$, where U is an element in $\mathfrak{A}_1$, $V.\Phi$ for the map $A \to \Phi(AV)$ and $V.\Phi.U$ for the map $A \to \Phi(UAV)$.

2.1.3. DEFINITION. Let Φ be a bounded linear map from a von Neumann algebra R into a C^*-algebra $\mathfrak{A}$. Φ is said to admit a *polar decomposition* if there exists a partial isometry U in R with $U^*U = E$, $UU^* = F$ such that $\Phi . U$ is a positive linear map (denoted by $|\Phi|$) with its support E and $|\Phi| . U^* = \Phi$.

When $\mathfrak{A} = \mathcal{C}$, we have the usual polar decomposition for all ultraweakly continuous linear functionals (see §1.14 in [15]). In the following we exhibit a necessary and sufficient condition for a linear map Φ from R into $\mathfrak{A}$ continuous under the ultraweak topology in R and norm topology in $\mathfrak{A}$ to admit a polar decomposition.

2.1.4. THEOREM. *Let Φ be an ultraweak-norm continuous map from R into $\mathfrak{A}$. Then Φ admits a polar decomposition if and only if there is an element V in $(R)_1$ (the unit ball of R) such that $\varphi \circ \Phi(V) = \mathrm{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)|$ for all pure states φ of $\mathfrak{A}$. In the following proof we refer to this condition as the "norm-condition" and say that V is "norming" for Φ.*

PROOF. Suppose that Φ admits a polar decomposition. Then there is a partial isometry U in R such that $\Phi . U\ (= |\Phi|)$ is positive. By Lemma 2.1.1 this implies that $\varphi \circ |\Phi|(1) = \mathrm{Sup}_{\|A\| \leqslant 1} |\varphi \circ |\Phi|(A)|$ for all pure states φ of $\mathfrak{A}$. Hence

$$\varphi \circ \Phi(U) = \operatorname*{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(UA)| \geqslant \operatorname*{Sup}_{\|B\| \leqslant 1} |\varphi \circ \Phi(UU^* B)|$$
$$= \operatorname*{Sup}_{\|B\| \leqslant 1} |\varphi \circ |\Phi|(U^* B)| = \operatorname*{Sup}_{\|B\| \leqslant 1} |\varphi \circ \Phi(B)|$$
$$\geqslant \varphi \circ \Phi(U).$$

This establishes the norm-condition for Φ.

The proof of the sufficiency of the norm-condition is of the same spirit as the proof of the polar decomposition for ultraweakly continuous linear functionals.

Let S be the subset of $(R)_1$ consisting of all elements V in $(R)_1$ that are norming for Φ. Then S is nonnull and ultraweakly closed, hence, ultraweakly compact. Moreover S is convex. By the Kreĭn-Milman theorem [19, p. 362], S has an extreme point U_0. We claim that U_0 is also an extreme point in $(R)_1$. If $U_0 = \frac{1}{2}(V_1 + V_2)$, V_1, $V_2 \in (R)_1$, then $\Phi(U_0) = \frac{1}{2}(\Phi(V_1) + \Phi(V_2))$, and for all pure states φ of $\mathfrak{A}$,

$$\alpha_\varphi = \varphi \circ \Phi(U_0) = \tfrac{1}{2}(\varphi \circ \Phi(V_1) + \varphi \circ \Phi(V_2)).$$

By the norm-condition $\alpha_\varphi \geqslant |\varphi \circ \Phi(V_i)|$ for $i = 1, 2$, and all pure states φ of $\mathfrak{A}$. Thus

$$\alpha_\varphi \leqslant \tfrac{1}{2}\{|\varphi \circ \Phi(V_1)| + |\varphi \circ \Phi(V_2)|\} \leqslant \tfrac{1}{2}\{\alpha_\varphi + \alpha_\varphi\} = \alpha_\varphi.$$

Hence $\alpha_\varphi = \varphi \circ \Phi(V_i)$ for $i = 1, 2$ and all pure states φ of $\mathfrak{A}$. Therefore $V_i \in S$, $i = 1, 2$, and, since U_0 is extreme in S, we have $U_0 = V_1 = V_2$. Hence V_0 is extreme in $(R)_1$ and V_0 is a partial isometry [8].

