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CIRCLE ACTIONS ON SIMPLY CONNECTED 4-MANIFOLDS

BY

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ABSTRACT. Locally smooth S^1-actions on simply connected 4-manifolds are studied in terms of their weighted orbit spaces. An equivariant classification theorem is proved, and the weighted orbit space is used to compute the quadratic form of a given simply connected 4-manifold with S^1-action. This is used to show that a simply connected 4-manifold which admits a locally smooth S^1-action must be homotopy equivalent to a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.

1. **Introduction.** In this article we investigate locally smooth actions of the circle group S^1 on simply connected 4-manifolds. Our techniques are essentially geometric and are motivated by the work of P. Orlik and F. Raymond [9], [10], and H. Seifert [11]. To each locally smooth S^1-action on an oriented simply connected closed 4-manifold we associate the orbit space "weighted" with certain isotropy information. §§3–6 are devoted to the equivariant classification of these actions. Two such actions are orientation-preserving equivariantly homeomorphic if and only if their weighted orbit spaces are isomorphic. Furthermore, each legally weighted simply connected 3-manifold gives rise to an S^1-action on a simply connected 4-manifold which is constructed by means of equivariant plumbing and other pasting techniques. This gives a complete account of such actions and generalizes results of [4]. In the case of an action which is free except for a finite fixed point set, these results follow from work of Church and Lamotke [3].

In §7 we turn to the question of extending S^1-actions on simply connected 4-manifolds to T^2-actions, and it is shown that an S^1-action extends if and only if the weighted orbit space has a relatively simple format.

In §8 the information contained in the weighted orbit space is used to give a recipe for the computation of the quadratic form of a simply connected 4-manifold with given S^1-action. The main result of this section is that a simply connected 4-manifold which admits a locally smooth S^1-action is homotopy equivalent to a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.

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2. **Preliminaries.** In this section we fix the notation and terminology which we shall use. Recall that an action of a compact Lie group G on a manifold N is called locally smooth if each $x \in N$ has a slice which is a disk on which the action of the isotropy group G_x is equivalent to an orthogonal action. All group actions which we discuss are assumed to be locally smooth and effective. See [1] for further terminology.

Throughout this paper M will denote a simply connected oriented 4-manifold with an action of the circle group S^1. For any subset X of M, X^* denotes its image in the orbit space M^* and $p\colon M \to M^*$ is the orbit map. Furthermore, if we are given a set X^* in M^*, we let $X = p^{-1}(X^*)$ when no confusion is caused by this notation. We let F be the fixed point set of M, E the union of the exceptional orbits, and P the union of the principal orbits.

All homology and cohomology used in this paper has $\mathbf{Z}$ coefficients. We shall often allow ourselves to confuse a loop in M with the element in $\pi_1(M)$ or $H_1(M)$ which the loop represents.

The Orlik-Raymond-Seifert classification of S^1-actions on 3-manifolds [9], [11] will be needed in §§3 and 5. We outline the aspects of the theory which will be of import there and refer the reader to [9] for further details. An S^1-action on a closed oriented 3-manifold W is determined up to orientation-preserving equivariant homeomorphism by a collection of invariants $\{b; (o, g, \bar{h}, 0); (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ as follows. The orbit space W^* is a surface of genus g with $\bar{h}$ boundary components each corresponding to a circle of fixed points in W, and W^* is oriented so that followed by the natural orientation on the orbits the orientation of W is obtained. There are n exceptional orbits each of which is assigned a pair of integers (α_i, β_i) called Seifert invariants.

If $E^* = \{x_1^*, \ldots, x_n^*\}$ choose disjoint closed 2-disk neighborhoods V_i^* of the x_i^*. If $x_i \in p^{-1}(x_i^*)$ there is a closed 2-disk slice S_i at x_i such that $S_i^* = V_i^*$. Orient S_i so that its intersection number with the oriented orbit $p^{-1}(x_i^*)$ is $+1$ in the solid torus V_i. On ∂V_i let m_i be an oriented boundary curve of S_i and let h_i be an oriented orbit. If the isotropy group at x_i is $\mathbf{Z}_{\alpha_i}$ an oriented section q_i of the action on ∂V_i is specified up to homology by the homology relation $m_i \sim \alpha_i q_i + \beta_i h_i$ where α_i and β_i are relatively prime and $0 < \beta_i < \alpha_i$. The Seifert invariants (α_i, β_i) determine V_i up to orientation-preserving equivariant homeomorphism. If the orientation on V_i is reversed, the Seifert invariants become $(\alpha_i, \alpha_i - \beta_i)$ and if q_i' is the new section, $q_i' \sim -q_i - h_i$. The action of the isotropy group $\mathbf{Z}_{\alpha_i}$ on S_i is orientation-preserving equivariantly homeomorphic to an action of $\mathbf{Z}_{\alpha_i}$ on D^2:

$$\frac{2\pi}{\alpha_i} \times (r, \theta) \to \left(r, \theta + \frac{2\pi\nu_i}{\alpha_i} \right).$$

The pair $[\alpha_i, \nu_i]$ are called the orbit invariants of $p^{-1}(x_i^*)$ and satisfy $\beta_i\nu_i \equiv 1 \pmod{\alpha_i}$.

The integer b is the obstruction to extending the section $q_1 \cup \cdots \cup q_n$ over all of $W^* - \cup \mathring{V}_i^*$. So if $\bar{h} \neq 0$ we have $b = 0$. If $\bar{h} = 0$ let V_0^* be a 2-disk in W^* disjoint from $\cup_{i=1}^n V_i^*$. If m_0 is an oriented meridian on ∂V_0 and q is the restriction to ∂V_0 of the oriented section $q_1 \cup \cdots \cup q_n$ extended over $W^* - \cup_{i=0}^n \mathring{V}_i^*$ then $q \sim -m_0 + bh$, where h is an oriented orbit. Reversing the orientation of the action on W changes the invariants to $\{-n - b; (o, g, \bar{h}, 0); (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_n, \alpha_n - \beta_n)\}$. Finally, we note that the free action $\{b; (o, 0, 0, 0)\}$ is the principal S^1-bundle over S^2 with Euler number $-b$ (see [8, p. 25]).

3. **The weighted orbit space.** In this section we describe the orbit structure of M^4.

PROPOSITION (3.1). (a) *The orbit space M^* is a simply connected 3-manifold with $\partial M^* \subset F^*$.*

(b) *The set $F^* - \partial M^*$ is finite, and F^* is nonempty.*

(c) *The closure of E^* is a collection of polyhedral arcs and simple closed curves in M^*. The components of E^* are open arcs on which orbit types are constant, and these arcs have closures with distinct endpoints in $F^* - \partial M^*$.*

PROOF. If $x \in E$ the isotropy group at x is a finite cyclic group which acts as a group of rotations on some 3-disk slice S_x at x with axis of rotation $E \cap S_x$. Then S_x^* is a neighborhood of x^* and the pair $(S_x^*, E^* \cap S_x^*)$ is homeomorphic to $(D^2 \times I, 0 \times I)$.

If $y \in F$ there is a closed 4-disk slice S_y at y on which the S^1-action is the cone at y of the action on the boundary 3-sphere ∂S_y. The only S^1-actions on S^3 are of the form

$$\{\pm 1; (o, 0, 0, 0)\}, \quad \{0; (o, 0, 1, 0)\}, \quad \{b; (o, 0, 0, 0); (\alpha, \beta)\} \quad \text{and}$$
$$\{-1; (o, 0, 0, 0); (\alpha, \beta), (\alpha', \beta')\}.$$

For these various cases we have $(S_y^*, E^* \cup F^*, F^*)$ homeomorphic to $(D^3, 0, 0)$, $(D^2 \times I, D^2 \times 0, D^2 \times 0)$, $(D^2 \times [-1, 1], 0 \times [0, 1], (0, 0))$, and $(D^2 \times [-1, 1], 0 \times [-1, 1], (0, 0))$ respectively. In the last case $\alpha \neq \alpha'$.

It is well known that M^* is simply connected. Since an $x^* \in P^*$ certainly has a 3-disk neighborhood in M^*, (a) is proved, and (b) follows from the formula $\chi(F) = \chi(M) = 2 + \operatorname{rank} H_2(M)$. A result of Montgomery and Yang [7, Lemma 2.3] implies that there is no simple closed curve in E^* on which orbit types are constant. This is proved in [7] for the case $M = S^4$, but the same proof holds for any simply connected M^4. Thus E^* consists of a collection of open arcs each of constant orbit type. Since P is open in M, $E^* \cup F^* = M^* - P^*$ is compact and (c) follows. $\square$

We remark that since M^* is simply connected, it follows from duality that the components of ∂M^* are 2-spheres.

(3.2) Since M is oriented, an orientation on M^* is determined, so that, followed by the natural orientation on the orbits, the orientation of M is obtained. Given an oriented submanifold X^* of M^* we use the above convention to orient X. We say that a codimension one submanifold of an oriented manifold is oriented by some given normal to mean that the orientation on the submanifold followed by the normal gives the orientation of the ambient manifold. The boundary of an oriented manifold is to be oriented by the inward normal unless it is explicitly stated to the contrary.

Now suppose $X^* \subset E^* \cup F^*$. We shall reserve the term "regular neighborhood" of X^* for those regular neighborhoods N^* of X^* which satisfy $N^* \cap (E^* \cup F^*) = X^*$.

(3.3) We assign to M^* the following orbit data.

(a) For each boundary component F_i^* of M^* choose a regular neighborhood $F_i^* \times [0, 1]$ and orient $F_i^* \times 1$ by the normal out of $F_i^* \times [0, 1]$. The restriction of the orbit map gives a principal S^1-bundle over $F_i^* \times 1$ and we assign to F_i^* the Euler number of this bundle [5]. This is clearly independent of our choice of collar. We shall call F_i^* a weighted 2-sphere.

