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DEFORMATIONS OF LIE SUBGROUPS

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ABSTRACT. We give rigidity and universality theorems for embedded deformations of Lie subgroups. If $K \subset H \subset G$ are Lie groups, with $H^1(K, g/h) = 0$, then for every C^∞ deformation of H, a conjugate of K lies in each nearby fiber H_s. If $H \subset G$ with $H^2(H, g/h) = 0$, then there is a universal "weak" analytic deformation of H, whose base space is a manifold with tangent plane canonically identified with Ker δ^1.

Introduction. This paper extends results of Richardson [4] on analytic deformations of Lie subgroups, to the C^∞ case. If $K \subset H \subset G$ are Lie groups, with $H^1(K, g/h) = 0$, then for every C^∞ deformation of H, a conjugate of K lies in each nearby fiber H_s. We also prove a partial converse, namely that if $H \subset G$ with $H^2(H, g/h) = 0$, then there is a universal "weak" analytic deformation of H, whose base space is a manifold with tangent plane canonically identified with Ker δ^1.

Notation. Throughout, $H \subset G$ being a Lie subgroup does not imply that H is topologically closed as a subset of G. Furthermore, the topological (resp. C^∞ or analytic) structure on H is that of an abstract Lie group, not the induced structure as a subset of G.

I. Rigidity.

Let S be a C^∞ (resp. analytic) manifold.

DEFINITION. A C^∞ (analytic) family of Lie groups parametrized by S is a C^∞ (analytic) manifold $\mathbf{H}$ together with a surjective submersion $\pi: \mathbf{H} \to S$, a section $e: S \to \mathbf{H}$, and a morphism $M: \mathbf{H} \times_S \mathbf{H} \to \mathbf{H}$, such that for each $s \in S$, the fiber $\pi^{-1}(s) = \mathbf{H}_s$ has a Lie group structure given by multiplication $M|\mathbf{H}_s \times \mathbf{H}_s$ and identity $e(s)$.

Let (S, s_0) be a pointed manifold and H a Lie group.

DEFINITION. An abstract C^∞ (analytic) deformation of H, parametrized by S, is a C^∞ (analytic) family of Lie groups $\mathbf{H}$ parametrized by S, together with an isomorphism $\mathbf{H}_{s_0} \xrightarrow{\sim} H$.

Let (S, s_0) be a pointed manifold. Let H be a Lie subgroup of a Lie group G.

DEFINITION. A weak embedded C^∞ (analytic) deformation of H, parametrized by S, consists of an abstract C^∞ (analytic) deformation $\mathbf{H}$ of H,

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parametrized by S, and a C^∞ (analytic) immersion $p\colon \mathbf{H} \to G \times S$, compatible with π, such that the restriction of p to each fiber $\mathbf{H}_s$ is a Lie group homomorphism onto its image $H_s \subset G \times \{s\}$, an isomorphism if $s = s_0$, and $H_{s_0} = H \times \{s_0\}$.

DEFINITION. A strong embedded C^∞ (analytic) deformation of H is a weak embedded C^∞ (analytic) deformation of H for which p is injective. (Thus p is a Lie group isomorphism on each fiber.)

REMARK. A strong embedded analytic deformation of H corresponds to Richardson's [4] concept of an analytic family of Lie subgroups of G.

Our definition of a weak deformation has the drawback that it is not symmetric with respect to points in the base space S; thus a weak embedded deformation of a subgroup $H = H_{s_0} \subset G$ is not necessarily a weak embedded deformation of a nearby fiber H_s. This is because p must be an isomorphism on the fiber $\mathbf{H}_{s_0}$ but only a homomorphism (covering map) on nearby fibers $\mathbf{H}_s$. This asymmetry is unavoidable if we are to prove any sort of universality theorem.

