ANISOTROPIC H^p , REAL INTERPOLATION, AND FRACTIONAL RIESZ POTENTIALS

BY

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ABSTRACT. We observe that the anisotropic variants of H^p interpolate by the real method in the usual manner. Using this fact we show that the corresponding fractional Riesz potentials and related operators perform an embedding in H^p , $p > 0$, analogous to the one for L^p , $p > 1$. We also state a theorem concerning the mapping properties of $f \rightarrow h * f$, where h is in $B_\alpha^{1,\infty}$, which hold only for a restricted range of p .

0. Introduction. Let $\{A_t\}_{t>0}$ be a continuous group of linear transformations on R^n with infinitesimal generator a , $A_t = t^a = \exp(a \log t)$. A tempered distribution, f , on R^n is in $H_a^{p,q}$, $0 < p \leq \infty$, $0 < q \leq \infty$, if and only if $\sup_{t>0} |\phi_t * f(x)|$ is in $L^{p,q}$, where ϕ is in $\mathcal{S}$, $\int \phi \neq 0$, $\phi_t(x) = t^{-\gamma} \phi(A_t^{-1}x)$ and γ is the trace of a .

In the case that a is the identity matrix it is known that

$$(H_a^{p_0,q_0}, H_a^{p_1,q_1})_{\theta,q} = H_a^{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1, 0 < q \leq \infty,$$

where $(\cdot, \cdot)_{\theta,q}$ denotes the interpolation space by the Lions-Peetre method.

In this note we observe that this result holds for all a satisfying $\langle ax, x \rangle > \langle x, x \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^n . Using this fact we show that the corresponding fractional Riesz potential and related operators perform an embedding in H_a^p , $p > 0$, analogous to the one for L^p , $p > 1$. In §3 we state a theorem concerning the mapping properties of the transformation $f \rightarrow k * f$, where k is in $B_\alpha^{1,\infty}$, which hold only for a restricted range of p .

Concerning complex interpolation of the H_a^p spaces, some results have been announced in [1].

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In this paper all functions are complex valued and all integrals are over R^n unless denoted otherwise. As is customary, the symbol C will be used

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generically to denote constants appearing in certain estimates and need not be the same at different occurrences.

1. Preliminaries. Let $\{A_t\}_{t>0}$ be a continuous group of linear transformations on R^n with infinitesimal generator a , $A_t = t^a = \exp(a \log t)$. We assume that a satisfies $\langle ax, x \rangle \geq \langle x, x \rangle$ for all $x \in R^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. If $x \neq 0$, let $\rho(x)$ denote the unique positive number t such that $|A_t^{-1}x| = 1$, $\rho(0) = 0$. It follows that the function $x \rightarrow \rho(x)$ is continuous on R^n , is in $C^\infty(R^n \setminus \{0\})$, and satisfies

- (i) $\rho(x) = 1$ iff $|x| = 1$.
- (ii) $\rho(A_t x) = t\rho(x)$,
- (iii) $\rho(x + y) \leq \rho(x) + \rho(y)$,
- (iv) $|\{x: \rho(x) \leq r\}| = (\omega_n/n)r^\gamma$,

where γ is the trace of a , ω_n is the area of $\{x: |x| = 1\}$, and if A is a measurable set then $|A|$ denotes its volume. ρ is called the a metric and $B(x, d) = \{y: \rho(x - y) \leq d\}$. For more details concerning this metric see [2], [7], and [12].

Given a function ϕ on R^n , $\pi_t^a \phi$, ϕ_t , and $\pi_t^{a^*} \phi$ are defined by $\pi_t^a \phi(x) = \phi(A_t x)$, $\phi_t(x) = t^{-\gamma} \phi(A_t^{-1} x)$, and $\pi_t^{a^*} \phi(x) = \phi(A_t^* x)$ where a^* and A_t^* are the adjoints of a and A_t , respectively. For a tempered distribution f , $\pi_t^a f$ is defined by $\langle \pi_t^a f, \phi \rangle = \langle f, t^{-\gamma} \pi_t^{a^*} \phi \rangle$ for $\phi \in \mathcal{S}$, where $\langle f, \phi \rangle$ denotes the distribution f acting on ϕ . $\pi_t^{a^*} f$ is defined analogously. f is said to be homogeneous of degree α with respect to a , $-\infty < \alpha < \infty$, if $\pi_t^a f = t^\alpha f$.