(b) If $x^* \in F^* - (\partial M^* \cup \mathrm{cl}\, E^*)$ let B^* be a polyhedral 3-disk neighborhood of x^* with $B^* - x^* \subset P^*$. Restriction of the orbit map gives a principal S^1-bundle over ∂B^* with total space a 3-sphere. Orient ∂B^* by the normal out of B^* and assign to x^* the Euler number, ± 1, of the bundle.

(c) Suppose L^* is a simple closed curve in $E^* \cup F^*$. To each component of E^* in L^* we assign Seifert invariants. First fix an orientation on L^*; this induces an orientation on each component J^* of E^* in L^*. Let y^* be an endpoint of $\mathrm{cl}\, J^*$ and let B^* be a polyhedral 3-disk neighborhood of y^* such that $B^* \cap (E^* \cup F^*) = B^* \cap L^*$ is an arc and $B^* \cap F^* = y^*$. If ∂B^* is oriented by the normal with direction J^* then ∂B is an oriented 3-sphere. Assign to J^* the Seifert invariants (α, β) of the orbit in ∂B with image in J^*. The covering homotopy theorem of Palais [1] implies that this definition is independent of the choices made.

The weights assigned to L^* consist of the orientation and the Seifert invariants. We abbreviate this system of weights by $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ where the order of the (α_i, β_i) is determined up to a cyclic permuation, and we call L^* a weighted circle. If the orientation of L^* is reversed each (α_i, β_i) becomes $(\alpha_i, \alpha_i - \beta_i)$ and we regard the resulting weighted circle as equivalent to the first.

(d) If A^* is an arc which is a component of $E^* \cup F^*$, we orient A^* and assign Seifert invariants as in (c). If y^* is the initial or final point of A^* and B^* is a small 3-disk neighborhood of y^*, then proceeding as in (c) ∂B has the

S^1-action $\{b; (o, 0, 0, 0); (\alpha, \beta)\}$. Assign this integer b to y^*. We call A^* a weighted arc and write the weight system as $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$. Reversing the orientation on A^* changes the weight system to $[-1 - b''; (\alpha_n, \alpha_n - \beta_n), \ldots, (\alpha_1, \alpha_1 - \beta_1); -1 - b']$ which we regard as equivalent to the original weight system on A^*.

The oriented orbit space M^* together with the above collection of weights is called a *weighted orbit space*.

(3.4) An *isomorphism* of weighted orbit spaces M_1^* and M_2^* is an orientation-preserving homeomorphism which carries the weights of M_1^* isomorphically onto the weights of M_2^*.

LEMMA (3.5). (a) *If* (α_i, β_i) *and* $(\alpha_{i+1}, \beta_{i+1})$ *are the Seifert invariants assigned to adjacent arcs in some weighted arc or circle then*

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{vmatrix} = \pm 1.$$

(b) *If* $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ *is a weighted arc then* $b'\alpha_1 + \beta_1 = \pm 1$ *and* $b''\alpha_n + \beta_n = \pm 1$. (*So for* $i = 1$ *or* n, $\beta_i = 1$ *or* $\alpha_i - 1$, *and* b' *and* b'' *can only take on the values* 0 *or* -1.)

PROOF. (a) If $y^* \in F^*$ belongs to the closure of both arcs involved and if B^* is a small polyhedral 3-disk containing y^* then the action on ∂B is $\{-1; (o, 0, 0, 0); (\alpha_i, \beta_i), (\alpha_{i+1}, \alpha_{i+1} - \beta_{i+1})\}$. The result now follows from [9, Theorem 4]. Similar considerations prove (b). $\square$

PROPOSITION (3.6). *Let* W^* *be a regular neighborhood in* M^* *of a weighted arc or circle. The weights determine* $W = p^{-1}(W^*)$ *up to orientation-preserving equivariant homeomorphism.*

PROOF. We give the proof for the case of a weighted circle; the proof in the other case is similar. Let K^* and L^* be isomorphically weighted circles with weight systems $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$, and let U^* and W^* be the respective regular neighborhoods. By deforming an orientation-preserving homeomorphism we obtain an isomorphism $f^*: U^* \to W^*$.

In U^* let $y_i^* \in F^*$ be the initial point of the arc with Seifert invariants (α_i, β_i) and choose x_i^* in the interior of this arc. Let I_i^* denote the interval $[x_{i-1}^*, x_i^*]$ of K^*. Representing U^* as $D^2 \times K^*$ we let $B_i^* = D^2 \times I_i^*$. Using the covering homotopy theorem of Palais along with the fact that the action is locally smooth, one sees that the action on B_i is orientation-preserving equivalent to the cone at y_i of the action on ∂B_i, the equivalence being the identity on ∂B_i. Similarly the action on $p_W^{-1}(f^*(B_i^*))$ is equivalent to the cone of the action on its boundary.

Let $V_i^* = D^2 \times x_i$ and $Y_i^* = f^*(V_i^*)$ be oriented by K^* and L^*. Since f^* is a weighted isomorphism there is an orientation-preserving equivariant homeomorphism $g_1\colon \cup V_i \to \cup Y_i$. The induced g_1^* is also orientation-preserving so there is an orbit-type preserving isotopy of g_1^* to $f^*| \cup V_i^*$. Lifting this isotopy equivariantly we may suppose that $g_1^* = f^*$.

Let q_i be the canonical section on ∂V_i. It follows from the classification of S^1-actions on S^3 that on $\partial B_i - \operatorname{int}(V_{i-1} \cup V_i)$ we have $q_{i-1} \sim -q_i' - h_i'$ where q_i' is the section of ∂V_i oriented as a submanifold of ∂B_i. Thus $q_{i-1} \sim -q_i' - h_i' \sim q_i$, so the section $q_{i-1} \cup q_i$ extends to a section on $\partial B_i - \operatorname{int}(V_{i-1} \cup V_i)$. Hence $q_1 \cup \cdots \cup q_n$ extends to a section over ∂U^*. Since $g_1(q_i)$ is homologous to the canonical section over ∂Y_i, also $g_1(q_1) \cup \cdots \cup g_1(q_n)$ extends to a section over ∂W^*. Using these sections g_1 extends to an orientation-preserving equivariant homeomorphism $\partial U \cup \cup V_i \to \partial W \cup \cup Y_i$. We now obtain the desired equivalence $g\colon U \to W$ via the cone structure. □

(3.7). *Addendum.* The equivalence g is constructed so that $g^*|\partial U^* = f^*|\partial U^*$.

COROLLARY (3.8). *If L^* is a weighted circle in M^* with regular neighborhood W^* then $\partial W \cong T^3$ with the free S^1-action* $\{0; (o, 1, 0, 0)\}$.

PROOF. The orbit map $p\colon \partial W \to \partial W^* \cong T^2$ is a principal S^1-bundle which, according to the proof of (3.6), admits a cross-section. □

PROPOSITION (3.9). *If $A^* = [b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ is a weighted arc in M^* with regular neighborhood W^*, the S^1-action restricted to ∂W is* $\{b'' - b'; (o, 0, 0, 0)\}$.

PROOF. Let $F^* = \{y_1^*, \ldots, y_{n+1}^*\}$ and write $W^* = \cup B_i^*$ where B_i^* is a polyhedral 3-disk containing y_i^*, $B_i^* \cap (E^* \cup F^*)$ is an arc containing no fixed point other than y_i^*, and $B_i^* \cap B_{i+1}^* = \partial B_i^* \cap \partial B_{i+1}^*$ is a 2-disk meeting A^* at a single point.

Letting q_i be the section on $\partial(B_i \cap B_{i+1})$ as in (3.6) we have $q_i \sim q_{i+1}$ for $i = 1, \ldots, n$. On $\operatorname{cl}(\partial B_1 - B_2)$ let m_0 be an oriented meridional curve. Since the S^1-action on ∂B_1 is $\{-b' - 1; (o, 0, 0, 0); (\alpha_1, \alpha_1 - \beta_1)\}$ we have $m_0 \sim -q_1' - (b' + 1)h_1' \sim q_1 - b'h_1 \sim q_2 - b'h_2 \sim \cdots \sim q_n - b'h_n$. The action on ∂B_{n+1} is $\{b''; (o, 0, 0, 0); (\alpha_n, \beta_n)\}$ so $q_n \sim -m_{n+1}' + b''h_{n+1}'$. We have $m_0 \sim -m_{n+1}' + (b'' - b')h_{n+1}'$. So the action on ∂W is as claimed. □

(3.10) Since $\{b; (o, 0, 0, 0)\} \cong L(b, 1)$, if $b' = b''$ in (3.9) then $\partial W \cong L(0, 1) = S^2 \times S^1$. It is easily seen using (3.5)(a) that this must be the case unless some $(\alpha_i, \beta_i) = (2, 1)$. If $b' \neq b''$ then $\partial W \cong L(\pm 1, 1) = S^3$.

4. Equivariant plumbing. The equivariant plumbing of 2-disk bundles over S^2 is used in this section to form building blocks which will be used in §5 to construct manifolds with S^1-actions and in §7 to construct manifolds with T^2-actions. We quickly describe our notation which is borrowed from [8]. See also [5] and [2].

(4.1) Write $S^2 = B_1 \cup B_2$ as the union of its upper and lower hemispheres. Coordinatize $B_i \times D_i^2$ $(i = 1, 2)$ using polar coordinates on B_i and D_i. For relatively prime integers u_i and v_i define an S^1-action by $S^1 \times B_i \times D_i \to B_i \times D_i$, $\phi \times (r, \gamma, s, \delta) \to (r, \gamma + u_i\phi, s, \delta + v_i\phi)$. Then if $u_2 = -u_1$ and $v_2 = -\omega u_1 + v_1$ we obtain $Y_\omega = B_1 \times D_1 \cup_G B_2 \times D_2$ by means of the equivariant pasting G: $\partial B_1 \times D_1 \to \partial B_2 \times D_2$, $G(1, \gamma, s, \delta) = (1, -\gamma, s, -\omega\gamma + \delta)$. The resulting manifold Y_ω is the D^2-bundle over S^2 with Euler number ω; i.e. ω is the self-intersection number of the zero section of Y_ω.