Now we define $|\Phi|(A) = \Phi(U_0 A)$ and verify that $|\Phi| \geqslant 0$. In fact, for any pure state φ of $\mathfrak{A}$,

$$\varphi \circ |\Phi|(1) = \varphi \circ \Phi(U_0) = \operatorname*{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)| \geqslant \operatorname*{Sup}_{\|A\| \leqslant 1} |\varphi \circ \Phi(U_0 A)|$$
$$= \operatorname*{Sup}_{\|A\| \leqslant 1} |\varphi \circ |\Phi|(A)| \geqslant |\varphi \circ |\Phi|(1)|.$$

Hence $\varphi \circ |\Phi|(1) = \operatorname{Sup}_{\|A\| \leqslant 1} |\varphi \circ |\Phi|(A)|$, and $|\Phi|$ is positive by Lemma 2.1.1. Suppose $|\Phi|$ has support E. If $E' = U_0^* U_0$ and $F' = U_0 U_0^*$, then

$$|\Phi|(1 - E') = |\Phi|(1) - |\Phi|(E') = \Phi(U_0) - \Phi(U_0 E')$$
$$= \Phi(U_0) - \Phi(U_0) = 0.$$

Hence $1 - E' \leqslant 1 - E$ and $E \leqslant E'$ by the definition of support E of $|\Phi|$. Let U be $U_0 E$; then $U^* U = E U_0^* U_0 E = E E' E = E$, and $\Phi(U) = \Phi(U_0 E)$ $= |\Phi|(E) = |\Phi|(1) = \Phi(U_0)$. Thus U is norming for Φ. Note that $\Phi . U$

$= \Phi . U_0 E = |\Phi| . E = |\Phi|$. Thus $\Phi . U = |\Phi|$ and the support of $|\Phi|$ is the initial space of the partial isometry U.

Finally we prove $|\Phi| . U^* = \Phi$. The proof here is a modification of a proof given by Kadison [10] in establishing the polar decomposition theorem for ultraweakly linear functionals. It is sufficient to show that $\varphi \circ |\Phi| . U^* = \varphi \circ \Phi$ for all pure states φ of $\mathcal{A}$. For a projection E in R acting on a Hilbert space $\mathcal{H}$ let C_E be a projection onto a closed subspace of $\mathcal{H}$ spanned by $\{Ax | A \in R, E(x) = x\}$. Clearly C_E commutes with all elements in R. It also commutes with all elements in the commutant R' of R. Hence C_E is in the center of R, and $E \leqslant C_E$. C_E is called the *central carrier* of E in R. Let Q be $I - C_{I-E}$ $(\leqslant E)$ and $F = UU^*$. Then

$$Q = QEQ = QU^* UQ = (UQ)^*(UQ).$$

Since $C_{I-F} C_{I-E} = 0$ (see [9, Theorem 1]) we have $I - F \leqslant I - C_{I-E} = Q$ and $I - Q \leqslant F$. Hence,

$$\begin{aligned}(U(I - Q))(U(I - Q))^* &= (I - Q)UU^*(I - Q) = (I - Q)F(I - Q) \\ &= I - Q.\end{aligned}$$

Set $V = UQ + (I - Q)$ and $W = U^*(I - Q) + Q$. We observe that $V^* V = I = W^* W$ and $W^* V = U$. Define $\eta_0(A)$ to be $\varphi \circ \Phi(W^* A V)$. Note that

$$\begin{aligned}\eta_0(1) &= \varphi \circ \Phi(W^* V) = \varphi \circ \Phi(U) = \sup_{\|A\| \leqslant 1} |\varphi \circ \Phi(A)| \\ &\geqslant \sup_{\|A\| \leqslant 1} |\varphi \circ \Phi(W^* A V)| = \|\eta_0\|.\end{aligned}$$