$D = (D_1, \dots, D_n)$ is the usual differential operator $D_i = \partial/\partial x_i$, $i = 1, \dots, n$. If A is a linear transformation with matrix (a_{ij}) with respect to the usual coordinate system in R^n , then AD is the differential operator with components $(AD)_i = \sum_{j=1}^n a_{ij} D_j$, $i = 1, \dots, n$. $\otimes_m AD$ is the tensor of rank m whose components we index by $\nu_m = (i_1, \dots, i_m)$, $i_j = 1, \dots, n$, and $(\otimes_m AD)_{\nu_m} = (AD)_{i_1} \dots (AD)_{i_m}$. Given a tensor, T , of rank m whose components, T_{ν_m} , are numbers, we define the norm of T , $\|T\|$, by the formula $\|T\| = \max_{\nu_m} |T_{\nu_m}|$.

We list several facts which can be easily verified by direct computation.

(1.1) If A and B are linear transformations on R^n then

$$\left(\otimes_m AD \right)_{\nu_m} f(Bx) = \left(\otimes_m AB^* Df \right)_{\nu_m} (Bx).$$

(1.2) If f is homogeneous of degree α with respect to a then so is $(\otimes_m A_{\rho(x)}^* D)_{\nu_m} f(x)$.

(1.3) $(\pi_t^a f)^\wedge = t^{-\gamma} \pi_t^{a^*} \hat{f}$ where $\hat{\cdot}$ denotes the Fourier transform.

(1.4) If f is homogeneous of degree α with respect to a then $\hat{f}$ is homogeneous of degree $-\alpha - \gamma$ with respect to a^* . Furthermore, if $-\gamma < \alpha < 0$ and if $f \in L_{loc}^1 \cap C^\infty(R^n \setminus \{0\})$, then $\hat{f}$ is in $L_{loc}^1 \cap C^\infty(R^n \setminus \{0\})$.

Given a tempered distribution, f , consider the following:

(1.5) $f^+(x) = \sup_{t>0} |\phi_t * f(x)|$ and $f^+ \in L^p$ for some $\phi \in \mathcal{S}$ with $\int \phi = 1$.

(1.6) $u_N(x) = \sup_{y,t} |\phi_t * f(y)| (1 + \rho(x-y)/t)^{-N}$ and $u_N \in L^p$ for some $\phi \in \mathcal{S}$ with $\int \phi = 1$.

(1.7) $f^*(x) = \sup_{\phi \in A_N} \sup_{|x-y| < 10t} |\phi_t * f(y)|$ and $f^* \in L^p$ where

$$A_N = \left\{ \phi \in \mathcal{S} : \sum_{|v| < N} \int (1 + \rho(x))^N |D^v \phi(x)| dx < 1 \right\}.$$

THEOREM 1. *If $N > \gamma/p + 1$, $0 < p < \infty$, then (1.5), (1.6), and (1.7) are equivalent. Furthermore, the L^p norms of f^+ , u_N , and f^* are equivalent.*

In the case that a is the identity, this result is contained in [5]. The equivalence of (1.5) and (1.6) is contained in [2, Theorem 2.4]. The proof that (1.6) implies (1.7) is completely analogous to the case when a is the identity (see [5, p. 185]) and we will not repeat it here. The fact that (1.7) implies (1.5) is obvious.

Given a tempered distribution, f , we say that f is in $H_a^{p,q}$ if and only if f^+ is in $L^{p,q}$ and $\|f\|_{H_a^{p,q}} = \|f^+\|_{L^{p,q}}$ where $L^{p,q}$ is the usual Lorentz space. (See [11] for a nice exposition of Lorentz spaces.) Note that $H_a^{p,p} = H_a^p$.

As pointed out in [5] and apparent in [4], f^* is useful for obtaining estimates of $\int f(x)\phi(x) dx$ for $\phi \in \mathcal{S}$ and $f \in H_a^p$. More precisely,

$$\left| \int f(x)\phi(x) dx \right| \leq N(\phi; x_0, d) \min_{\rho(x_0-y) < 10d} f^*(y),$$

where

$$N(\phi; x_0, d) = \min\{A > 0: \phi(x) = A\psi_d(x - x_0), \psi \in A_N\}$$

$$\approx \sum_{0 \leq m < N} \sum_{r_m} \int \left(1 + \frac{\rho(x - x_0)}{d}\right)^N |(\otimes A_d^* D)_{r_m} \phi(x)| dx.$$

The following lemmas are basic in the study of these spaces.