Given Y_{ω_1} and Y_{ω_2} with $u_{2,1} = v_{1,2}$ and $v_{2,1} = u_{1,2}$ (or $u_{2,1} = -v_{1,2}$ and $v_{2,1} = -u_{1,2}$) we may equivariantly plumb together Y_{ω_1} and Y_{ω_2} with sign $+1$ (sign -1) by identifying $B_{2,1} \times D_{2,1}$ with $B_{1,2} \times D_{1,2}$ by means of the equivariant homeomorphism $(r, \gamma, s, \delta) \to (s, \delta, r, \gamma)$ $((r, \gamma, s, \delta) \to (s, -\delta, r, -\gamma))$. The resulting manifold $Y_{\omega_1} \square Y_{\omega_2}$ then has an induced S^1-action.

Similarly one may define (effective) T^2-actions on Y_ω by means of integers u_i, v_i, w_i, and t_i, where

$$\begin{vmatrix} u_i & w_i \\ v_i & t_i \end{vmatrix} = \pm 1.$$

The corresponding T^2-action on $B_i \times D_i$ is given by $(\phi, \theta) \times (r, \gamma, s, \delta) \to (r, \gamma + u_i\phi + w_i\theta, s, \delta + v_i\phi + t_i\theta)$. The pasting G defined above will be T^2-equivariant if also $w_2 = -w_1$ and $t_2 = -\omega w_1 + t_1$. The plumbing $Y_{\omega_1} \square Y_{\omega_2}$, say with sign $+1$, may then be constructed T^2-equivariantly if also $w_{2,1} = t_{1,2}$ and $t_{2,1} = w_{1,2}$.

(4.2) For the S^1-action on $B_i \times D_i$ described in (4.1), suppose neither u_i nor v_i is 0. The orbit space $(B_i \times D_i)^*$ of the action may be viewed as the suspension of D^2 in $\mathbf{R}^3$, i.e. as $\{(\rho, \theta, z) | z \in [-1, 1], 0 \leqslant \rho \leqslant 1 - |z|\}$, with orbit map $p(r, \gamma, s, \delta) = (rs, u\delta - v\gamma, r - s)$. The orientation induced on $(B_i \times D_i)^*$ from the action is just the orientation $(B_i \times D_i)^*$ inherits as a submanifold of $\mathbf{R}^3$.

Following (3.3) we assign weights to the arc $0 \times [-1, 1]$ in $(B_i \times D_i)^*$. It will be convenient here to use orbit invariants rather than Seifert invariants. Orient the arc from -1 to $+1$. To compute the invariants on $0 \times (0, 1]$ use the outward normal to orient $\partial(B_i \times D_i)^*$. This orientation lifts to the orientation by the inward normal on $\partial(B_i \times D_i)$, and then the slice $(1, 0) \times D_i$ at $(1, 0, 0, 0)$ has orientation (sgn u_i) times the usual orientation on D_i (so as to have intersection number $+1$ with an oriented orbit). It now follows from the definition that the orbit invariants are $[|u_i|, (\operatorname{sgn} u_i)v_i]$. Similarly the orbit invariants assigned to $0 \times [-1, 0)$ are $[|v_i|, -(\operatorname{sgn} v_i)u_i]$. Note that if

$u_i > 2$ or $v_i > 2$ then changing the sign on exactly one of the u_i or v_i gives the opposite orbit invariants.

If instead of p we view the map $p'(r, \gamma, s, \delta) = (rs, v\gamma - u\delta, r - s)$ as the orbit map, then the weighted orbit space $(B_i \times D_i)^*$ is the same as that given above except that it has the opposite orientation.

(4.3) We now catalogue some fundamental S^1 and T^2-actions on the disk bundles Y_ω. According to (4.1) S^1 and T^2-actions on Y_ω are described by giving the matrix

$$\begin{pmatrix} u_1 & u_2 & w_1 & w_2 \\ v_1 & v_2 & t_1 & t_2 \end{pmatrix}$$

which satisfies certain conditions. The weighted orbit space Y^* of the S^1-action may be computed using (4.2) and the descriptions in §2. Throughout, $\varepsilon = \pm 1$, n is an arbitrary integer, and pairs (α, β) consist of relatively prime integers $0 < \beta < \alpha$.

(a) Suppose

$$\varepsilon' = \begin{vmatrix} \alpha' & \beta' \\ \alpha & \beta \end{vmatrix} = \pm 1, \qquad \varepsilon'' = \begin{vmatrix} \alpha & \beta \\ \alpha'' & \beta'' \end{vmatrix} = \pm 1,$$

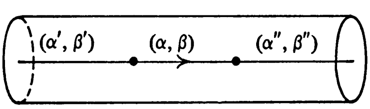
and

$$\omega = \varepsilon'\varepsilon'' \begin{vmatrix} \alpha' & \beta' \\ \alpha'' & \beta'' \end{vmatrix}.$$

Then

$$\begin{pmatrix} \varepsilon\alpha & -\varepsilon\alpha & \varepsilon(b + n\alpha) & -\varepsilon(\beta + n\alpha) \\ \varepsilon\varepsilon'\alpha' & -\varepsilon\varepsilon''\alpha'' & \varepsilon\varepsilon'(\beta' + n\alpha') & -\varepsilon\varepsilon''(\beta'' + n\alpha'') \end{pmatrix}$$

describes actions on Y_ω with $Y_\omega^* \cong D^3$:



(b) Suppose

$$\varepsilon'' = \begin{vmatrix} \alpha & \beta \\ \alpha'' & \beta'' \end{vmatrix} = \pm 1, \qquad b\alpha + \beta = \pm 1,$$

$$\varepsilon' = \begin{vmatrix} 1 & |b| \\ \alpha & \beta \end{vmatrix},$$

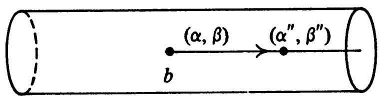
and

$$\omega = \varepsilon'\varepsilon'' \begin{vmatrix} 1 & |b| \\ \alpha'' & \beta'' \end{vmatrix}.$$

Then

$$\begin{pmatrix} \varepsilon\alpha & -\varepsilon\alpha & \varepsilon(\beta+n\alpha) & -\varepsilon(\beta+n\alpha) \\ \varepsilon\varepsilon' & -\varepsilon\varepsilon''\alpha'' & \varepsilon\varepsilon'(|b|+n) & -\varepsilon\varepsilon''(\beta''+n\alpha'') \end{pmatrix}$$

describes actions on Y_ω with $Y_\omega^* \cong D^3$:



(c) If

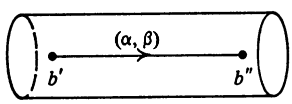
$$b'\alpha+\beta=\pm 1, \qquad b''\alpha+\beta=\pm 1, \qquad \varepsilon'=\begin{vmatrix} 1 & |b'| \\ \alpha & \beta \end{vmatrix},$$

$$\varepsilon''=\begin{vmatrix} \alpha & \beta \\ 1 & |b''| \end{vmatrix}, \quad \text{and} \quad \omega=\varepsilon'\varepsilon''\begin{vmatrix} 1 & |b'| \\ 1 & |b''| \end{vmatrix},$$

then

$$\begin{pmatrix} \varepsilon\alpha & -\varepsilon\alpha & \varepsilon(b+n\alpha) & -\varepsilon(b+n\alpha) \\ \varepsilon\varepsilon' & -\varepsilon\varepsilon'' & \varepsilon\varepsilon'(|b'|+n) & -\varepsilon\varepsilon''(|b''|+n) \end{pmatrix}$$

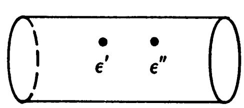
defines actions on Y_ω with $Y_\omega^* = D^3$:



(d) Let $\varepsilon', \varepsilon'' = \pm 1$ and $\omega = -\varepsilon' - \varepsilon''$. Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon n & -\varepsilon n \\ -\varepsilon\varepsilon' & \varepsilon\varepsilon'' & -\varepsilon\varepsilon'(n+\varepsilon') & \varepsilon\varepsilon''(n-\varepsilon'') \end{pmatrix}$$

defines actions on Y_ω; $Y_\omega^* \cong D^3$:



(e) If

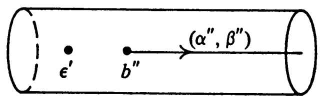
$$b''\alpha''+\beta''=\pm 1, \qquad \varepsilon'=\pm 1,$$

$$\varepsilon''=\begin{vmatrix} 1 & |b''| \\ \alpha'' & \beta'' \end{vmatrix}, \quad \text{and} \quad \omega=\varepsilon''\alpha''-\varepsilon'$$

then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon n & -\varepsilon n \\ -\varepsilon\varepsilon' & -\varepsilon\varepsilon''\alpha'' & -\varepsilon\varepsilon'(n+\varepsilon') & -\varepsilon\varepsilon''\left(\beta'' + (n - |b''|)\alpha''\right) \end{pmatrix}$$

defines actions on Y_ω with $Y_\omega^* \cong D^3$:



(f) Suppose

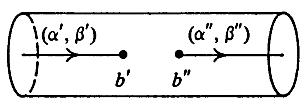
$$b'\alpha' + \beta' = \pm 1, \qquad b''\alpha'' + \beta'' = \pm 1, \qquad \varepsilon' = \begin{vmatrix} \alpha' & \beta' \\ 1 & |b'| \end{vmatrix},$$

$$\varepsilon'' = \begin{vmatrix} 1 & |b''| \\ \alpha'' & \beta'' \end{vmatrix}, \quad \text{and} \quad \omega = \varepsilon'\alpha' + \varepsilon''\alpha''.$$

Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon(|b'|+n) & -\varepsilon(|b'|+n) \\ \varepsilon\varepsilon'\alpha' & -\varepsilon\varepsilon''\alpha'' & \varepsilon\varepsilon'(\beta' + n\alpha') & -\varepsilon\varepsilon''\left(\beta'' + (n + |b'| - |b''|)\alpha''\right) \end{pmatrix}$$

defines actions on Y_ω with $Y_\omega^* \cong D^3$:



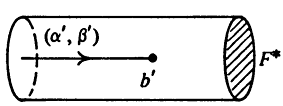
(g) Suppose

$$b'\alpha' + \beta' = \pm 1, \qquad \varepsilon' = \begin{vmatrix} \alpha' & \beta' \\ 1 & |b'| \end{vmatrix}, \quad \text{and} \quad \omega = \varepsilon'\alpha'.$$

Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon(|b'|+n) & -\varepsilon(|b'|+n) \\ \varepsilon\varepsilon'\alpha' & 0 & \varepsilon\varepsilon'(\beta' + n\alpha') & -\varepsilon \end{pmatrix}$$

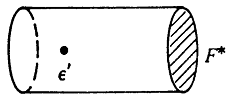
defines actions on Y_ω, and $Y_\omega^* = D^3$:



(h) Let $\varepsilon' = \pm 1$ and $\omega = -\varepsilon'$. Then

$$\begin{pmatrix} \varepsilon & -\varepsilon & \varepsilon & -\varepsilon n \\ -\varepsilon\varepsilon' & 0 & -\varepsilon\varepsilon'(n+\varepsilon') & -\varepsilon \end{pmatrix}$$

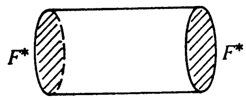
describes actions on Y_ω with $Y_\omega^* \cong D^3$:



(i) The matrix

$$\begin{pmatrix} \varepsilon & -\varepsilon & n & -n \\ 0 & 0 & \delta & \delta \end{pmatrix}$$

for $\delta = \pm 1$ defines actions on Y_0 with $Y_0^* \cong D^3$:



(j) For ω arbitrary and $\delta = \pm 1$ actions on Y_ω are defined by

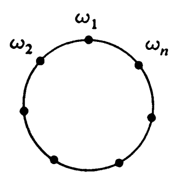
$$\begin{pmatrix} 0 & 0 & \delta & -\delta \\ \varepsilon & \varepsilon & n & -\omega\delta + n \end{pmatrix},$$

and $Y_\omega^* \cong S^2 \times I$ with $E^* \cup F^* = F^* = S^2 \times 0$ with weight ω.

LEMMA (4.4). *Let L^* be the weighted circle $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ in M^*, and for $i = 1, \ldots, n$ let*

$$\omega_i = \begin{vmatrix} \alpha_{i-1} & \beta_{i-1} \\ \alpha_i & \beta_i \end{vmatrix} \begin{vmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{vmatrix} \begin{vmatrix} \alpha_{i-1} & \beta_{i-1} \\ \alpha_{i+1} & \beta_{i+1} \end{vmatrix}$$

(*identifying* $-1 \equiv n$ *and* $n + 1 \equiv 0$). *Then an equivariant plumbing according to the graph*



yields a 4-manifold W with S^1-action and weighted orbit space $D^2 \times S^1$ and $E^* \cup F^* = 0 \times S^1$ isomorphic to L^*. Furthermore, this S^1-action extends to a T^2-action. Each plumbing, except perhaps for the last, $Y_{\omega_n} \square Y_{\omega_1}$ may be chosen to have sign $+1$.

PROOF. This follows immediately from (4.3)(a). Note that in this instance we may always choose $n = 0$ in (4.3)(a); so all the plumbings may be constructed T^2-equivariantly. $\square$

LEMMA (4.5). *For the W constructed in* (4.4), *$E \cup F$ is a strong deformation retract of W.*

PROOF. It is a general fact that the resultant manifold of a plumbing strong deformation retracts onto the union of the zero sections. See [2, V.2.2.]. □

(4.6) Let $S_1^*, \ldots, S_t^*$ be the collection of all the weighted sets of M^* other than the weighted circles. If there are any weighted boundary components of M^* we list these at the end. For each $i = 1, \ldots, t-1$ let γ_i^* be an arc in M^* running from S_i^* to S_{i+1}^* such that the interior of the arc lies in P^* and such that if S_i^* is a weighted arc, γ_i^* begins at the endpoint of S_i^*, and if S_{i+1}^* is a weighted arc, γ_i^* ends at the initial point of S_{i+1}^*.

Let R^* be a regular neighborhood of $\bigcup S_i^* \cup \bigcup \gamma_i^*$. The following lemma is a direct result of the constructions in (4.3).

LEMMA (4.7). *By means of an equivariant linear plumbing* (*with each plumbing of sign* $+1$) *there may be constructed a* 4-*manifold* W *with* S^1-*action and weighted orbit space isomorphic to* R^*. *Furthermore, this action extends to a* T^2-*action on* W. □

5. **Constructions.** Let N^* be a regular neighborhood of $E^* \cup F^*$ in M^* and let $\hat{M}^* = \mathrm{cl}(M^* - N^*)$. For a boundary component T_i^* of $\hat{M}^*$ denote by e_i the Euler number of the principal S^1-bundle $T_i \to T_i^*$. If T_i^* is the boundary of a regular neighborhood of a weighted sphere or point then e_i is just that weight. Corollary (3.8) implies that each T_i^* bounding a regular neighborhood of a weighted circle has $e_i = 0$. Finally, if T_i^* bounds a regular neighborhood of a weighted arc $[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ it follows from (3.9) that the S^1-action on T_i is $\{b' - b''; (o, 0, 0, 0)\}$; so $e_i = b'' - b'$.

LEMMA (5.1). $\Sigma e_i = 0$.

PROOF. Let $\xi \in H^2(\hat{M}^*)$ be the Euler class of the principal S^1-bundle $\hat{M} \to \hat{M}^*$, and let $\mu \in H^3(\hat{M}^*, \partial\hat{M}^*)$ and $\mu_i \in H^2(T_i^*)$ be the generators corresponding to the orientation. We have the exact sequence $H^2(\hat{M}^*) \to^{\rho} H^2(\bigcup T_i^*) \to^{\delta} H^3(\hat{M}^*, \partial\hat{M}^*)$. Now $H^2(\bigcup T_i^*) = \bigoplus H^2(T_i^*)$ and $0 = \delta\rho(\xi) = \delta(e_1\mu_1, \ldots, e_n\mu_n) = (\Sigma e_i)\mu$; so $\Sigma e_i = 0$. □

(5.2) We define a *legally weighted simply connected 3-manifold* to be an oriented simply connected compact 3-manifold X^* along with the data:

(a) an integer a_i assigned to each boundary component of X^*,

(b) a finite collection of points in $\operatorname{int} X^*$ with each assigned an integer $b_i = \pm 1$, and

(c) a collection of weighted arcs and circles in $\operatorname{int} X^*$ as in (3.3) and satisfying the criteria of Lemma (3.5). To each weighted arc $A_i^* = [b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$ the integer $c_i = b'' - b'$ is assigned.

Furthermore, at least one of the above collections must be nonempty, and we require $\Sigma a_i + \Sigma b_i + \Sigma c_i = 0$.

The definition of isomorphism carries over from the class of weighted orbit spaces to the class of legally weighted simply connected 3-manifolds. It follows from (3.1) and (5.1) that the weighted orbit space M^* of an S^1-action on a simply connected M^4 is legally weighted.

Now fix a legally weighted simply connected X^*. We shall construct a simply connected 4-manifold M with S^1-action such that M^* is isomorphic to X^*.

CONSTRUCTION (5.3). In analogy with (4.6) connect all the weighted points, spheres, and arcs of X^* with arcs and take a regular neighborhood R^*. If each boundary component of X^* is capped off with a 3-disk then X^* becomes a homotopy 3-sphere, and the union of R^* with these disks is a 3-disk. Thus $X_1^* = \mathrm{cl}(X^* - R^*)$ is a homotopy 3-disk and $\partial X_1^* = \partial R^* - \partial X^*$ is a 2-sphere. Using Lemma (4.7) we obtain a 4-manifold R with S^1-action and weighted orbit space which we identify with R^*. If e is the Euler number of the principal S^1-bundle $\partial R \to \partial R^* - \partial X^*$ obtained by restricting the action, then the argument of (5.1) shows $e + \Sigma a_i + \Sigma b_i + \Sigma c_i = 0$; so $e = 0$ since X^* is legally weighted. Thus $\partial R \to \partial R^* - \partial X^*$ is a trivial S^1-bundle.

Let $L_1^*, \ldots, L_m^*$ be the weighted circles of X^* and let Q_i^* be a regular neighborhood of L_i^* in X_1^*. Let $X_2^* = \mathrm{cl}(X_1^* - \cup Q_i^*)$ and equip $X_2 = X_2^* \times S^1$ with the S^1-action by translation in the second factor and orbit map π: $X_2 \to X_2^*$. There is an orientation-reversing equivariant homeomorphism σ: $\partial R \to \pi^{-1}(R^* \cap X_2^*)$ with $\sigma^* = \mathrm{id}$. It is easily seen that σ is unique up to vertical equivariant isotopy, i.e. equivariant isotopy over the identity, so the 4-manifold $R \cup_\sigma X$ is well defined up to orientation-preserving equivariant homeomorphism.

We view $L_1^* \cup \cdots \cup L_m^*$ as a link in the simply connected 3-manifold X_1^* and $\pi_1(X_2^*)$ as the group of this link. Let s be the section s: $X_2^* \to X_2^* \times 0 \subset X_2$. If R^* is empty then $\pi_1(R \cup_\sigma X_2) = s_\# \pi_1(X_2^*) \times \pi_1(S^1)$. If R^* is non-empty then it follows from Van Kampen's Theorem that $\pi_1(R \cup_\sigma X_2) = s_\# \pi_1(X_2^*)$, for R is the result of a linear plumbing and thus is simply connected [2, V.2.10] and $R \cap F \neq \varnothing$. In particular, if there are no weighted circles, $\pi_1(R \cup_\sigma X) \approx \pi_1(X_1^*) = 1$.