Recall the definition of Lie group cohomology [3]:

Let $\mathcal{C}^r(A, B)$ denote the space of C^r functions from A to B, where $r = \infty$ or ω, C^ω denoting $\mathbf{R}$-analytic as usual. Consider the complex

$$g/h \xrightarrow{\delta^0} \mathcal{C}^r(K, g/h) \xrightarrow{\delta^1} \mathcal{C}^r(K \times K, g/h) \xrightarrow{\delta^2} \mathcal{C}^r(K \times K \times K, g/h) \ldots,$$

where

$$(\delta^0 f)(x) = x \circ f - f,$$

$$(\delta^1 f)(x, y) = x \circ f(y) - f(xy) + f(x),$$

$$(\delta^2 f)(x, y, z) = x \circ f(y, z) - f(xy, z) + f(x, yz) - f(x, y),$$

and $x \circ f$ denotes the adjoint representation $(d/dt)x(exp\ tf)x^{-1}|_{t=0}$. Then $H^i(K, g/h) = \operatorname{Ker}(\delta^i)/\operatorname{Im}(\delta^{i-1})$. The cohomology space $H^i(K, g/h)$ is independent of r (the order of differentiability).

We may now state and prove our first result.

THEOREM 1. *Suppose $K \subset H \subset G$ are Lie groups. Let $(\mathbf{H}, S)$ be a weak embedded C^∞ deformation of H. Suppose the factor group K/K^0 (where K^0 is the connected component of the identity in K) is finitely generated, and suppose $H^1(K, g/h) = 0$. Then there is an open neighborhood U of s_0 in S and a C^∞ map $b\colon U \to G$ such that $K \subset b(s)H_s b(s)^{-1}$ for each $s \in U$, and $b(s_0) = e$.*

(Here g, h, and k denote the Lie algebras of G, H, and K.)

PROOF. The key to extending Richardson's results to the C^∞ case is in the use of the finite-dimensional implicit function theorem in place of Artin's theorem [1].

Choose $h^\perp$ a vector subspace of g complementary to h. Define the *normal displacement function* Ψ: $U' \to h^\perp$ (where U' is a neighborhood of $H \times \{s_0\}$ in $H \times S$) by the properties: For all $(x, s) \in U'$,

(a) $\Psi(x, s_0) = 0$.

(b) $(\exp \Psi(x, s))x \in H_s$.

(c) The map $(x, s) \to p^{-1}((\exp \Psi(x, s))x, s)$ is a C^∞ map of U' onto an open neighborhood of $\mathbf{H}_{s_0}$ in $\mathbf{H}$, whose derivative at (e, s_0) is a linear isomorphism.

See Richardson [4] for a proof of existence and uniqueness, via the implicit function theorem.

Since g/h is finite dimensional, so is $\delta^0(g/h) = \operatorname{Im} \delta^0 \subset \mathcal{C}^\infty(K, g/h)$. We can choose finitely many points y_i, $1 < i < m$ (noncanonically), and a linear map Z: $(g/h)^m \to \operatorname{Im} \delta^0$, such that Z is a one-sided inverse to the evaluation map, in the sense that for $f \in g/h$, $Z(\langle(\delta^0 f)(y_i)\rangle) = \delta^0 f$.

Construct a new weak embedded C^∞ deformation H' of H, parametrized by $G \times S$, whose abstract fiber is given by $\mathbf{H}'_{(x,s)} = \mathbf{H}_s$ while the embedded fiber is given by $H'_{(x,s)} = xH_s x^{-1}$, $x \in G$, $s \in S$; p' is p composed with a conjugation by x. Let Ψ': $T' \to h^\perp$ denote the associated normal displacement function, where T' is a neighborhood of $K \times (e, s_0)$ in $K \times (G \times S)$.

Define a map j: $T \to \operatorname{Im} \delta^0$ by $j(x, s) = Z(\langle[\Psi'(y_i, (x, s))]\rangle)$, where T is a neighborhood of (e, s_0) in $G \times S$. The partial derivative of j with respect to G at (e, s_0) is the surjective map $g \to g/h \xrightarrow{\delta^0} \operatorname{Im} \delta^0$, and $j(e, s_0) = 0$. By the finite-dimensional implicit function theorem, there is a neighborhood U of s_0 in S and a C^∞ function b: $U \to G$ such that $j(b(s), s) = 0$ identically in s, and $b(s_0) = e$.