LEMMA 1. *Let Ω be an open set properly contained in R^n . There is a collection $\{B(x_i, d_i)\}$ of balls with the following properties:*

(1) *They are pairwise disjoint and $\Omega = \bigcup_{i=1}^{\infty} B(x_i, 2d_i)$.*

(2) *If Ω^c denotes the complement of Ω , then $B(x_i, 10d_i) \cap \Omega^c \neq \emptyset$ for every i .*

(3) *Let $B_i = B(x_i, 3r_i)$. Then $B_i \subset \Omega$ for all i ; and if $x \in \Omega$ and n_x denotes the number of B_i 's which contain x then n_x is bounded by a constant, C_0 , which depends only on a .*

(4) *Let ψ be a C^∞ function with support in $\{x: \rho(x) < 3\}$ and identically one for $\rho(x) < 2$. If $\psi_i(x) = \psi(A_d^{-1}(x - x_i))$, then $\chi_\Omega(x) \leq \sum \psi_i(x) \leq C_0 \chi_\Omega(x)$ where C_0 is the constant of (3). Furthermore, if $\phi_i(x) = \psi_i(x) [\sum_j \psi_j(x)]^{-1}$, then $\sum \phi_i = \chi_\Omega$ and $N(\phi_i, x_i, d_i) \leq C$, where C depends only on a and ψ .*

Lemma 1 is a generalization of the Whitney covering lemma and is a consequence of well-known facts concerning Vitali families in [9, Theorem 2.1]. The construction of the desired covering can be found in the proof of Lemma 1 of [7]. Analogous lemmas have been used by several authors (e.g. see [3]).

LEMMA 2. Let $f \in H^p \cap L^1$ and $s > 0$ be given. Let $\Omega = \{x: f^*(x) > s\}$ and let $\{B_i\}$ be the covering and $\{\phi_i\}$ the partition of unity of Ω as in Lemma 1. Then f may be written as the sum of two functions, g_s and b_s , where

- (1) $\|g_s\|_{L^\infty} \leq Cs$,
- (2) $b_s = \sum b_i$ where
 - (i) $b_i(x) = [f(x) - P_i(x)]\phi_i(x)$, P_i is a polynomial of degree $\leq N$ and $\|P_i\|_{L^\infty(B_i)} \leq Cs$.
 - (ii) $\int P(x)b_i(x) dx = 0$ for all polynomials of degree $\leq N$.
 - (iii) $\|b_i\|_{H_a^p}^p \leq C \int_{B_i} |f^*(x)|^p dx$ and, hence, $\|b_s\|_{H_a^p}^p \leq C \int_\Omega |f^*(x)|^p dx$.

This is essentially the main result of [4] in the case that a is the identity. The proof in our case is the same, *mutatis mutandis*.

As in [4], an immediate consequence of Lemma 2 is an interpolation theorem for $H_a^{p,q}$ spaces by the real method. (See [6] for the basic definitions, properties, and consequences.)

THEOREM 2. For $0 < p_0 < \infty$, $0 < \theta < 1$, and $0 < q \leq \infty$, $(H_a^{p_0}, L^\infty)_{\theta,q} = H_a^{p,q}$ where $1/p = (1 - \theta)/p_0$.

Using reiteration (see [6]) we get the following

COROLLARY. Suppose θ , p_i , and q_i satisfy $0 < \theta < 1$, and $0 < p_i < \infty$, $0 < q_i \leq \infty$, $i = 0, 1$. Then $(H_a^{p_0, q_0}, H_a^{p_1, q_1})_{\theta, q} = H_a^{p, q}$ where $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $0 < q \leq \infty$.