Using Lemma (4.4) construct for each L_i^* a 4-manifold Q_i with S^1-action and weighted orbit space identified with Q_i^*. Let p_i: $Q_i \to Q_i^*$ be the orbit map. It follows from (3.8) that there is an orientation-reversing equivariant homeomorphism τ_i: $\partial Q_i \to \pi^{-1}(\partial Q_i^*)$ with $\tau_i^* = \mathrm{id}$. The maps τ_i are not unique and a choice must be made here. Let M be the 4-manifold obtained by pasting each Q_i to $R \cup_\sigma X$ via τ_i.

We now describe the possibilities. On each ∂Q_i^* choose a meridional curve m_i^* and a longitudinal curve l_i^* (l_i^* bounds in $X_1^* - L_i^*$) both passing through some point z_i^*. Orient l_i^* and m_i^* so that l_i^* followed by m_i^* gives the orientation of ∂Q_i^* as a boundary component of X_2^*. Let $z_i' = s(z_i^*)$ and $z_i \in \tau_i^{-1}(z_i')$. Then $\pi_1(\partial Q_i, z_i) = \langle q_i \rangle \times \langle l_i \rangle \times \langle h_i \rangle$ where q_i is a section over m_i^*, l_i is a section over l_i^*, and h_i is an oriented orbit containing z_i. Also $\pi_1(\pi^{-1}(\partial Q_i^*), z_i') = \langle m_i' \rangle \times \langle l_i' \rangle \times \langle h_i' \rangle$ where $m_i' = s(m_i^*)$, $l_i' = s(l_i^*)$, and h_i' is the orbit through z_i'. The attaching τ_i is determined up to vertical equivariant isotopy by the relations $\tau_{i_\#}(h_i) = h_i'$, $\tau_{i_\#}(l_i) = l_i' h_i'^{r_i}$, and $\tau_{i_\#}(q_i) = m_i' h_i'^{s_i}$ for integers r_i and s_i. Actually, τ_i is determined by the corresponding homology relations.

Next we show that $\pi_1(M) = 1$. This has already been done in the case where $M = R \cup_\sigma X_2$. Recall that $\pi_1(R \cup_\sigma X_2)$ is generated by $s_\# \pi_1(X_2^*)$ and an oriented orbit, and by Lemma (4.5), $\pi_1(Q_1, z_1) = \langle k_1 \rangle \approx \mathbf{Z}$. The inclusion induced $i_\#: \pi_1(\partial Q_1, z_1) \to \pi_1(Q_1, z_1)$ satisfies $i_\#(h_1) = 1$ and $i_\#(l_1) = k_1$. Now m_1^* bounds a disk in Q_1^* which meets L_1^* in one point, say in the open arc with weight (α, β). So $1 = i_\#(q_1)^\alpha i_\#(h_1)^\beta = i_\#(q_1)^\alpha$. Since it has finite order in $\pi_1(Q_1, z_1)$, $i_\#(q_1) = 1$.

Van Kampen's Theorem yields the relations $h_1' = h_1 = 1$, $1 = q_1 = m_1' h_1'^{s_1} = m_1'$, and $k_1 = l_1 = l_1' h_1'^{r_1} = l_1' \in j_\# \pi_1(X_2^*, z_1^*)$ in $\pi_1(R \cup_\sigma X_2 \cup_{\tau_1} Q_1, z_1)$ where j is s composed with the inclusion into $R \cup_\sigma X_2 \cup_{\tau_1} Q_1$. Let G_1 be the subgroup of $\pi_1(X_2^*, z_1^*)$ generated by the meridians on Q_1^*. Since

$$j_\#(m_1^*) = m_1' = 1,$$

it follows easily that $j_\#(G_1) = 1$. Thus $\pi_1(R \cup_\sigma X_2 \cup_{\tau_1} Q_1, z_1)$ is $\pi_1(X_2^*, z_1^*)$ modulo the normal closure of G_1.

The link group $\pi_1(X_2^*)$ is generated by the meridians lying on the ∂Q_i^* and sewing in Q_j^* kills those meridians lying on ∂Q_j^*; so we finally obtain $\pi_1(M) = 1$. Thus M is a simply connected 4-manifold with S^1-action and weighted orbit space isomorphic to X^*. In (5.6) we show that M is independent of the choices involved in attaching the Q_i.

LEMMA (5.4). *Given any integer n, there is an equivariant homeomorphism Φ: $Q_i \to Q_i$ with $\Phi^* = \mathrm{id}$ such that $\Phi(q_i) \sim q_i$ and $\Phi(l_i) \sim l_i h_i^n$.*

PROOF. Identify Q_i^* with $D^2 \times L_i^*$ and in L_i^* choose a closed interval J^* which does not meet F^*. The covering homotopy theorem of Palais implies that there is an equivariant homeomorphism

$$f: p_i^{-1}(D^2 \times y^*) \times I \to p_i^{-1}(D^2 \times J^*)$$

with $f(p_i^{-1}(D^2 \times y^*) \times 0) = p_i^{-1}(D^2 \times y^*)$ where y^* is the initial point of J^*. The desired homeomorphism is

$$\Phi(x) = \begin{cases} x, & x \notin p_i^{-1}(D^2 \times J^*), \\ (2\pi n t) \cdot x, & x = f(v, t). \end{cases} \qquad \square$$

LEMMA (5.5). *For $i, k = 1, \ldots, m$ let ε_{ki} be the linking number of L_k^* and L_i^**

in X_1^*. If $f\colon \bigcup_1^m \partial Q_i^* \to S^1$ satisfies

$$\deg(f|l_i^*) = \sum_k \varepsilon_{ki}\deg(f|m_k^*)$$

for each i then f may be extended to a map $X_2^* \to S^1$.

PROOF. Let $\zeta \in H_3(X_2^*, \partial X_2^*)$ be the fundamental class and let λ and μ be the inclusions $\bigcup \partial Q_i^* \to^{\lambda} \partial X_2^* \to^{\mu} X_2^*$. Since ∂X_1^* is a 2-sphere we have isomorphisms s and t such that

$$\begin{array}{ccc} H^1(X_2^*, \partial X_1^*) & \xrightarrow{\ s\ } & H^1(X_2^*) \\ \mu^* \downarrow & & \downarrow \mu^* \\ H^1(\partial X_2^*, \partial X_1^*) & \xrightarrow{\ t\ } & H^1(\partial X_2^*) \end{array}$$

commutes.

The elements m_i^* and l_i^* form a basis for $H_1(\partial X_2^*)$. (We shall tacitly denote the isomorphism λ_* by equality.) Let $\{m_i^*\}$ and $\{l_i^*\}$ denote the elements of the dual basis for $H^1(\partial X_2^*)$. Since X_2^* is the complement of a regular neighborhood of the link $\bigcup L_i^*$ in the homotopy 3-disk X_1^*, $H_1(X_2^*)$ is free abelian with basis consisting of the $\mu_* m_i^*$. Let $[\mu_* m_i^*]$ denote the elements of the dual basis for $H^1(X_2^*)$.

By [12, 5.6.20] we have the commutative diagram:

$$\begin{array}{ccccc} H^1(X_2^*, \partial X_1^*) & \xrightarrow{\ \mu^*\ } & H^1(\partial X_2^*, \partial X_1^*) & \xrightarrow{\ t\ } & H^1(\partial X_2^*) \\ \cap\zeta \downarrow & & \cap\partial\zeta \downarrow & & \downarrow \cap\partial\zeta \\ H_2(X_2^*, \bigcup\partial Q_i^*) & \xrightarrow{\ \partial\ } & H_1(\bigcup\partial Q_i^*) & = & H_1(\partial X_2^*) \end{array}$$

Thus $\partial(s^{-1}[\mu_* m_i^*] \cap \zeta) = t\mu^* s^{-1}[\mu_* m_i^*] \cap \partial\zeta = \mu^*[\mu_* m_i^*] \cap \partial\zeta$. Also we have $\{m_i^*\} \cap \partial\zeta = l_i^*$ and $\{l_i^*\} \cap \partial\zeta = -m_i^*$.

Each longitude l_i^* spans a Seifert surface S_i in X_i^*, and by general position we may suppose $S_1 \cap Q_j^*$ consists of a finite number of disjoint 2-disks. It follows that in $H_1(X_2^*)$ we have $\mu_* l_i^* = \sum_j \varepsilon_{ij}\mu_* m_j^*$. The homomorphism μ^* is the transpose of μ_*, so $\mu^*[\mu_* m_i^*] = \{m_i^*\} + \sum_k \varepsilon_{ki}\{l_k^*\}$. Thus $\mu^*[\mu_* m_i^*] \cap \partial\zeta = l_i^* - \sum_k \varepsilon_{ki} m_k^*$.