This implies that η_0 is a positive normal linear functional on R. Let R acting on $\mathcal{H}$ be the universal normal representation of R (the direct sum of all representations induced by the normal states of R), and let $\eta_0/\|\eta_0\|$ be $\omega_z R$ with z a unit vector in $\mathcal{H}$, where $\omega_z|_R(A) = (Az, z)$ for all $A \in R$. Note that

$$\varphi \circ \Phi(A) = \varphi \circ \Phi(W^* WAV^* V) = \eta_0(WAV^*) = \|\eta_0\|(AV^* z, W^* z).$$

So we have

$$\|\eta_0\| = \varphi \circ \Phi(U) = \|\eta_0\|(UV^* z, W^* z).$$

Since $\|V^* z\| \leqslant 1$, we have, by Schwarz's inequality,

$$1 \leqslant \|UV^* z\| \, \|W^* z\| \leqslant \|V^* z\| \, \|W^* z\| \leqslant 1.$$

Thus $\|V^* z\| = \|W^* z\| = 1$. Furthermore, we consider $\eta(A) = \varphi \circ |\Phi|(A)$ which is also a positive normal linear functional of R. Hence we have

$$\begin{aligned}\varphi \circ (|\Phi| . U^*)(A) &= (\varphi \circ |\Phi|)(U^* A) = (\varphi \circ \Phi)(UU^* A) \\ &= (\varphi \circ \Phi)(FA) = \|\eta_0\| (AV^* z, FW^* z)\end{aligned}$$

for all A in R. In particular,

$$\begin{aligned}\|\eta_0\| (UV^* z, FW^* z) &= \varphi \circ (|\Phi| . U^*)(U) = \varphi \circ |\Phi|(U^* U) \\ &= \varphi \circ |\Phi|(E) = \varphi \circ \Phi(U) = \|\eta_0\|.\end{aligned}$$

This implies that $(UV^* z, FW^* z) = 1$. And by Schwarz's inequality we have $1 \leqslant \|UV^* z\| \, \|FW^* z\| \leqslant \|FW^* z\| \leqslant 1$. Hence $\|FW^* z\| = \|W^* z\| = 1$. Because F is a projection we have $FW^* z = W^* z$. Therefore

$$\varphi \circ (|\Phi| . U^*)(A) = \|\eta_0\| (AV^* z, FW^* z) = \|\eta_0\| (AV^* z, W^* z) = (\varphi \circ \Phi)(A)$$

for all $A \in R$. This is true for all pure states φ of $\mathcal{C}$. Hence $|\Phi| . U^* = \Phi$. Q.E.D.

Let Φ be as in Theorem 2.1.4. For each pure state φ of $\mathcal{C}$, $\varphi \circ \Phi$ is an ultraweakly linear functional on R. Since $(R)_1$ is ultraweakly compact, the set S_φ of elements T in $(R)_1$ such that $(\varphi \circ \Phi)(T) = \|\varphi \circ \Phi\|$ is nonnull. The norm-condition says $\bigcap_\varphi S_\varphi \neq \varnothing$ for all pure states φ of $\mathcal{C}$. This condition is not easily satisfied, as some examples will show.

We give several examples of bounded linear maps, some which admit polar decompositions and some which do not.

2.1.5. Examples. (i) Let Φ be an ultraweak-norm continuous linear map from R into $\mathcal{C}$. If there is a V in $(R)_1$ such that $\Phi(V) = \|\Phi\| I_2$, where I_2 is the identity element in $\mathcal{C}$, then, by straightforward computation, the norm-condition is satisfied (and V is norming for Φ). Hence such a Φ admits a polar decomposition. In particular, if Φ is a linear isometry of R onto itself, there is an element V (in $(R)_1$) such that $\Phi(V) = I$.

(ii) We construct a bounded linear map Φ from M_4 into $C(\{1,2\})$ for which the image of $(M_4)_1$ does not contain $\|\Phi\| I_2$, where I_2 is the identity function on $\{1,2\}$. Nevertheless Φ admits a polar decomposition.