2. Estimates for Riesz potentials of order α . The Riesz potential operator of order α , $-\infty < \alpha < \infty$, is the transformation $f \rightarrow R_\alpha * f$ where $\hat{R}_\alpha(\xi) = \rho_*(\xi)^{-\alpha}$ and ρ_* is the a^* metric. From (1.4) it follows that if $0 < \alpha < \gamma$ then R_α is in $L_{loc}^1 \cap C^\infty(R^n \setminus \{0\})$ and homogeneous of degree $\alpha - \gamma$ with respect to a . Using (1.2), it follows that $(\otimes_m A_{\rho(x)}^* D)R_\alpha(x)$ enjoys the same properties and, hence, $\|(\otimes_m A_{\rho(x)}^* D)R_\alpha(x)\| \leq C\rho(x)^{\alpha-\gamma}$ for $m = 0, 1, 2, \dots$, where C is a constant which depends only on m . The above inequality allows us to obtain H^p estimates for $R_\alpha * f$.

LEMMA 3. Suppose $K \in L_{loc}^1 \cap C^\infty(R^n \setminus \{0\})$ and for some α , $0 < \alpha < \gamma$,

$$(2.1) \quad \left\| \otimes_m A_{\rho(x)}^* DK(x) \right\| \leq C_m (\rho(x))^{\alpha-\gamma}$$

for all m , where C_m depends only on m . If $\psi \in C^\infty$, $\text{supp } \psi \subset \{x: |x| \leq 1\}$, $\int \psi = 1$, and $f \in H_a^p \cap C^\infty$, $0 < p < 1$, then

$$(2.2) \quad \left| \left\{ x: \sup_{\varepsilon > 0} |\psi_\varepsilon * K * f(x)| > t \right\} \right| \leq (Ct^{-1} \|f\|_{H^p_t})^q$$

for $t > 0$, where $1/q = 1/p - \alpha/\gamma$ and C is independent of f .

PROOF. First observe that if $2\rho(y) \leq \rho(x)$ then

$$(2.3) \quad \left\| \left(\bigotimes_m A_{\rho(x)}^* D \right) K(x-y) \right\| \leq C_m \rho(x)^{\alpha-\gamma}$$

for fixed x or for fixed y . Define $K_\varepsilon = \psi_\varepsilon * K$, then K_ε enjoys property (2.1), with C_m independent of ε .

To see this, if $2\varepsilon < \rho(x)$ write

$$\begin{aligned} \left\| \bigotimes_m A_{\rho(x)}^* D K_\varepsilon(x) \right\| &\leq \int_{\rho(y) < \varepsilon} |\psi_\varepsilon(y)| \left\| \left(\bigotimes_m A_{\rho(x)}^* D \right) K(x-y) \right\| dy \\ &\leq C_m \|\psi\|_{L^1} \rho(x)^{\alpha-\gamma}. \end{aligned}$$

If $2\varepsilon > \rho(x)$, write

$$\left\| \bigotimes_m A_{\rho(x)}^* D K_\varepsilon(x) \right\| \leq \int_{\rho(y) < \varepsilon} |K(y)| \left\| \left(\bigotimes_m A_{\rho(x)}^* D \right) \psi_\varepsilon(x-y) \right\| dy.$$

Now,

$$\begin{aligned} \left\| \left(\bigotimes_m A_{\rho(x)}^* D \right) \psi_\varepsilon(x-y) \right\| &= \left\| \left(\bigotimes_m A_{\rho(x)/\varepsilon}^* D \psi \right)_\varepsilon(x-y) \right\| \\ &\leq C \varepsilon^{-\gamma} \left(\frac{\rho(x)}{\varepsilon} \right)^m \max_{1 \leq j \leq m} \sup_x \left\| \bigotimes_j D \psi(x) \right\| \end{aligned}$$

where C depends only on m and n . Hence

$$\left\| \bigotimes_m A_{\rho(x)}^* K_\varepsilon(x) \right\| \leq C' \varepsilon^{-\gamma} \int_{\rho(y) < \varepsilon} |K(y)| dy$$

where C' depends only on m , n , and ϕ . Now $\int_{\rho(y) < \varepsilon} |K(y)| dy \leq C\varepsilon^\alpha$ and this, together with the previous inequality, implies the desired result.