The obstruction to extending f is $\delta f^*(\iota) \in H^2(X_2^*, \partial Q_i^*)$ where ι is a generator of $H^1(S^1)$ [12, 8.1.12]. Computing the Kronecker index we have

$$\left\langle \delta f^*(\iota), s^{-1}[\mu_* m_i^*] \cap \zeta \right\rangle = \left\langle f^*(\iota), \partial\left(s^{-1}[\mu_* m_i^*] \cap \zeta\right)\right\rangle$$
$$= \left\langle f^*(\iota), l_i^* - \sum_k \varepsilon_{ki} m_k^* \right\rangle = \left\langle \iota, f_*\left(l_i^* - \sum_k \varepsilon_{ki} m_k^*\right)\right\rangle$$
$$= \deg(f|l_i^*) - \sum_k \varepsilon_{ki} \deg(f|m_k^*),$$

and the obstruction $\delta f^*(\iota)$ vanishes if and only if $\langle \delta f^*(\iota), s^{-1}[\mu_* m_i^*] \cap \zeta\rangle = 0$ for each i, since the $s^{-1}[\mu_* m_i^*] \cap \zeta$ form a basis for $H_2(X_2^*, \cup \partial Q_i^*)$. □

THEOREM (5.6). *Construction* (5.3) *is independent of the choice of attaching maps.*

PROOF. Let τ_1 and τ_2 be equivariant embeddings: $\cup_{i=1}^m \partial Q_i \to \partial X_2$ with $\tau_1^* = \tau_2^* = \mathrm{id}$. Say that $\tau_k(l_i) \sim l_i' + r_{ik} h_i'$ and $\tau_k(q_i) \sim m_i' + s_{ik} h_i'$ for $k = 1, 2$. Choose a section K' over $\cup \partial Q_i^*$ oriented so that $K' \cap p_i^{-1}(l_i^*) \sim l_i + a_i h_i$ and $K' \cap p_i^{-1}(m_i^*) \sim q_i + b_i h_i$ for some integers a_i and b_i. Using Lemma (5.4) choose an equivariant homeomorphism Ψ: $\cup Q_i \to \cup Q_i$ such that $\Psi^* = \mathrm{id}$ and for each i,

$$\Psi(q_i) \sim q_i \quad \text{and} \quad \Psi(l_i) \sim l_i - \left[a_i + r_{i1} - \sum_j \varepsilon_{ji}(b_j + s_{j1})\right] h_i.$$

Then $K = \Psi(K')$ is a section over $\cup \partial Q_i^*$ with

$$\tau_1(K) \cap \pi^{-1}(m_i^*) \sim m_i' + (b_i + s_{i1}) h_i'$$

and

$$\tau_1(K) \cap \pi^{-1}(l_i^*) \sim l_i' + \left(\sum_j \varepsilon_{j1}(b_j + s_{j1})\right) h_i'.$$

Let f: $\cup \partial Q_i^* \to S^1$ be the map such that $(x^*, f(x^*)) \in \tau_1(K) \subset X_2 = X_2^* \times S^1$. Then

$$\deg(f|l_i^*) = \sum_j \varepsilon_{ji}(b_j + s_{j1}) = \sum_j \varepsilon_{ji} \deg(f|m_j^*),$$

and by Lemma (5.5) f extends, defining a section K_1 over X_2^* which extends $\tau_1(K)$.

Now in a similar fashion choose an equivariant homeomorphism Φ: $\cup Q_i \to \cup Q_i$ such that $\Phi^* = \mathrm{id}$ and for each i,

$$\Phi(q_i) \sim q_i \quad \text{and} \quad \Phi(l_i) \sim l_i + \left[(r_{i1} - r_{i2}) + \sum_j \varepsilon_{ji}(s_{j2} - s_{j1})\right] h_i.$$

Then $\Phi(K)$ is a section over $\cup \partial Q_i^*$ and $\tau_2 \Phi(K)$ extends to a section K_2 over all of X_2^*.

Let $F\colon X_2 \to X_2$ be the equivariant homeomorphism such that $F^* = \mathrm{id}$ and $F(K_1) = K_2$. Then F and Φ combine to give an equivariant homeomorphism $(\bigcup Q_i) \cup_{\tau_1} X_2 \to (\bigcup Q_i) \cup_{\tau_2} X_2$ which is orientation-preserving since it is the identity on the orbit spaces. We have already seen that up to vertical equivariant isotopy there is just one way to attach R to X_2. This completes the proof. □

6. **Equivariant classification.** In this section we show that the weighted orbit space M^* characterizes M up to orientation-preserving equivariant homeomorphism.

LEMMA (6.1). *Let S^1 act freely on the 3-manifold A with orbit space $A/S^1 \cong S^2$, and let $f\colon A \to A$ be an equivariant homeomorphism satisfying $f^* = \mathrm{id}$. Then there is a vertical equivariant isotopy of f with the identity of A.*

PROOF. Define $\Phi\colon A \to S^1$ by $\Phi(z) \cdot z = f(z)$. Then Φ is continuous and for $\theta \in S^1$, $\Phi(\theta \cdot z) \cdot (\theta \cdot z) = f(\theta \cdot z) = \theta \cdot f(z) = \theta \cdot (\Phi(z) \cdot z) = \Phi(z) \cdot (\theta \cdot z)$. Since the action is free, $\Phi(\theta \cdot z) = \Phi(z)$; so Φ is constant on orbits. Now $\pi_1(A)$ is cyclic, generated by an oriented orbit, so $\Phi_\#(\pi_1(A)) = 1$. Thus Φ lifts to a map $\tilde{\Phi}$ to the universal cover:

$$\begin{tikzcd} & \mathbf{R} \arrow[d, "e"] \\ A \arrow[ur, "\tilde{\Phi}"] \arrow[r, "\Phi"'] & S^1 \end{tikzcd}$$

Then $H_t(z) = e(t\tilde{\Phi}(z)) \cdot z$ is the desired equivariant isotopy. □

THEOREM (6.2). *Let M_1 and M_2 be oriented simply connected closed 4-manifolds with S^1-actions. If the weighted orbit spaces of these actions are isomorphic, M_1 and M_2 are orientation-preserving equivariantly homeomorphic.*

PROOF. Identify M_1^* and M_2^* by means of the given isomorphism, and let $p_i\colon M_i \to M^*$ be the orbit maps. Fix regular neighborhoods Q_j^* of the weighted circles L_j^* of M^*, and choose a regular neighborhood N^* of $(E^* \cup F^*) - \bigcup L_j^*$. We let

$$X^* = \mathrm{cl}\left(M^* - \bigcup Q_j^*\right), \quad Y^* = \mathrm{cl}(M^* - N^*),$$

and

$$Z^* = \mathrm{cl}\left(Y^* - \bigcup Q_j^*\right).$$

Since Y^* is obtained from M^* by removing a collar of ∂M^* and a finite number of disjoint 3-disks, $\pi_1(Y^*) = 1$.

Consider the principal S^1-bundles $p_i|Z_i \to Z^*$. It follows from (3.8) that the restriction of p_i to each ∂Q_j is a trivial S^1-bundle projection which may be extended over Q_j^*. We obtain in this manner principal S^1-bundles p_i': $Y_i' \to Y^*$ which restrict over Z^* to $p_i|Z_i \to Z_i^*$. Let ξ_i be the Euler class of p_i': $Y_i' \to Y^*$. We have the exact sequence $H^2(Y^*, \partial Y^*) \to H^2(Y^*) \to^{\rho} H^2(\partial Y^*)$ and $H^2(Y^*, \partial Y^*) \approx H_1(Y^*) = 0$, so ρ is injective. Now

$$H^2(\partial Y^*) = \bigoplus H^2(B_k^*)$$

where the B_k^* are the components of ∂Y^*, and the jth coordinate of $\rho\xi_i$ is η_{ik}, the Euler class of the restriction of p_i' to B_k^*. The η_{ik} are determined from the weighting of M_i^*, so $\eta_{1k} = \eta_{2k}$ for each k, and $\rho\xi_1 = \rho\xi_2$. Thus $\xi_1 = \xi_2$ and there is an S^1-bundle equivalence $Y_1' \to Y_2'$ over Y^*. This restricts to an equivalence $Z_1 \to Z_2$ over Z^*.

The action on M_i over a collar of ∂M^* is obtained essentially by spreading out the principal orbit bundles and collapsing orbits over ∂M^* to points. (See [1, V.10.1].) The action over a regular neighborhood of a weighted point is just the cone of the action over the boundary of that neighborhood. These facts, along with (3.6), (3.9), and (6.1) imply that the equivalence $Z_1 \to Z_2$ gives rise to an orientation-preserving equivariant homeomorphism f_X: $X_1 \to X_2$.

Also (3.6) provides an orientation-preserving equivariant homeomorphism f_Q: $\cup p_1^{-1}(Q_j^*) \to \cup p_2^{-1}(Q_j^*)$ such that $f_Q^*|\partial Q_j^* = \mathrm{id}$ for each j. The map $\tau = f_X^{-1}f_Q$: $\cup p_1^{-1}(\partial Q_j^*) \to \partial X_1$ is an equivariant homeomorphism with $\tau^* = \mathrm{id}$. Define h by $h|\cup Q_i = f_Q$, $h|X_1 = f_X$. Then h is an orientation-preserving equivariant homeomorphism $\cup Q_i \cup_\tau X_1 \to M_2$. But by (5.6) $\cup Q_i \cup_\tau X_1$ is orientation-preserving equivariantly homeomorphic to M_1. □

COROLLARY (6.3). *Orientation-preserving equivariant homeomorphism classes of S^1-actions on oriented simply connected closed* 4-*manifolds are in bijective correspondence with isomorphism classes of legally weighted simply connected* 3-*manifolds.* □

7. Torus actions.

THEOREM (7.1). *Let S^1 act on the simply connected* 4-*manifold M. The action extends to an action of $T^2 = S^1 \times S^1$ if and only if*

(i) *M^* is not a counterexample to the Poincaré conjecture, and*

(ii) *if M^* contains a weighted circle L^*, then $M^* \cong S^3$, $L^* = E^* \cup F^*$, and L^* is unknotted in M^*.*

PROOF. Suppose that the action extends to a T^2-action. Applying an automorphism of T^2, if necessary, we may suppose that the given S^1-action is the action of the subgroup $0 \times S^1$ of T^2. Then $S^1 \times 0$ acts effectively on $M/(0 \times S^1) = M^*$. This implies (i). If points of M lie in the same T^2-orbit they have the same isotropy groups under the induced $0 \times S^1$-action. It follows that $S^1 \times 0$ pointwise fixes $(E^* \cup F^*) - \partial M^*$.