Construct yet another weak embedded C^∞ deformation $\tilde{\mathbf{H}}$ of H, parametrized by U, with abstract fiber $\tilde{\mathbf{H}}_s = \mathbf{H}_s$ and embedded fiber $\tilde{H}_s = b(s)H_s b(s)^{-1}$, so that $\tilde{p}$ is again p composed with a conjugation by $b(s)$. The associated normal displacement function $\tilde{\Psi}$ satisfies $Z(\langle[\tilde{\Psi}(y_i, s)]\rangle) = 0$ identically in s.

We wish to prove that $(\tilde{\mathbf{H}}, U)$ is "trivial along K", i.e. $K \times U \subset \tilde{p}\tilde{\mathbf{H}}$. It suffices to show that $k \subset \tilde{h}_s$ (the Lie algebra of the fiber $\tilde{H}_s$), and that for a finite set $\{x_i\}_{1<i<p} \subset K$ generating K/K^0, we have $x_i \in \tilde{H}_s$. Defining $\tilde{\alpha}_s \in \operatorname{Hom}(k, h^\perp)$ by the condition $z + \tilde{\alpha}_s(z) \in \tilde{h}_s$ for each $z \in k$ (which determines $\tilde{\alpha}_s$ uniquely, after perhaps restricting U), we are in turn reduced to proving that if α: $U \to \operatorname{Hom}(k, h^\perp) \times (h^\perp)^p$ is defined by $\alpha(s) = (\tilde{\alpha}_s, \langle\tilde{\Psi}(x_i, s)\rangle)$, then α is identically 0.

Suppose this is not the case. Then there exists $\{s_n\}$, a sequence of points in U converging to s_0, such that $\alpha(s_n) \neq 0$. View $\operatorname{Hom}(k, h^\perp) \times (h^\perp)^p$ as $\mathbf{R}^N$ with Euclidean norm $|\ |$. If $\varepsilon_n = |\alpha(s_n)|$, then $\alpha(s_n)/\varepsilon_n$ is a sequence of points on the (compact) $(N - 1)$-sphere. Choosing a subsequence if necessary, we find $\alpha(s_n)/\varepsilon_n$ converges to a point $\alpha_0 \in S^{N-1} \subset \mathbf{R}^N$, representing $(\gamma_0, \langle\gamma_i\rangle) \in \operatorname{Hom}(k, h^{\perp}) \times (h^{\perp})^p$. This α_0 will determine an element of the cohomology.

CONSTRUCTION. Define $\gamma \in \mathcal{C}^{\infty}(K, g/h)$ as follows:

(1) For $a \in k$,

$$\gamma(\exp a) = \left[(d/dt)\left(\exp(a + t\gamma_0(a))\exp(-a)\right)\big|_{t=0}\right]$$

(where [ ] denotes class in g/h).

(2) For x_i one of the generators of K/K^0, $\gamma(x_i) = [\gamma_i]$.

(3) For x, y such that $\gamma(x), \gamma(y)$ are already defined, $\gamma(xy) = \gamma(x) + [x \circ \gamma(y)]$.

As $\exp k$ and $\{x_i\}$ generate K, this defines γ on all of K.

LEMMA 1. *With γ so defined, for each $x \in K$ we have $\gamma(x) = \lim(1/\varepsilon_n)[\tilde{\Psi}(x, s_n)]$. In particular, this limit exists.*

The proof is a straightforward but tedious computation, and shall be omitted.

Granting this lemma, we see that γ is well defined. By properties (1) and (3) it is $\mathbf{R}$-analytic; by (3) again γ lies in the kernel of δ^1: $\mathcal{C}^{\infty}(K, g/h) \to \mathcal{C}^{\infty}(K \times K, g/h)$. So by assumption on vanishing cohomology, γ lies in the image of δ^0. But, as $Z(\langle\tilde{\Psi}(y_i, s_n)\rangle) = 0$ for all n large enough, we have, by the lemma, $Z(\langle\gamma(y_i)\rangle) = 0$. By our choice of Z and y_i, this means $\gamma = 0$. This contradicts $\alpha_0 \neq 0$, proving our theorem. Q.E.D. THEOREM 1.