Suppose that $\|f\|_{H^p_t} = 1$ and let $\Omega = \{x: f^*(x) > s\}$ where $s = t^{q/p}$. Note that $|\Omega| \leq Cs^{-p} = Ct^{-q}$. Write $f = g_s + b_s$ as in Lemma 2. Choose p_1 and q_1 so that $1/p_1 - 1/q_1 = \alpha/\gamma$ and $1 < p_1 < \infty$ and $1 < q_1 < \infty$, and recall that the transformation $f(x) \rightarrow \sup_{\varepsilon > 0} |K_\varepsilon * f(x)|$ maps L^{p_1} into L^{q_1} . Write

$$\left| \left\{ x: \sup_{\varepsilon > 0} |K_\varepsilon * g_s(x)| > t \right\} \right| \leq C(t^{-1} \|g_s\|_{L^{p_1}})^{q_1}.$$

Using the fact that $\|g_s\|_{L^{p_1}}^{p_1} \leq s^{p_1-p} \|g_s\|_{L^p}^p \leq s^{p_1-p}$ and the relations between p , q and p_1 , q_1 and s , t , it follows that

$$(2.4) \quad \left| \left\{ x: \sup_{\varepsilon > 0} |K * g_s(x)| > t \right\} \right| \leq Ct^{-q}.$$

Now write

$$I = \left| \left\{ x \in \Omega^c : \sup_{\varepsilon > 0} |K_\varepsilon * b_s(x)| > t \right\} \right| \leq t^{-p} \int_{\Omega^c} \left(\sup_{\varepsilon > 0} |K_\varepsilon * b_s(x)| \right)^p dx,$$

where Ω^c denotes the complement of Ω . Recall that $b_s = \sum b_i$ as in Lemma 2 and the fact that $p < 1$, to write

$$I \leq t^{-p} \sum \int_{\Omega^c} \left(\sup_{\varepsilon > 0} |K_\varepsilon * b_i(x)| \right)^p dx.$$

To obtain an estimate on $K_\varepsilon * b_i(x)$ assume, without loss of generality, that $x_i = 0$, take x and ε fixed, and write $K_\varepsilon(x - y) = Q(y) + F(x, y)$ where $Q(y)$ is a polynomial of degree less than N and

$$F(x, y) = \int_0^1 \frac{(1 - \tau)^{N-1}}{(N-1)!} \frac{d^N}{d\tau^N} K_\varepsilon(x - \tau y) d\tau.$$

Now, recall that $\int Q(y) b_i(y) dy = 0$ and, hence,

$$\begin{aligned} |K_\varepsilon * b_i(x)| &= \left| \int F(x, y) b_i(y) dy \right| \\ &\leq \left| \int F(x, y) \phi_i(y) f(y) dy \right| + \left| \int F(x, y) \phi_i(y) P_i(y) dy \right| \\ &= I_1 + I_2. \end{aligned}$$

Since $|P_i(y)| \leq Cs$ for $y \in \text{supp } \phi_i \subset \{y : \rho(y) \leq 3d_i\}$,

$$I_2 \leq Cs \int_{\rho(y) \leq 3d_i} |F(x, y)| dy = Cs d_i^\gamma \int_{\rho(y) \leq 3} |F(x, A_d y)| dy.$$

Observe that

$$\frac{d^N}{d\tau^N} K_\varepsilon(x + \tau y) = \sum_{\nu_N} \left\{ \left(\bigotimes_N A_{\rho(x)}^* D K_\varepsilon \right)_{\nu_N} (x - \tau y) \right\} \left(\bigotimes_N A_{\rho(x)}^{-1} y \right)_{\nu_N}$$

and, hence, using 2.3, the fact that

$$\sup_{\rho(y) \leq 3} \left\| \left(\bigotimes_N A_{\rho(x)}^* D K_\varepsilon \right) (x - \tau A_d y) \right\| \leq C \rho(x)^{\alpha - \gamma}$$

and

$$\sup_{\rho(y) \leq 3} \left\| \bigotimes_N A_{d_i/\rho(x)} y \right\| \leq C \left[\frac{d_i}{\rho(x)} \right]^N$$

implies that

$$I_2 \leq Cs [d_i/\rho(x)]^{N+\gamma} \rho(x)^\alpha.$$

Recall that

$$I_1 \leq N(F(x, \cdot) \phi_i, d_i, 0) \min_{\rho(y) \leq 10d_i} f^*(y).$$

To estimate $N(F(x, \cdot) \phi_i, d_i, 0)$, using the change of variables $y \rightarrow A_d y$ in its "formula", it follows that it is bounded by

