THE HUYGENS PROPERTY FOR THE HEAT EQUATION

BY

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Dedicated to R. P. Boas for his 65th birthday

ABSTRACT. This note summarizes several criteria which guarantee that a solution of the heat equation should also have the semigroup property described in equation (1.2) below. In particular, it corrects a mistake in an earlier proof of one of these.

1. Introduction. A function $u(x, t)$ belongs to class H in a strip $0 < t < c$ if it has continuous second order derivatives there and satisfies the heat equation

$$(1.1) \quad \partial^2 u / \partial x^2 = \partial u / \partial t.$$

It has the Huygens property or belongs to class H^0 there if, in addition, for every t and δ such that $0 < \delta < t + \delta < c$,

$$(1.2) \quad u(x, t + \delta) = k(x, t) * u(x, \delta) = \int_{-\infty}^{\infty} k(x - y, t) u(y, \delta) dy,$$

the integral converging absolutely. Here $k(x, t)$ is the source solution of equation (1.1),

$$k(x, t) = (4\pi t)^{-1/2} \exp[-x^2 / (4t)], \quad 0 < t.$$

It is a familiar fact that class H is larger than H^0 [1, p. 163]. Since relation (1.2) is so important in the applications of the heat equation it is evidently desirable to have convenient characterizations of class H^0 .

In this note we give several criteria for admission to class H^0 . In [1, p. 162] it was shown that any function defined by a convergent Poisson transform,

$$(1.3) \quad u(x, t) = k(x, t) * d\alpha(x) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y),$$

belongs to H^0 . A necessary and sufficient condition for the representation (1.3) with $\int_{-\infty}^{\infty} k(x, c) |d\alpha(x)| < \infty$ was stated in [2, p. 235]. However, in the proof there given there is a hiatus,¹ not easily filled. A primary purpose of the

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¹ It was shown that the function $w_h(x, t)$ belongs to H before applying Lemma 6.2 to it.

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present note is to provide a valid (and simplified) proof of that theorem, using a different method.

2. The criteria. In this section we state four of the theorems to be proved.

THEOREM 2.1. *If $u(x, t) \in H$, $0 < t < c$, and if*

$$(2.1) \quad |u(x, t)| \leq M \exp Ax^2, \quad 0 < t < c \leq 1/(4A),$$

then

$$(2.2) \quad u(x, t) = k(x, t) * \varphi(x), \quad 0 < t < c,$$

where

$$(2.3) \quad |\varphi(x)| \leq M \exp Ax^2, \quad -\infty < x < \infty.$$

As noted in the introduction, it follows from (2.2) that $u(x, t) \in H^0$, $0 < t < c$.

THEOREM 2.2. *If $u(x, t) \in H$, $0 < t < c$, then a necessary and sufficient condition for the representation (2.2), (2.3) is that*

$$(2.4) \quad |u(x, t)| \leq M k(x, t) * \exp Ax^2, \quad 0 < t < c \leq 1/(4A).$$

The convolution (2.4) is equal to [1, p. 171]

$$k(x, t) * \exp Ax^2 = (1 - 4At)^{-1/2} \exp[Ax^2/(1 - 4At)], \\ 0 < t < 1/(4A).$$

THEOREM 2.3. *If $u(x, t) \in H$, $0 < t < c$, and if*

$$(2.5) \quad \int_{-\infty}^{\infty} |u(x, t)| \exp[-Ax^2] dx \leq M, \quad 0 < t < c \leq 1/(4A),$$

then

$$(2.6) \quad u(x, t) = k(x, t) * d\alpha(x), \quad 0 < t < c,$$

where

$$(2.7) \quad \int_{-\infty}^{\infty} k(x, c) |d\alpha(x)| < \infty.$$

THEOREM 2.4. *If $u(x, t) \in H$, $0 < t < c$, then a necessary and sufficient condition for the representation (2.6), (2.7) is that*

$$(2.8) \quad \int_{-\infty}^{\infty} |u(x, t)| k(x, c - t) dx \leq M, \quad 0 < t < c.$$

We show at once that Theorem 2.1 is a corollary of Theorem 2.2. We have listed it first because it is usually the simplest condition to apply. We have only to note that

$$\exp Ax^2 < (1 - 4At)^{-1/2} \exp[Ax^2/(1 - 4At)], \quad 0 < t < 1/(4A),$$

since $1 < 1/(1 - 4At)$ and $1 > (1 - 4At)^{1/2}$.