II. **Universality.** Let us now consider the case $K = H$. We have seen, in the proof of Theorem 1, that any weak embedded deformation of H gives rise, infinitesimally, to an element of $\operatorname{Ker} \delta^1$: $\mathcal{C}(H, g/h) \to \mathcal{C}(H \times H, g/h)$. We would like to prove a converse, namely, that in the case $H^2(H, g/h) = 0$, there is a weak embedded deformation $(\mathbf{H}, S)$ of H whose base space S is a manifold with tangent plane, at s_0, naturally identified with all of $\operatorname{Ker} \delta^1$; i.e. any infinitesimal deformation can be realised as a tangent to a weak deformation. To this end we must develop some more machinery.

Suppose $H \subset G$ has finitely generated factor group H/H^0, so that $\exp h$ and $\{x_1, \ldots, x_p\}$ generate H. Suppose $\pi_1(H^0, e)$ is generated by $\{p_j | j \in J\}$, where each p_j is of the form $\prod_{i=1}^{n_j} \exp a_{ij} = e$, $a_{ij} \in h$. Suppose the relations $\{q_m | m \in M\}$ generate all relations on H/H^0, where each q_m is of the form

$$\left(\prod_{k=1}^{N_m} (x_{km}^{\varepsilon_{km}})\right)\left(\prod_{k=1}^{N'_m} \exp b_{km}\right) = e,$$

with $x_{km} \in \{x_1, \ldots, x_p\}$, $\varepsilon_{km} \in \mathbf{Z}$, and $b_{km} \in h$. Then, as in the proof of Theorem 1, any weak embedded deformation $(\mathbf{H}, S)$ determines locally (i.e. after restricting S) a map $\alpha: S \to \operatorname{Hom}(h, h^{\perp}) \times (h^{\perp})^p$, denoted $\alpha(s) = (\alpha_{0s}, \langle\alpha_{is}\rangle)$, $s \in S$, such that for all $z \in h$, $z + \alpha_{0s}(z) \in \tilde{h}_s$, the Lie algebra of the fiber H_s, and $(\exp \alpha_{is})x_i \in H_s$. We claim that such a map α must satisfy the following four *consistency equations*:

$$[\tilde{h}_s, \tilde{h}_s] \subset \tilde{h}_s \quad \left(\text{where } \tilde{h}_s = \{z + \alpha_{0s}(z) | z \in h\}\right), \tag{1}$$

$$\tilde{x}_{is} \circ \tilde{h}_s \subset \tilde{h}_s \quad (\text{where } \tilde{x}_{is} = (\exp \alpha_{is})x_i), \tag{2}$$

$$\exp^{-1}\left\{ \prod_{i=1}^{n_j} \exp(a_{ij} + \alpha_{0s}(a_{ij})) \right\} \in \tilde{h}_s, \tag{3}$$

$$\exp^{-1}\left\{ \left(\prod_{k=1}^{N_m} (\tilde{x}_{kms}^{e_{km}}) \right) \left(\prod_{k=1}^{N_m'} \exp(b_{km} + \alpha_{0s}(b_{km})) \right) \right\} \in \tilde{h}_s \tag{4}$$

identically in s, where the $\exp^{-1}$ in equations (3) and (4) are chosen to be 0 at $s = s_0$.

Equations (1) and (2) are necessary in order that H_s be a Lie group at all. Equations (3) and (4) follow from the existence and uniqueness of the normal displacement function $\tilde{\Psi}$ (see [4]). All these consistency equations are **R**-analytic, and thus (by Oka's coherency theorem in the **C**-analytic case) they may be replaced locally by a finite set of **R**-analytic equations R_i.

LEMMA 2. *Let $\overline{R_i}$ denote the first-order (in α_{0s}, α_{is}) terms of the equations R_i. Then the variety determined by $\overline{R_i}$ is naturally identified with* $\operatorname{Ker} \delta^1: \mathcal{C}(H, g/h) \to \mathcal{C}(H \times H, g/h)$.