$$Cd_i^\gamma \left(\max_{0 \leq m \leq N} \sup_{\rho(y) < 3} \left\| \left(\bigotimes_m D \right) \left(\bigotimes_N A_{\rho(x)}^* DK_\varepsilon \right) (x - \tau A_d y) \right\| \right) \\ \times \left(\max_{0 \leq m \leq N} \sup_{\rho(y) < 3} \left\| \bigotimes_m D \bigotimes_N A_{d/\rho(x)} y \right\| \right),$$

where $0 < \tau < 1$. Since $\bigotimes_m D = \bigotimes_m A_{\rho(x)}^* A_{\rho(x)}^{-1} D$, we have

$$\left\| \left(\bigotimes_m D \right) \left(\bigotimes_N A_{\rho(x)}^* DK_\varepsilon \right) (x - \tau A_d y) \right\| \\ = \tau^m \left\| \left(\bigotimes_m A_{d/\rho(x)}^* A_{\rho(x)} D \bigotimes_N A_{\rho(x)} DK_\varepsilon \right) (x - \tau A_d y) \right\| \\ \leq C' \tau^m \|A_{d/\rho(x)}\|^m \left\| \left(\bigotimes_{N+m} A_{\rho(x)}^* DK_\varepsilon \right) (x + \tau A_d y) \right\| \\ \leq C \rho(x)^{-\gamma+\alpha}$$

for $\rho(y) < 3$, where C depends only on m and n . Clearly

$$\sup_{\rho(y) < 3} \left\| \bigotimes_m D \bigotimes_N A_{d/\rho(x)} y \right\| \leq C \left(\frac{d_i}{\rho(x)} \right)^N,$$

and since $\min_{\rho(y) < 10d_i} f^*(y) \leq s$, it follows that $I_1 \leq Cs(d_i/\rho(x))^{N+\gamma}\rho(x)^\alpha$.

Since the estimates for I_1 and I_2 are independent of ε , we have

$$\sup_{\varepsilon > 0} |K_\varepsilon * b_i(x)| \leq Cs d_i^{N+\gamma} (\rho(x - x_i))^{\alpha-N-\gamma}.$$

Hence

$$\int_{\Omega^c} \left(\sup_{\varepsilon > 0} |K_\varepsilon * b_i(x)| \right)^p dx \leq \int_{\rho(x-x_i) > 8d_i} \dots dx \leq Cs^p d_i^{\alpha p + \gamma}.$$

Since $|B_i| = Cd_i^\gamma$ and $s^p = t^q$, substituting the last inequality into the estimate for I gives us

$$I \leq Ct^{q-p} \sum |B_i|^{1+\alpha p/\gamma} \leq Ct^{q-p} \left(\sum |B_i| \right)^{1+\alpha p/\gamma}.$$

Since $\sum |B_i| \leq C' |\Omega| \leq Ct^{-q}$, using the relation between p and q it follows that

$$(2.5) \quad \left| \left\{ x \in \Omega^c: \sup_{\varepsilon > 0} |K_\varepsilon * b_i| > t \right\} \right| \leq Ct^{-q}.$$

Inequalities (2.4) and (2.5) imply that

$$\left| \left\{ x: \sup_{\varepsilon > 0} |K_\varepsilon * f(x)| > 2t \right\} \right| \leq Ct^{-q}$$

where C is a constant independent of f , $\|f\|_{H_t^p} = 1$. Since Lemma 3 follows from the last equality, this completes its proof.

As a consequence of Lemma 3 and Theorem 2 we have

THEOREM 3. *Let f be a distribution in H_a^p , $0 < p < \gamma/\alpha$, and let K and ψ be as in Lemma 3. Then $K * F = \lim_{\epsilon \rightarrow 0} K * \psi_\epsilon * f$ exists in the sense of distributions, is independent of the choice of ψ , and is in H^q , $1/p - 1/q = \alpha/\gamma$. Furthermore, the mapping $f \rightarrow K * f$ is bounded from H^p into H^q , $1/p - 1/q = \alpha/\gamma$.*

In the case a is the identity and $K(x) = |x|^{\alpha-n}$, a variant of Theorem 3 seems to be known. See [8, p. 50].