The example $u(x, t) = k(x, t) * \exp Ax^2$ shows that condition (2.1) is not necessary for the representation (2.2), (2.3). For, it has that representation; yet (2.1) is not satisfied for any constant M since

$$\lim_{|x| \rightarrow \infty} \exp[Ax^2/(1 - 4At) - Ax^2] = \infty, \quad 0 < t < 1/(4A).$$

Theorem 2.2 is proved in [3, p. 206].

3. The main result. We proceed now to the proof of Theorem 2.4, which was Theorem 6.1 of [2]. The necessity of condition (2.8) is immediate. For, if

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y),$$

this integral converges at $x = 0, t = c$ by (2.7) and, hence, defines a function of class H for $0 < t < c$ [1, p. 64]. Moreover,

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x, t)| k(x, c - t) dx &\leq \int_{-\infty}^{\infty} k(x, c - t) dx \int_{-\infty}^{\infty} k(x - y, t) |d\alpha(y)| \\ &= \int_{-\infty}^{\infty} |d\alpha(y)| \int_{-\infty}^{\infty} k(x, c - t) k(x - y, t) dx \\ &= \int_{-\infty}^{\infty} k(y, c) |d\alpha(y)| < \infty. \end{aligned}$$

We have used Fubini's theorem, applicable by virtue of (2.7). Thus (2.8) is satisfied.

For the converse we record a preliminary result.

LEMMA 3. *If $a > 1, h > 0$, then*

$$\exp[(|x| + h)^2] \leq \exp[ah^2/(a - 1)] \exp ax^2, \quad -\infty < x < \infty.$$

This is obvious if we note that the maximum value of the polynomial $(1 - a)x^2 + 2xh + h^2$ for $0 \leq x < \infty$ is attained at $x = h/(a - 1)$.

Now define

$$w_h(x, t) = \frac{1}{2h} \int_{x-h}^{x+h} u(y, t) dy, \quad 0 < t < c.$$

Direct differentiation shows that $w_h(x, t) \in H, 0 < t < c$. We show that $w_h(x, t) \in H^0$ also.

By (2.8) we have

$$\begin{aligned}
 |w_h(x, t)| &\leq \frac{[\pi(c-t)]^{1/2}}{h} \int_{x-h}^{x+h} |u(y, t)| k(y, c-t) \exp\left[\frac{y^2}{4(c-t)}\right] dy \\
 (3.1) \quad &\leq \frac{[\pi c]^{1/2}}{h} \exp\left[\frac{(|x|+h)^2}{4(c-t)}\right] \int_{-\infty}^{\infty} |u(y, t)| k(y, c-t) dy, \\
 &\frac{1}{2h} \int_{x-h}^{x+h} |u(y, t)| dy \leq \frac{M[\pi c]^{1/2}}{h} \exp\left[\frac{(|x|+h)^2}{4(c-t)}\right].
 \end{aligned}$$

Applying Lemma 3 with some constant $a > 1$, we have

$$(3.2) \quad |w_h(x, t)| \leq \frac{M[\pi c]^{1/2}}{h} \exp\left[\frac{ah^2}{4(a-1)(c-t)}\right] \exp\left[\frac{ax^2}{4(c-t)}\right].$$

If we restrict t to the interval $0 < t < c/a$, then we may replace $c-t$ in the first exponent (3.2) by $\delta = c - (c/a)$ to obtain

$$(3.3) \quad |w_h(x, t)| \leq N \exp\left[\frac{ax^2}{4(c-t)}\right], \quad N = \frac{M[\pi c]^{1/2}}{h} \exp\left[\frac{ah^2}{4(a-1)\delta}\right].$$

The right-hand side of inequality (3.3) is less than

$$Nk(x, t) * \exp\left[\frac{ax^2}{(4c)}\right] = N[1 - (at)/c]^{-1/2} \exp\left[\frac{ax^2}{4(c-at)}\right],$$

so that we may apply Theorem 2.2 with $A = a/(4c)$. We conclude that $w_h(x, t) \in H^0$ in $0 < t < c/a$ for every $a > 1$ and, hence, in $0 < t < c$.