Let $\{x_1, x_2, x_3, x_4\}$ be an orthonormal basis for $\mathcal{C}^4$. Let $P_{1,2}$ ($P_{3,4}$) be the projection in M_4 with range space generated by $\{x_1, x_2\}$ ($\{x_3, x_4\}$) and P_i be the projection in M_4 with range space generated by x_i, $i = 1, 2, 3, 4$. We define two partial isometries U_1 (U_2) as follows:

$$U_1 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 1 & 0 & \\ 0 & 1 & 0 \end{array}\right], \qquad U_2 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 1 & 0 & \\ 0 & 0 & 0 \end{array}\right]$$

with initial space $U_1^* U_1 = P_{1,2}$ ($U_2^* U_2 = P_1$) and final space $U_1 U_1^* = P_{3,4}$ ($U_2 U_2^* = P_3$). Let φ be the positive linear functional on M_4 defined by

$\varphi(A) = \mathrm{tr}(P_{1,2}AP_{1,2})$, where tr is the trace function on M_4. Let $(P_1.\varphi.P_1)(A)$ be $\varphi(P_1AP_1)$ and define η_1, η_2 by

$$\eta_1 = \varphi.U_1^*, \qquad \eta_2 = (P_1.\varphi.P_1).U_2^*.$$

It is clear that $\{\varphi, U_1\}$ $(\{P_1.\varphi.P_1, U_2\})$ is a polar decompsotion for η_1 (η_2), and $\|\eta_1\| = 2$, $\|\eta_2\| = 1$. We define a linear map Φ from M_4 into $C(\{1,2\})$ by

$$\Phi(A)(1) = \eta_1(A), \qquad \Phi(A)(2) = \eta_2(A),$$

and observe that $\|\Phi\| = \max\{\|\eta_1\|, \|\eta_2\|\} = 2$. For $A \in M_4$,

$$\begin{aligned}\Phi.U_1(A)(1) &= \eta_1(U_1A) = \varphi(A),\\ \Phi.U_1(A)(2) &= \eta_2(U_1A) = (P_1.\varphi.P_1)(U_2^*U_1A)\\ &= (P_1.\varphi.P_1)(U_2^*U_2A) = (P_1.\varphi.P_1)(A).\end{aligned}$$

Hence $\Phi.U_1$ is positive and it has $U_1^*U_1$ as support. Also note $(\Phi.U_1).U_1^* = \Phi$. Suppose that there exists a V in $(M_4)_1$ such that $\Phi(V) = 2I_2$; then $|\Phi|(U_1^*V) = 2I_2$. Let W be $U_1^*V (\in (M_4)_1)$, and let W' be $(W + W^*)/2$. Then $2I_2 = |\Phi|(W') \leqslant |\Phi|(I)$. But

$$\begin{aligned}|\Phi|(I)(2) &= \eta_2(U_1) = (P_1.\varphi.P_1)(I) = 1,\\ 2I_2(2) &= |\Phi|(W')(2) = 2.\end{aligned}$$

Thus no such V exists.

(iii) An example in which the norm-condition is not satisfied.

If X is a set consisting of a finite number of points, say, 10 points $\{x_1, \ldots, x_{10}\}$, l_{10}^∞ is a commutative von Neumann algebra. Since l_{10}^∞ is finite dimensional, the weak-operator topology on l_{10}^∞ is identical with the norm topology. Let Φ be a map from l_{10}^∞ into itself defined by

$$\begin{aligned}\Phi(f)(1) &= f(1),\\ \Phi(f)(n) &= f(n) - f(n-1), \qquad 2 \leqslant n \leqslant 10.\end{aligned}$$

Then Φ is linear and $\|\Phi\| = 2$.