Let p': $M^* \to M^*/(S^1 \times 0) = M/T^2$ and π: $M \to M/T^2$ be the orbit maps. Then M/T^2 is a 2-disk whose boundary consists of the T^2-orbits of M with nontrivial isotropy [10]; so $p'(E^* \cup F^*) = \pi(E \cup F) \subset \partial(M/T^2)$. This proves (ii).

Conversely, suppose that conditons (i) and (ii) are satisfied. If M^* contains no weighted circle let R^* be the neighborhood of $E^* \cup F^*$ which was described in (4.6). By (4.7) there is a 4-manifold W with T^2-action whose weighted orbit space under the induced $0 \times S^1$-action is isomorphic to R^*. The action of $0 \times S^1$ on ∂W is free, and it follows from (5.3) that $\partial W \cong S^2 \times S^1$. Any T^2-action on $S^2 \times S^1$ has exactly two nonprincipal orbits each of which has a circle group as isotropy group. So both nontrivial isotropy groups on ∂W must be $S^1 \times 0$.

By condition (i), $X^* = \mathrm{cl}(M^* - R^*) \cong D^3$, so there is an obvious componentwise action of T^2 on $X^* \times S^1$. Again the nontrivial isotropy groups on $\partial(X^* \times S^1)$ are each $S^1 \times 0$. If we are careful in the choice of orientation of the S^1-action on X^*, there is an orientation-reversing T^2-equivariant homeomorphism h: $\partial W \to \partial(X^* \times S^1)$. Now $W \cup_h (X^* \times S^1)$ is easily seen to be a simply connected T^2-manifold, and the induced $0 \times S^1$-action has weighted orbit space isomorphic to M^*. So M is S^1-equivariantly homeomorphic to $W \cup_h (X^* \times S^1)$, and the S^1-action on M extends to a T^2-action via this homeomorphism.

If L^* is a weighted circle in M^* choose a regular neighborhood Q^* of L^*. By (4.4) there is a 4-manifold W with T^2-action whose weighted orbit space under the induced $0 \times S^1$-action is isomorphic to Q^*, and the T^2-action restricted to $\partial W \cong T^3$ is free. Conditions (i) and (ii) imply that

$$X^* = \mathrm{cl}(M^* - Q^*) \cong D^2 \times S^1.$$

Letting S^1 act freely on X^* induces a free T^2-action on $X^* \times S^1$. Again with careful choice there is an orientation-reversing T^2-equivariant homeomorphism g: $\partial W \to \partial(X^* \times S^1)$, and we form the T^2-manifold $W \cup_g (X^* \times S^1)$. The orbit space $W \cup_j (X^* \times S^1)/T^2$ is a 2-disk and the action is free over the interior of the disk. It follows from [10] that $W \cup_g (X^* \times S^1)$ is a simply connected 4-manifold which as above is S^1-equivariantly homeomorphic with M. This extends the action on M. $\square$

8. **Quadratic forms and homotopy type.** Let S^1 act on the simply connected 4-manifold M. We shall compute the quadratic form of M using the information contained in the weighted orbit space M^*, and this, in turn, will be used to show that M is homotopy equivalent to a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.

(8.1) Let R^* be the set described in (4.6); so R^* is a neighborhood of $(E^* \cup F^*) - \bigcup L_i^*$ where the L_i^* are the weighted circles of M^*. Using (3.6) and (4.7) we identify R with the resultant 4-manifold of an equivariant linear plumbing

$$\underset{\omega_1}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{\omega_2}{\bullet}\!\!-\!\!-\!\!- \;\cdots\; -\!\!-\!\!-\underset{\omega_t}{\bullet}$$

If ∂M^* has m components and $(F^* - \partial M^*) \cap R^*$ contains l points then $t = 2m + l - 1$.

The matrix B_0 of this plumbing is the $t \times t$ matrix with ijth entry

$$(B_0)_{ij} = \begin{cases} \omega_i, & i = j, \\ 1, & i = j \pm 1, \\ 0, & \text{otherwise}, \end{cases}$$

since each plumbing involved has sign $+1$. Now if $\phi\colon \mathbf{Z}^t \to \mathbf{Z}^t$ is the linear map with matrix B_0 then $H_1(\partial R) \approx \operatorname{coker} \phi$ [5, 8.2]. But it was shown in (5.3) that $\partial R \cong S^2 \times S^1$; so $\det B_0 = 0$.

If B is a square matrix we shall denote by B^- the matrix obtained from B by striking out the last row and column. If $t \geqslant 2$, then B_0^- is the matrix of the equivariant linear plumbing

$$\underset{\omega_1}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{\omega_2}{\bullet}\!\!-\!\!-\!\!- \;\cdots\; -\!\!-\!\!-\underset{\omega_{t-1}}{\bullet}$$

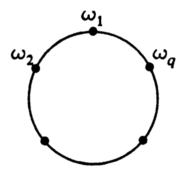
whose resultant manifold we call V. According to [5, 8.6], ∂R is homeomorphic to the lens space $L(p, q)$ where

$$\begin{pmatrix} p & a \\ -q & b \end{pmatrix} = \begin{pmatrix} -\omega_t & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -\omega_1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\omega_t & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p' & a' \\ -q' & b' \end{pmatrix},$$

and $\partial V \cong L(p', q')$. We have $p = 0$ since $\partial R \cong S^2 \times S^1$; so $q' = -\omega_t p'$. Each matrix in the above product has determinant 1, hence $p'(b' - \omega_t a') = p'b' + a'q' = 1$, and $p' = \pm 1$. Thus $\partial V \cong L(\pm 1, q') \cong S^3$. Using [5, 8.2] once more we see that $|\det B_0^-|$ is the order of $H_1(\partial V)$, and so $\det B_0^- = \pm 1$. The zero sections $S_1, \ldots, S_{t-1}$ of $Y_{\omega_1}, \ldots, Y_{\omega_{t-1}}$ have intersection matrix B_0^-; so considered as elements of $H_2(M)$, $S_1, \ldots, S_{t-1}$ are independent. If there are no weighted circles in M^* then $\chi(F) = 2m + l$; so $\operatorname{rk} H_2(M) = 2m + l - 2 = t - 1$. In this case $S_1, \ldots, S_{t-1}$ also generate $H_2(M)$, and B_0^- is the matrix of the quadratic form of M.

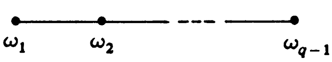
Now $\partial R^* - \partial M^* \cong S^2$, and attaching a 3-disk to R^* along this 2-sphere we obtain a simply connected 3-manifold which we weight by the weights of R^*. There is a simply connected 4-manifold Y with S^1-action and $Y^* = R^* \cup D^3$. Since Y^* satisfies the hypotheses of (7.1) the S^1-action on Y extends to an action of T^2. It now follows that Y is a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$ since any simply connected 4-manifold with T^2-action admits such a connected sum decomposition [10]. The matrix B_0^- of the quadratic form of Y must then be congruent over $\mathbf{Z}$ to a direct sum of copies of matrices (1), (-1), and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ [6].

(8.2) Let $L_1^*, \ldots, L_k^*$ be the weighted circles of M^* with regular neighborhoods $Q_1^*, \ldots, Q_k^*$. Fix j and suppose $L_j^* = \{(\alpha_1, \beta_1), \ldots, (\alpha_q, \beta_q)\}$. We may identify Q_j with the resultant manifold of an equivariant plumbing according to the graph



where each plumbing except perhaps for $Y_{\omega_q} \square Y_{\omega_1}$ has sign $+1$. Index the ω_i so that the exceptional orbits lying in the zero section S_i of Y_{ω_i} are of type $\mathbf{Z}_{\alpha_i}$.

A legally weighted simply connected 3-manifold X^*, is obtained by embedding Q_j^* in S^3 as an unknotted solid torus, and taking $E^* \cup F^* = L_j^*$. There is a simply connected 4-manifold X with S^1-action and weighted orbit space X^*, and since X has q fixed points, rk $H_2(X) = q - 2$. Thus $S_1, \ldots, S_{q-1}$ are linearly dependent when considered as elements of $H_2(X)$; so their intersection matrix B has determinant 0. But B is the matrix of the linear plumbing



The boundary plumbing yields $S^2 \times S^1$ [5, 8.2], and as in (8.1) det $B^- = \pm 1$ if $q - 1 \geqslant 2$. Again (7.1) applies to X, so the action extends to a T^2-action, and B^-, the matrix of the quadratic form of X, is congruent over $\mathbf{Z}$ to a direct sum of copies of matrices (1), (-1), and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that if $E^* \cup F^* = L_j^*$ $(k = 1)$, then B^- is the matrix of the quadratic form of M.

(8.3) Let L_j^* be as given in (8.2) and suppose that we have $L_{j+1}^* = \{(\alpha_1', \beta_1'), \ldots, (\alpha_r', \beta_r')\}$. As before we view Q_{j+1} as the resultant manifold of an equivariant plumbing with zero sections $S_1', \ldots, S_r'$. In M we let $x \in S_{q-1} \cap S_q \subset L_j$ and $y \in S_r' \cap S_1' \subset L_{j+1}$. Choose an arc Γ_j^* in M^* which has endpoints x^* and y^* and such that $\Gamma_j^* - \{x^*, y^*\} \subset P^*$. Then Γ_j is a 2-sphere in M. There is a 4-disk D which is a linear slice to the action at x, and we may suppose $\partial D \cap \Gamma_j$ is a single orbit. The intersection number $\Gamma_j \cdot S_{q-1}$ is the linking number in ∂D of the principal orbit $\Gamma_j \cap \partial D$ with the $\mathbf{Z}_{\alpha_{q-1}}$-orbit $S_{q-1} \cap \partial D$. Thus $\Gamma_j \cdot S_{q-1} = \delta\alpha_q$ for $\delta = \pm 1$. Also $\Gamma_j \cdot S_q = \varepsilon\alpha_{q-1}$,

$\varepsilon = \pm 1$. Similar considerations show that $\gamma_j = \Gamma_j \cdot \Gamma_j = \pm\, \alpha_{q-1}\alpha_q \pm \alpha_r'\alpha_1'$. This determines γ_j modulo 2 which is all that will be needed. We let λ_j be $\Gamma_j \cdot S_1'$.