PROOF. As in the proof of Theorem 1, an element of $\operatorname{Ker} \delta^1$ is determined by its action on h and on $\{x_i\}$. Given a point $\alpha \in \operatorname{Hom}(h, h^\perp) \times (h^\perp)^p$, we can try to construct $\gamma \in \mathcal{C}(H, g/h)$ as before. We will succeed if and only if the construction is self-consistent, i.e. if for each $x \in H$, any two ways of assigning a value to $\gamma(x)$ give the same result. But a way of assigning a value to $\gamma(x)$ is equivalent to an expression of x as the product of x_i's and exponentials of Lie algebra elements. The relations $\overline{R_i}$ are precisely what is needed to insure this self-consistency. Q.E.D. LEMMA 2.

The relations R_i define an analytic subvariety S_H of $\operatorname{Hom}(h, h^\perp) \times (h^\perp)^p$, and each weak embedded deformation $(\mathbf{H}, \underline{S})$ determines a map f from S to S_H, such that for $s \in S$, $\overline{H_{f(s)}} \subset H_s$, where $\overline{H_{f(s)}}$ is the subgroup of G generated by $\tilde{h}_{f(s)}$ and $\tilde{x}_{if(s)}$.

This correspondence from weak embedded deformations to maps of S to S_H is unfortunately not one-one. Lack of injectivity arises because connected components of **H** which do not intersect the special fiber do not affect the map into S_H; thus a deformation $\mathbf{H}_1$ with these extra components determines the same map as another deformation $\mathbf{H}_2$ without the extra components.

The correspondence is surjective, as shown by the following.

LEMMA 3. *Let $H \subset G$ and S_H be as above. Let (S, s_0) be a pointed analytic manifold, and $f: (S, s_0) \to (S_H, 0)$ an analytic map. After restricting S to a neighborhood of s_0, there is a weak embedded analytic deformation* $\mathbf{H}$ *of H, parametrized by S, for which the above construction yields $f: (S, s_0) \to (S_H, 0)$.*

PROOF. To fix notation, let the composite map

$$S \xrightarrow{f} S_H \xrightarrow{\text{incl.}} \operatorname{Hom}(h, h^{\perp}) \times (h^{\perp})^p$$

send $s \in S$ to $(a_{0s}, \langle a_{is} \rangle)$.

We construct the deformation $\mathbf{H}$, following Douady and Lazard [2]. To begin with, we consider only H^0, the connected component of the identity in H. Construct an analytic family of Lie algebras $\mathbf{h}$ over S, whose fiber $\underline{h}_s$ over $s \in S$ is isomorphic to the Lie algebra $h_s = \{z + a_{0s}(z) | z \in h\}$. Let D be a neighborhood of 0 in $\mathbf{h}$, with each fiber D_s a neighborhood of 0 in $\underline{h}_s$, such that exp is 1-1 on D_s. Let D^n be the nth fiber product of D with itself, with fiber $D_s^n = (D_s)^n = (D^n)_s$.

Define a map $V_n: D^n \to G$ by $V_n(\underline{b}_1, \ldots, \underline{b}_n; s) = \prod_{i=1}^n \exp b_i$, where $\underline{b}_i \in \underline{h}_s$ corresponds under the isomorphism to $b_i \in h_s$ considered as a Lie subalgebra of g. Let $k = \dim h$. $\bar{F}_n =_{\text{def}} V_n^{-1}(e)$ is an analytic subvariety of D^n; as such, it is the locally finite union of irreducible components. Let F_n be the union of those irreducible components of $\bar{F}_n$ of codimension $k = \dim h$. These are precisely the components which, when projected to S, map to open subsets of S. We find that F_n is a manifold, not just a variety. Under the natural inclusion $i: D^n \to D^{n+1}$ given by $i(\underline{b}_1, \ldots, \underline{b}_n; s) = (\underline{b}_1, \ldots, \underline{b}_n, \underline{0}; s)$, we find that F_n also includes into F_{n+1} by the same map: $iF_n \subset F_{n+1}$. In fact, $F_n = i^{-1}F_{n+1}$. Then $\bigcup_n D^n$ has the structure of a family of free groups over S, and $\bigcup_n F_n$ is a family of normal subgroups. For each $s \in S$, we have $\bigcup_n D_s^n / \bigcup_n F_{ns}$ a connected Lie group, with Lie algebra isomorphic to h_s. Let $\mathbf{H}' = \bigcup_n D^n / \bigcup_n F_n$. According to Douady and Lazard [2], $\mathbf{H}'$ will be a Hausdorff family of groups over S, and thus an abstract deformation in the present sense, if F_n satisfies two conditions:

(1) Axiom of semicontinuity of relations: For each point $(\underline{b}_1, \ldots, \underline{b}_n; s) \in F_n$, there are analytic maps $\tilde{b}_i$ of a neighborhood of s in S to D, with $(\tilde{b}_1(s'), \ldots, \tilde{b}_n(s'); s') \in F_n$ identically in s' in the neighborhood of s, with $\tilde{b}_i(s) = \underline{b}_i$. This is immediate by our choice of F_n as being of codimension k. (This is where we use the fact that S is a manifold.)

(2) Closure of relations: F_n is closed in D^n. This is true as F_n is a locally finite union of analytic subvarieties of D^n.

By the *consistency equations* (1) and (3), we find that the fiber of $\mathbf{H}'$ over s_0 (the special fiber) is in fact isomorphic to H^0 and not some covering space thereof; relation (3) ensured that $(\underline{a}_{1j}, \ldots, \underline{a}_{n_j j}, \underline{0}; s_0)$ in fact lay in an irreducible component F'_{n_j+1} of codimension k for each relation $(\prod_{i=1}^{n_j} \exp a_{ij})$ in $\pi_1(H^0, e)$.

The factor group H/H^0 presents little trouble. In the consistency equation (4), set

$$q_{ms} = \exp^{-1}\left\{\left(\prod_{k=1}^{N_m} (\tilde{x}_{kms}^{\epsilon_{km}})\right)\left(\prod_{k=1}^{N'_m} \exp(b_{km} + \alpha_{0s}(b_{km}))\right)\right\} \in h_s,$$

let $\underline{q}_{ms}$ be the corresponding element of $\underline{h}_s$, and let $\underline{Q}_{ms} = \exp \underline{q}_{ms} \in \mathbf{H}'_s$. Then append to $\mathbf{H}'$ a collection of generators $\{\underline{x}_{is} | 1 < i < p, s \in S\}$ with analytic structure given by that of S, with the action $\underline{x}_{is} \circ \underline{h}_s$ given by that of $\tilde{x}_{is}$ on h_s (see consistency equation (2)), and relations given by

$$\left(\prod_{k=1}^{N_m} (\underline{x}_{kms}^{\epsilon_{km}})\right)\left(\prod_{k=1}^{N'_m} \exp \underline{b}_{kms}\right) = \underline{Q}_{ms}.$$

The result is a new family of groups $\mathbf{H}$ over S. Along with a projection p from $\mathbf{H}$ to G, given by p agreeing with V_n on D^n, and $p(\underline{x}_{is}) = \tilde{x}_{is}$, we obtain a weak embedded analytic deformation. Finally we calculate that p, restricted to the fiber over s_0, is an isomorphism onto H, and that p is an embedding elsewhere. Thus $\mathbf{H}$ is the desired deformation. Q.E.D. LEMMA 3.

In general, S_H itself is not the base space of a weak embedded analytic deformation, as S_H is only a variety and not a manifold. Even if we were to expand the definitions to allow deformations to be parametrized by varieties, we would still be unable to build a weak deformation parametrized by S_H, as the following example (based on one by Douady and Lazard [2, IV.5]) indicates.

EXAMPLE. Let $G = \mathrm{GL}(4, \mathbf{R})$, and $H = \mathbf{R}^3$, embedded as

$$\left\{\left.\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}\right| x, y, z \in \mathbf{R}\right\}.$$

Let $S = \{(\mathrm{a, b, c, d}) \in \mathbf{R}^4 | cd = 0\}$. Let $\tilde{h}_{(a,b,c,d)}$ be generated by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & d & -a & 0 \\ 0 & b & d & 0 \\ -ab & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 \\ 0 & -bc & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -c & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -ac & 0 \end{bmatrix}.$$

We cannot build a Hausdorff fiber product $\mathbf{H}$ corresponding to this map from S to S_H. Reason: for $a, b, d > 0$, the fiber over $(a, b, 0, d)$ is of necessity simply connected. As d approaches 0, the fiber approaches the universal cover of E^2; thus the fiber over $(a, b, 0, 0)$ must also be simply connected. On the other hand, for $a, b, c > 0$, the fiber over $(a, b, c, 0)$ is either S^3 or SO^3, and if we exponentiate $4\pi/ab$ times the first Lie algebra generator, we must return to e in the fiber over $(a, b, c, 0)$. Letting c approach 0; the same must hold true for the fiber over $(a, b, 0, 0)$, an impossibility.