We show next that for any fixed f in l_{10}^∞ with $\|f\| \leqslant 1$, there exist an integer n_f with $1 \leqslant n_f \leqslant 10$ and an element g_f in l_{10}^∞ with $\|g_f\| \leqslant 1$, such that $\Phi(f)(n_f) \ngeqslant |\Phi(g_f)(n_f)|$ (so that f is not norming for Φ). We observe that

$$\begin{aligned}f(10) &= \Phi(f)(10) + f(9)\\ f(9) &= \Phi(f)(9) + f(8)\\ &\vdots\\ f(2) &= \Phi(f)(2) + f(1);\end{aligned}$$

hence

$$f(10) = \sum_{n=2}^{10} \Phi(f)(n) + f(1).$$

Since $\|f\| < 1$ and $-1 \leqslant |f(10)| \leqslant 1$, there must be an integer n_f, $2 < n_f < 10$, such that $\Phi(f)(n_f) \not\geqslant 2$. We define g_f by

$$g_f(n_f - 1) = -1,$$

$$g_f(n_f) = 1,$$

$$g_f(n) = 0, \qquad n_f - 1 \neq n \neq n_f, \; 1 \leqslant n \leqslant 10.$$

Since

$$\Phi(g_f)(n_f) = g_f(n_f) - g_f(n_f - 1) = 1 - (-1) = 2,$$

it follows that

$$\Phi(f)(n_f) \not\geqslant \Phi(g_f)(n_f) = 2.$$

2.1.6. COROLARY. *Let R be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_1$ and Φ be a bounded linear map from R into $B(\mathcal{H}_2)$. If Φ admits a polar decomposition with $|\Phi|$ completely positive, then there exists a norm preserving extension of Φ to $B(\mathcal{H}_1)$ into $B(\mathcal{H}_2)$, i.e., there exists a Λ mapping $B(\mathcal{H}_1)$ into $B(\mathcal{H}_2)$ with $\|\Lambda\| = \|\Phi\|$ and $\Lambda|R = \Phi$.*

PROOF. Let $\{|\Phi|, U\}$ be a polar decomposition for Φ, and assume that $|\Phi|$ is completely positive. By 1.4.2, $|\Phi|$ extends to a completely positive linear map Λ' on $B(\mathcal{H}_1)$. We claim that $\|\Lambda'\| = \| \, |\Phi| \, \|$. Choose Ψ and V as in Theorem 1.1.3 such that $\Lambda'(A) = V^* \Psi(A) V$, $A \in B(\mathcal{H}_1)$. Then $\|\Lambda'\| \leqslant \|V^*\| \, \|V\| = \|V^* V\| = \|\Lambda'(I_1)\|$. Since $I_1 \in R$ it follows that $\|\Lambda'\| \leqslant \| \, |\Phi| \, \|$. If we let $\Lambda = \Lambda' . U^*$, then $\Lambda|R = \Phi$ since $\Lambda|R = |\Phi| . U^*$. Moreover $\|\Lambda\| \leqslant \|\Lambda'\| = \| \, |\Phi| \, \| = \|\Phi\|$. Q.E.D.

2. **Uniqueness of polar decomposition.** The norm condition says that the existence of a polar decomposition is guaranteed by the existence of a partial isometry U which is a common partial isometry in the polar decomposition for all ultraweakly continuous linear functionals $\varphi \circ \Phi$ (φ a pure state of $\mathfrak{A}$). This makes it plausible that the polar decomposition, if it exists, is unique. We prove this in Theorem 2.2.2. A lemma is needed. Let Φ be an ultraweak-norm continuous linear map from R into $\mathfrak{A}$.

2.2.1. LEMMA. *Let $\Phi = \Psi . U$, $\Psi \geqslant 0$, $\|U\| \leqslant 1$, and $\|\varphi \circ \Phi\| = \|\varphi \circ \Psi\|$ for all pure states φ of $\mathfrak{A}$. If there exists a V in $(R)_1$ such that $\Phi . V \geqslant 0$ and $\|\varphi \circ \Phi . V\| = \|\varphi \circ \Psi\|$ for all pure states φ of $\mathfrak{A}$, then $\Phi . V = \Psi$.*

PROOF. It suffices to show $\varphi \circ \Phi . V = \varphi \circ \Psi$ for all pure states φ of $\mathfrak{A}$. Let