Since α_{q-1} and α_q are relatively prime we may choose integers m and n with $\varepsilon m\alpha_{q-1} + \delta n\alpha_q = 1$. If we let T_j be the 2-cycle $nS_{q-1} + mS_q$ on Q_j then $\Gamma_j \cdot T_j = 1$. The intersection matrix B_j of $S_1, \ldots, S_{q-2}, T_j$ has determinant 0 since these elements are dependent in $H_2(X)$, and if $q > 3$ the $(s-t)$th entry of B_j is

$$(B_j)_{st} = \begin{cases} \omega_s, & s = t \leqslant q-2, \\ m^2\omega_q + n^2\omega_{q-1} + 2nm, & s = t = q-1, \\ 1, & (s,t) = (s, s \pm 1) \text{ and } s \neq q-1, t \neq q-1, \\ \pm m, & (s,t) = (1, q-1) \text{ or } (q-1, 1), \\ n, & (s,t) = (q-2, q-1) \text{ or } (q-1, q-2), \\ 0, & \text{otherwise.} \end{cases}$$

If $q = 2$ then $B_j = (0)$, and if $q = 3$ then

$$B_j = \begin{pmatrix} \omega_1 & n \pm m \\ n \pm m & m^2\omega_3 + n^2\omega_2 + 2nm \end{pmatrix}.$$

Now B_j^- is the matrix B^- of (8.2); so B_j^- is congruent over $\mathbf{Z}$ to a direct sum of matrices (1), (-1), and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(8.4) Suppose that $R^* \neq \varnothing$ and that $S_1, \ldots, S_t$ are the zero sections as in (8.1). Further suppose that we have $L_1^* = \{(\alpha_1', \beta_1'), \ldots, (\alpha_r', \beta_r')\}$ with zero sections S_i'. Let $x \in (S_t - S_{t-1}) \cap F$ and $y \in S_r' \cap S_1'$. If Γ_0^* is an arc in M^* with endpoints x^* and y^* such that $\Gamma_0^* - \{x^*, y^*\} \subset P^*$ then Γ_0 is a 2-sphere in M. We have $\Gamma_0 \cdot S_t = 1$ and $\gamma_0 = \Gamma_0 \cdot \Gamma_0 = \pm\, \alpha \pm \alpha_r'\alpha_1'$ where S_t has orbits of type $\mathbf{Z}_\alpha$. (We let $\alpha = 0$ if $S_t \subset F$ and $\alpha = 1$ if $S_t \cap P \neq \varnothing$.) Finally, let $\lambda_0 = \Gamma_0 \cdot S_1'$.

(8.5) We now have B_0, an intersection matrix of 2-cycles in R, and $B_1, \ldots, B_k$, intersection matrices of 2-cycles in $Q_1, \ldots, Q_k$. For each $i = 0, \ldots, k$ let d_i denote the dimension of the matrix B_i. Define the matrix $C_i' = B_i \oplus (\gamma_i) \oplus B_{i+1} \oplus \cdots \oplus B_{k-1} \oplus (\gamma_{k-1}) \oplus B_k^-$, for $i = 0, \ldots, k-1$ ($i = 1, \ldots, k-1$ if $R^* = \varnothing$) where (γ_i) is the 1×1 matrix with entry γ_i. Further for $i = 0, \ldots, k-1$ define $C_i'' = O_i \oplus H_i \oplus O_{i+1}^- \oplus \cdots \oplus O_{k-1}^- \oplus H_{k-1} \oplus O_k^-$ where H_j is the 3×3 matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \lambda_j \\ 0 & \lambda_j & 0 \end{bmatrix}$$

and O_j is the $(d_j - 1) \times (d_j - 1)$ zero matrix. Also let $C_k' = B_k^-$ and $C_k'' = O_k$. The square matrices C_i' and C_i'' have the same dimension so we may form $C_i = C_i' + C_i''$.

Define $C = C_0$ if $R^* \neq \varnothing$ and $C = C_1$ if $R^* = \varnothing$. Then C is the intersection matrix of 2-cycles examined in (8.1)–(8.4). It is easy to check that C has $\operatorname{rk} H_2(M)$ rows and columns.

THEOREM (8.6). *The quadratic form of M has matrix C.*

THEOREM (8.7). *M is homotopy equivalent to a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.*

PROOF. We prove both theorems by showing that C is congruent over $\mathbf{Z}$ to a direct sum of matrices (1), (-1), and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. This implies that $\det C = \pm 1$, so $H_2(M)$ is generated by the 2-cycles for which C is the intersection matrix. This proves (8.6) and then (8.7) follows from the homotopy type classification of simply connected 4-manifolds [6].

Our proof proceeds by induction on $n(C)$, where $n(C) = k + 1$ if $R^* \neq \varnothing$ and $n(C) = k$ if $R^* = \varnothing$. If $n(C) = 1$ then (8.1)–(8.3) prove that C is congruent over $\mathbf{Z}$ to a direct sum of matrices (1), (-1), and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

So suppose $n(C) > 1$, and let $\varepsilon = 0$ if $R^* \neq \varnothing$ and $\varepsilon = 1$ if $R^* = \varnothing$. The operation of adding an integral constant times row i to row j and then that constant times column i to column j preserves the congruence class over $\mathbf{Z}$ of an integral matrix. Let us call this an elementary operation. In (8.1) and (8.2) we saw that $\det B_\varepsilon = 0$. So if d_ε, the dimension of B_ε, is 1 then $B_\varepsilon = (0)$. In this case C has the form

$$\begin{pmatrix} 0 & 1 & 0 & \\ 1 & \gamma_\varepsilon & \lambda_\varepsilon & O \\ 0 & \lambda_\varepsilon & & \\ & O & \multicolumn{2}{|c}{*} \end{pmatrix}$$

where the lower right-hand corner matrix is $C_{\varepsilon+1}$. The elementary row operation $-\lambda_\varepsilon$ times the first added to the third yields the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & \gamma_\varepsilon \end{pmatrix} \oplus C_{\varepsilon+1}$$

which is thus congruent to C.

If $d_\varepsilon \geqslant 2$ and $d = d_\varepsilon - 1$ there is a $d \times d$ integral invertible matrix E such that $E^t B_\varepsilon^- E$ is a direct sum of matrices (1), (-1), and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Let F be the matrix $E \oplus I$ where I is the identity matrix so that $E \oplus I$ has the same dimension as C. Then

$$F^tCF = \begin{bmatrix} x_{11} & \cdots & x_{1d} & z_1 & & & \\ \cdot & & \cdot & \cdot & & & \\ \cdot & & \cdot & \cdot & & \bigcirc & \\ \cdot & & \cdot & \cdot & & & \\ x_{d1} & \cdots & x_{dd} & z_d & & & \\ z_1 & \cdots & z_d & c & 1 & & \\ & & & 1 & \gamma_\epsilon & \lambda_\epsilon & \\ & \bigcirc & & & \lambda_\epsilon & \multicolumn{2}{|c}{*} \end{bmatrix}$$

where the lower right-hand corner matrix is $C_{\varepsilon+1}$ and x_{ij} is the ijth entry of $E'B_\varepsilon^- E$.

The rows (and columns) of $E'B_\varepsilon^- E$ are linearly independent, and each consists of $d - 1$ zeros and one 1. Thus a sequence of elementary operations may be performed on $F'CG$ to yield the congruent matrix

$$A_1 = \begin{bmatrix} x_{11} & \cdots & x_{id} & & & & \\ \cdot & & \cdot & & & & \\ \cdot & & \cdot & & & \bigcirc & \\ \cdot & & \cdot & & & & \\ x_{d1} & \cdots & x_{dd} & & & & \\ & & & c' & 1 & & \\ & & & 1 & \gamma_\epsilon & \lambda_\epsilon & \\ & \bigcirc & & & \lambda_\epsilon & \multicolumn{2}{|c}{*} \end{bmatrix}$$

where the lower right-hand corner is still $C_{\varepsilon+1}$.

If $G = E \oplus (1)$ then

$$G'B_\varepsilon G = \begin{bmatrix} x_{11} & \cdots & x_{1d} & z_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ x_{d1} & \cdots & x_{dd} & z_d \\ z_1 & \cdots & z_d & c \end{bmatrix}.$$

Performing the same elementary operations on $G'B_\varepsilon G$ that were performed on $F'CF$ results in the matrix $E'B_\varepsilon^- E \oplus (c')$, which is congruent over $\mathbf{Z}$ to B_ε. Thus $0 = \det B_\varepsilon = c' \det(E'B_\varepsilon^- E) = \pm c'$.

Now the elementary operation $-\lambda_\varepsilon$ times the d_εth added to the $(d_\varepsilon + 2)$nd performed on A_1 yields the congruent matrix

$$A_2 = E'B_\varepsilon^- E \oplus \begin{pmatrix} 0 & 1 \\ 1 & \gamma_\varepsilon \end{pmatrix} \oplus C_{\varepsilon+1}.$$

Note that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & \gamma_\varepsilon \end{smallmatrix}\right)$ is congruent to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if γ_ε is even and to $(1) \oplus (-1)$ if γ_ε is odd. There is clearly a simply connected 4-manifold with S^1-action whose matrix associated by (8.5) is $C_{\varepsilon+1}$. Since $n(C_{\varepsilon+1}) < n(C)$, the induction is completed. $\square$

(8.8) There are easily constructed examples of S^1-actions on simply connected 4-manifolds which are not equivariant connected sums of homotopy CP^2's, $-CP^2$'s, and $S^2 \times S^2$'s. The weighted orbit space S^3 with three weighted circles each linking the other two results in such an example.

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