From the definition of H^0 we have for $0 < \delta < t + \delta < c$,

$$\begin{aligned}
 (3.4) \quad w_h(x, t + \delta) &= \frac{1}{2h} \int_{-\infty}^{\infty} k(x-y, t) dy \int_{y-h}^{y+h} u(z, \delta) dz \\
 &= \frac{1}{2h} \int_{-\infty}^{\infty} u(z, \delta) dz \int_{z-h}^{z+h} k(x-y, t) dy.
 \end{aligned}$$

To justify this change in the order of integration we have only to apply (3.1) and to note that

$$\int_{-\infty}^{\infty} k(x-y, t) \exp\left[\frac{(|y|+h)^2}{4(c-\delta)}\right] dy < \infty,$$

since $t < c - \delta$. Now allow h to approach zero in (3.4):

$$u(x, t + \delta) = \int_{-\infty}^{\infty} u(z, \delta) k(x-z, t) dz,$$

provided Lebesgue's theorem on dominant convergence is applicable. Set

$$\begin{aligned}
 R_h(x, z, t) &= \frac{1}{2h} \int_{z-h}^{z+h} k(x-y, t) dy \\
 (3.5) \qquad &= \frac{[\pi t]^{-1/2}}{4h} \int_{z-x-h}^{z-x+h} \exp\left[\frac{-y^2}{(4t)}\right] dy.
 \end{aligned}$$

We must show that

$$\begin{aligned}
 (3.6) \qquad |R_h(x, y, t)| &\leq F(x, y, t), \quad 0 < h < 1, \\
 \int_{-\infty}^{\infty} |u(z, \delta)| F(x, z, t) dz &< \infty.
 \end{aligned}$$

For $|z-x| \leq 1$ it is sufficient to note that the integrand in (3.5) is bounded, so that we may take $F = [4\pi t]^{-1/2}$ there. For $|z-x| > 1$ and $|y-z+x| \leq h < 1$ we have $|y| > |z-x| - 1$, so that we may take

$$F(x, z, t) = [4\pi t]^{-1/2} \exp\left[\frac{-(|z-x|-1)^2}{(4t)}\right], \quad |z-x| > 1.$$

Thus, $F(x, z, t)$ is independent of h , and (3.6) follows from

$$\begin{aligned}
 (3.7) \quad \int_{|z-x|>1} |u(z, \delta)| k(z, c-\delta) \exp\left[\frac{z^2}{4(c-\delta)} - \frac{(|z-x|-1)^2}{4t}\right] dz \\
 < \infty.
 \end{aligned}$$

The exponential factor in (3.7) is a bounded function of z since $t < c - \delta$, so that hypothesis (2.8) establishes (3.7).

We have thus proved that $u(x, t) \in H^0$ in $0 < t < c$. The final step, to establish the Poisson representation (2.6), (2.7), is accomplished by a standard application of the Helly selection theorem, as in [2, p. 238].

4. Proof of Theorem 2.3. This theorem seems not to be an immediate corollary of Theorem 2.4. However, slight modifications in the previous proof will give the desired result. Define $w_h(x, t)$ as before. Inequalities (3.1) and (3.3) become

$$\begin{aligned}
 |w_h(x, t)| &\leq \frac{1}{2h} \int_{x-h}^{x+h} |u(y, t)| \exp[-Ay^2] \exp Ay^2 dy \leq \frac{M}{2h} \exp[A(|x|+h)^2] \\
 &\leq (M/2h) \exp[Aah^2/(a-1)] \exp Aax^2, \quad 0 < t < c.
 \end{aligned}$$

We now apply Theorem 2.1 and conclude that $w_h(x, t) \in H^0$ on $0 < t < c/a$ and, hence, on $0 < t < c$. As before, equation (3.4) holds after noting that

$$\int_{-\infty}^{\infty} k(x-y, t) \exp[A(|y|+h)^2] dy < \infty, \quad A < 1/(4t).$$

Inequality (3.7) becomes

$$\int_{|z-x|>1} |u(z, \delta)| \exp[-Az^2] \exp\left[Az^2 - \frac{(|z-x|-1)^2}{4t}\right] dz < \infty.$$

Allowing h to approach zero in (3.4) we again see that $u(x, t) \in H^0$ in $0 < t < c$, as desired. The proof is concluded as before.

We observe that condition (2.5) is not necessary for the representation (2.6), (2.7). For, if

$$d\alpha(x) = \exp[x^2/(4c)]/(1+x^2) dx,$$

then

$$\int_{-\infty}^{\infty} k(x, c) d\alpha(x) = (4\pi c)^{-1/2} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} < \infty.$$

But the integral (2.5), with $A = 1/(4c)$, becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} \exp\left[\frac{y^2/(4c)}{1+y^2}\right] dy \int_{-\infty}^{\infty} k(x-y, t) \exp[-x^2/(4c)] dx \\ &= [1+(t/c)]^{-1/2} \int_{-\infty}^{\infty} \exp\left[\frac{ty^2/4c(t+c)}{1+y^2}\right] dy = \infty, \quad 0 < t. \end{aligned}$$

This integral diverges for all $t > 0$, so that no constant M exists satisfying (2.5).

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