R acting on $\mathcal{H}$ be the universal normal representation of R. Then $\varphi \circ \Psi(A) = \|\varphi \circ \Psi\|(Ax, x)$ for all A in R, where x is a unit vector in $\mathcal{H}$. (Note that if $0 = \|\varphi \circ \Psi\| = \|\varphi \circ \Phi\|$, then $\varphi \circ \Phi . V = \varphi \circ \Psi = 0$. We may assume that $\|\varphi \circ \Psi\| \neq 0$.) Hence,

$$\begin{aligned}\|\varphi \circ \Psi\|(UVx, x) = \varphi \circ \Psi(UV) &= \varphi \circ \Phi(V) \\ &= \varphi \circ (\Phi . V)(1) = \|\varphi \circ \Phi . V\|,\end{aligned}$$

since $(\varphi \circ \Phi . V)$ is a positive linear functional. Thus $(x, V^* U^* x) = (UVx, x) = 1$. By Schwarz's inequality we have $x = V^* U^* x$. Thus, for $A \in R$,

$$\begin{aligned}(\varphi \circ \Phi . V)(A) &= \varphi \circ \Phi(VA) = (\varphi \circ \Psi . U)(VA) = (\varphi \circ \Psi)(UVA) \\ &= \|\varphi \circ \Psi\|(Ax, V^* U^* x) = \|\varphi \circ \Psi\|(Ax, x) = (\varphi \circ \Psi)(A). \quad \text{Q.E.D.}\end{aligned}$$

2.2.2. UNIQUENESS THEOREM OF POLAR DECOMPOSITION. *Let Φ be an ultraweak-norm continuous linear map from R into $\mathfrak{A}$. If $\Phi . U = \Psi$, $\Psi . U^* = \Phi$, and $\Phi . V = \Lambda$, $\Lambda . V^* = \Phi$, where Ψ and Λ are positive and have supports $E = U^* U$, $E' = V^* V$ respectively, then $\Psi = \Lambda$ and $U = V$.*

PROOF. For any pure states φ of $\mathfrak{A}$

$$\|\varphi \circ \Phi\| = \|\varphi \circ \Psi . U^*\| \leqslant \|\varphi \circ \Psi\| = \|\varphi \circ \Phi . U\| \leqslant \|\varphi \circ \Phi\|;$$

hence $\|\varphi \circ \Phi\| = \|\varphi \circ \Psi\|$. For the same reason $\|\varphi \circ \Phi\| = \|\varphi \circ \Lambda\|$. By Lemma 2.2.1, $\Psi = \Lambda$ because $\|\varphi \circ \Phi . U\| = \|\varphi \circ \Lambda\|$. Consequently, $E = E'$, i.e., $U^* U = V^* V$.

Next, we show that $U = V$. Let $W = U^* V$. Since the initial projection of U and V is E, we have $EWE = W$.

$$\Psi(W) = \Psi(U^* V) = \Psi . U^*(V) = \Phi(V) = \Phi . V(1) = \Lambda(1) = \Psi(1) \geqslant 0,$$

$$\Psi(W^*) = \Psi(W)^* = \Psi(1)^* = \Psi(1),$$

$$\begin{aligned}\Psi(W^* W) &= \Psi(V^* UU^* V) = \Psi . V^*(UU^* V) = \Phi(UU^* V) \\ &= \Phi . U(U^* V) = \Psi(U^* V) = \Psi(W) = \Psi(1).\end{aligned}$$

So,

$$\begin{aligned}\Psi((W - E)^*(W - E)) &= \Psi(W^* W - W^* E - EW + E) \\ &= \Psi(W^* W) - \Psi(W^* E) - \Psi(EW) + \Psi(E) \\ &= \Psi(1) - \Psi(W^*) - \Psi(W) + \Psi(E) \\ &= \Psi(1) - \Psi(1) - \Psi(1) + \Psi(1) = 0.\end{aligned}$$

Since Ψ is faithful on ERE, we have $W - E = 0$ and $W = E = U^* V$.

V is an isometry on the range of E, so is U. Since $E = U^*V$, $V = U$. Q.E.D.

Let Φ be in $B(\mathcal{C}_1, \mathcal{C}_2)$ and U_1, U_2 be in $\mathcal{C}_1$. If $\Phi \geqslant 0$, then $U.\Phi.U^* \geqslant 0$.