3. Remarks. To avoid introducing more notation, we consider the case where a is the identity matrix. In this case $H_a^p = H^p$.

Recall the definitions of space BMO and the homogeneous Besov spaces $B_{\alpha}^{p,q}$, $1 < p < \infty$, $1 < q < \infty$, and $-\infty < \alpha < \infty$. It is well known that the dual of H^p is $B_{n(1/p-1)}^{\infty,\infty}$ if $0 < p < 1$ and BMO if $p = 1$. (A nice account of these facts together with the definitions and properties of the Besov spaces may be found in [10].)

Define $\mathcal{L}^p$ for $-\infty < 1/p < \infty$ as follows:

$$\begin{aligned}\mathcal{L}^p &= B_{-n/p}^{\infty,\infty} && \text{if } 1/p < 0, \\ &= \text{BMO} && \text{if } 1/p = 0 \\ &= H^p && \text{if } 1/p > 0.\end{aligned}$$

Let $\phi \in \mathcal{S}$ be such that $\hat{\phi} = 0$ in a neighborhood of the origin and a neighborhood of infinity and satisfies $\int_0^\infty \hat{\phi}(t\xi) dt/t = 1$ for $\xi \neq 0$. Define $\phi_{\epsilon,r}$ by

$$\hat{\phi}_{\epsilon,r}(\xi) = \int_\epsilon^r \hat{\phi}(t\xi) \frac{dt}{t},$$

and note that if $f \in \mathcal{L}^p$ then $\lim_{\epsilon \rightarrow 0; r \rightarrow \infty} \phi_{\epsilon,r} * f = f$ in $\mathcal{L}^p$.

THEOREM 4. *Suppose $K \in B_{\alpha}^{1,\infty}$, $\alpha > 0$, and f is a distribution in $\mathcal{L}^p$, $1/p < 1 + \alpha/n$. Then $K * f = \lim_{\epsilon \rightarrow 0; r \rightarrow \infty} K * \phi_{\epsilon,r} * f$ exists in the sense of distributions, is independent of the choice of ϕ , and is in $\mathcal{L}^q$, $1/p - 1/q = \alpha/n$. Furthermore, the mapping $f \rightarrow K * f$ is bounded from $\mathcal{L}^p$ into $\mathcal{L}^q$, $1/p - 1/q = \alpha/n$.*

(Note that if K satisfies the hypothesis of Lemma 3 then K is in $B_{\alpha}^{1,\infty}$. Also observe that it follows from the arguments below that analogous results hold for $K \in B_{\alpha}^{p,\infty}$, $p > 1$.)

Theorem 4 is well known in the case that $1/p < 1$ (see [10]) and follows by duality for $1 < 1/p < 1 + \alpha/n$. In fact, if $(\mathcal{L}^p, \mathcal{L}^q)$ denotes the class of translation invariant continuous linear operators mapping $\mathcal{L}^p$ into $\mathcal{L}^q$ then, with a slight abuse of notation,

$$B_{\alpha}^{1,\infty} \subset (H^p, H^1) \subset (\text{BMO}, B_{\alpha}^{\infty,\infty}) \subset (L^{\infty}, B_{\alpha}^{\infty,\infty}) = B_{\alpha}^{1,\infty},$$

where $1/p = 1 + \alpha/n$.

The reason the duality argument fails in the case that $1/p > 1 + \alpha/n$ is, of course, the fact that $\mathcal{L}^q$, $1/q > 1$, is not a Banach space. To see that Theorem 4 cannot be extended to the case $1/p > 1 + \alpha/n$, consider the case $n = 1$, $1/p > 1 + \alpha$, and $1/q = 1/p - \alpha$. Let f and g be C^∞ functions with support in the interval $[-\frac{1}{10}, \frac{1}{10}]$ and define K by the formula $K(x) = \sum_{m=1}^{\infty} m^{-1/q} g(x+m)$. Clearly $K \in B_\alpha^{1,\infty}$ and, if sufficiently many moments of f are zero, f is in H^p . Moreover, if neither f nor g are identically zero,

$$\|K * f\|_{H^q}^q \geq \|K * f\|_{L^q}^q = \|g * f\|_{L^q}^q \sum_{m=1}^{\infty} \frac{1}{m} = \infty.$$

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