2.2.3. COROLLARY. *If $\{|\Phi|, U\}$ is the polar decomposition of Φ, then $|\Phi^*| = U.|\Phi|.U^*$ and $|\Phi^*| = \Phi^*.U^*$ with support UU^*.*

PROOF. For $A \in R$,

$$\Phi^*(A) = \Phi(A^*)^* = [|\Phi|.U^*(A^*)]^* = |\Phi|(U^*A^*)^*$$
$$= |\Phi|(AU) = |\Phi|(U^*UAU) = U.|\Phi|.U^*(UA).$$

Let F be UU^*, the final projection of U. Since $U.|\Phi|.U^* \geqslant 0$ and $U.|\Phi|.U^*(F) = |\Phi|(U^*FU) = |\Phi|(U^*U) = U.|\Phi|.U^*(1)$, hence $F \geqslant$ support of $U.|\Phi|.U^*$. If $F' < F$ (F' is a projection), then

$$U^*F'U < U^*FU = U^*U = E$$

and

$$U.|\Phi|.U^*(F') = |\Phi|(U^*F'U) < |\Phi|(E) = U.|\Phi|.U^*(1)$$

since Φ is faithful on ERE. Thus $F =$ support of $(U.|\Phi|.U^*)$ from uniqueness of the polar decomposition, $|\Phi^*| = U.|\Phi|.U^*$, and $\Phi^* = |\Phi^*|.U$. Moreover, $\Phi^*.U^* = |\Phi^*|.UU^* = |\Phi^*|.F = |\Phi^*|$. Q.E.D.

We can combine the polar decomposition with positive decomposition to obtain

2.2.4. THEOREM. *Let $\Phi \in B_{\mathrm{s.a.}}(R, \mathcal{C})$. If Φ admits a polar decomposition, then Φ is positively decomposable, i.e., $\Phi = \Phi_1 - \Phi_2$, where Φ_1, Φ_2 are positive linear maps. Furthermore, Φ_1 and Φ_2 have orthogonal supports and $|\Phi| = \Phi_1 + \Phi_2$; $\|\Phi_1\| \leqslant \|\Phi\|$ and $\|\Phi_2\| \leqslant \|\Phi\|$.*

As a part of the proof we derive a lemma of independent interest.

2.2.5. LEMMA. *If Φ is a selfadjoint linear map admitting a polar decomposition, then the partial isometry U in the polar decomposition is also selfadjoint.*

PROOF. By 2.2.2 and 2.2.3, since $|\Phi^*| = |\Phi|$, $\Phi = |\Phi|.U^*$ and $\Phi = |\Phi^*|.U$, we have $U^* = U$. Q.E.D.

PROOF OF THEOREM 2.2.4. Let $\{|\Phi|, U\}$ be the polar decomposition for Φ. Since $U^2 = U^*U = UU^* = E$, the spectrum of U is contained in $\{-1, 0, 1\}$ so that $U = E_1 - E_2$ where E_1, E_2 are two orthogonal projections. Note that if the spectrum of $U = \{0, 1\}$ (or $\{-1, 0\}$), then U (or $-U$) is E, the support of $|\Phi|$; and Φ is positive (or negative). By 2.2.3 we have

$$(1) \qquad |\Phi| = |\Phi^*| = U.|\Phi|.U = (E_1 - E_2).|\Phi|.(E_1 - E_2).$$

But $U^2 = (E_1 - E_2)^2 = E_1 + E_2$ is the support of $|\Phi|$; thus

$$|\Phi| = (E_1 + E_2).|\Phi|.(E_1 + E_2). \tag{2}$$

Equalities (1) and (2) imply that

$$E_1.|\Phi|.E_2 + E_2.|\Phi|.E_1 = 0 \quad \text{and} \quad |\Phi| = E_1.|\Phi|.E_1 + E_2.|\Phi|.E_2.$$

Hence

$$E_1.|\Phi| = E_1.(E_1.|\Phi|.E_1) + E_1.(E_2.|\Phi|.E_2) = E_1.|\Phi|.E_1 = |\Phi|.E_1.$$

Similarly

$$E_2.|\Phi| = E_2.|\Phi|.E_2 = |\Phi|.E_2.$$

Hence

$$\Phi = |\Phi|.U = |\Phi|.(E_1 - E_2) = E_1.|\Phi|.E_1 - E_2.|\Phi|.E_2,$$

where $E_1.|\Phi|.E_1$ and $E_2.|\Phi|.E_2$ are positive linear maps with orthogonal supports E_1, E_2 respectively. Also $\|E_i.|\Phi|.E_i\| \leqslant \|\,|\Phi|\,\| = \|\Phi\|$ for $i = 1, 2$. The proof is complete. Q.E.D.

2.2.6. DEFINITION. Let $\Phi \in B_{\text{s.a.}}(R, \mathfrak{A}).\Phi$ is said to be *Hahn-Jordan decomposable* (or to admit a *Hahn-Jordan decomposition*), if there exists two bounded positive linear maps Φ_1, Φ_2 in $B_{\text{s.a.}}(R, \mathfrak{A})$ with orthogonal supports such that $\Phi = \Phi_1 - \Phi_2$.

2.2.7. THEOREM. *If $\Phi \in B_{\text{s.a.}}(R, \mathfrak{A})$ and Φ is Hahn-Jordan decomposable, then Φ admits a polar decomposition. Furthermore, if E_1 and E_2 are the pair of orthogonal supports in the decomposition, then $|\Phi|$ has support $E_1 + E_2$ and $\Phi = |\Phi|.(E_1 - E_2)$ is the polar decomposition for Φ. With $\Phi_1 = \Phi.E_1$ and $\Phi_2 = -\Phi.E_2$, $\Phi = \Phi_1 - \Phi_2$ is the Hahn-Jordan decomposition of Φ.*

PROOF. Let $\Phi_1 - \Phi_2$ be a Hahn-Jordan decomposition for Φ with orthogonal supports E_1, E_2 for Φ_1, Φ_2 respectively. For a projection P in R we have $(\Phi_1 + \Phi_2)(P) = 0$ if and only if $\Phi_1(P) = 0 = \Phi_2(P)$, i.e., $PE_1 = PE_2 = 0$. Hence $\Phi_1 + \Phi_2$ has support $E_1 + E_2$. In addition, $E_1 - E_2$ is a partial isometry with initial and final projection $(E_1 - E_2)^2 = E_1 + E_2$. We have

$$(\Phi_1 + \Phi_2).(E_1 - E_2) = \Phi_1.E_1 - \Phi_2.E_2 = \Phi_1 - \Phi_2 = \Phi, \tag{1}$$

$$\begin{aligned} \Phi.(E_1 - E_2) &= (\Phi_1 - \Phi_2).(E_1 - E_2) = \Phi_1.E_1 + \Phi_2.E_2 \\ &= \Phi_1 + \Phi_2. \end{aligned} \tag{2}$$

Equalities (1) and (2) imply that $\{\Phi_1 + \Phi_2, E_1 - E_2\}$ is the polar decomposition of Φ. Finally,

$$\Phi.E_1 = (\Phi_1 - \Phi_2).E_1 = \Phi_1.E_1 = \Phi_1$$

and

$$-\Phi . E_2 = (-\Phi_1 + \Phi_2) . E_2 = \Phi_2 . E_2 = \Phi_2 . \quad \text{Q.E.D.}$$

2.2.8. COROLLARY. *If* $\Phi \in B_{\text{s.a.}}(R, \mathfrak{A})$ *and* Φ *admits a Hahn-Jordan decomposition, then* Φ *admits a unique such decomposition.*

PROOF. From 2.2.7, $|\Phi| = \Phi_1 + \Phi_2$ and $\Phi = \Phi_1 - \Phi_2$. Hence Φ_1, Φ_2 are uniquely determined by Φ and $|\Phi|$ which is, in turn, uniquely determined by Φ. Q.E.D.

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