In spite of these difficulties, the equations R_i retain enough of the structure of weak deformations to suit our purposes.

We may now state and prove our second major result.

THEOREM 2 (EMBEDDED UNIVERSALITY). *Let $H \subset G$ be Lie groups such that $H^2(H, g/h) = 0$, and the factor group H/H^0 is finitely generated (so that $V =_{\text{def}} \operatorname{Ker} \delta^1$: $\mathcal{C}(H, g/h) \to \mathcal{C}(H \times H, g/h)$ is finite dimensional). Then there is a weak embedded* $\mathbf{R}$*-analytic deformation* $(\mathbf{H}, S)$ *of H such that*:

(1) *S is an open neighborhood of* 0 *in* $V = \operatorname{Ker} \delta^1$.

(2) *The normal displacement function Ψ agrees with the inclusion map i: $S \to \operatorname{Ker} \delta^1$ to first order.*

(3) *Any other deformation* $(\mathbf{H}', S')$ *of H is locally equivalent to one obtained by pullback of a map f: $S' \to S$, in the sense that $H_{f(s)} \subset H'_s$ for all s near* 0 *in S'.*

PROOF. We construct a formal power series map from V into S_H which will give rise, by Artin's lemma, to an analytic map from S a neighborhood of 0 in V into S_H, whose derivative is nonsingular at the origin. Thus S_H will contain a manifold of dimension equal to that of $\operatorname{Ker} \delta^1$; by Lemma 2, S_H will itself be a manifold of the right dimension. Applying Lemma 3 to the identity map on S_H, we will obtain the desired deformation.

Let $V = \operatorname{Ker} \delta^1$; let $h^\perp$ be the previously chosen complement of h in g, and let $p_{h^\perp}$ denote the inverse of the natural projection $h^\perp \to g/h$; $p_{h^\perp}$ will also denote the composition of this inverse with the projection $g \to g/h$.

We construct F_i: $H \times V \to g$, f_i: $H \times V \to G$, and A_i: $H \times H \times V \to h$, sequences of $\mathbf{R}$-analytic maps, such that:

(1) F_i and A_i are polynomials of degree i in the V variables;

(2) F_i agrees with F_{i-1} up through terms of degree $i-1$ in the V variables, as does A_i with A_{i-1};

(3) $f_i(x, s) = (\exp F_i(x, s))x$;

(4) $A_i(e, *, *) = A_i(*, e, *) = A_i(*, *, 0) = F_i(e, *) = F_i(*, 0) = 0$;

(5) $\{f_i(x, s) | x \in H\}$ is closed under group multiplication to ith order, that is,

$$\exp^{-1}\left\{f_i(x, s) f_i(y, s) f_i(xy, s)^{-1} f_i(\exp A_i(x, y, s), s)\right\} = O(s^{i+1}).$$

Let $F_1(x, s) = p_{h^\perp}(s(x))$, considering $s \in \operatorname{Ker} \delta^1 \subset \mathcal{C}(H, g/h)$ as a map from H to g/h. Let $f_1 = (\exp F_1(x, s))x$. Set

$$A_1(x,y,s) = -(I - p_{h^\perp})(x \circ p_{h^\perp}s(y) - p_{h^\perp}s(xy) + p_{h^\perp}s(x)) = -(I - p_{h^\perp})(x \circ p_{h^\perp}s(y)) = -p_h(x \circ p_{h^\perp}s(y)),$$

where p_h denotes the projection $g \to h$ induced by $g = h + h^\perp$, and I is the identity.

Then F_1, f_1, A_1 satisfy the inductive hypotheses.

Suppose F_i, f_i, A_i satisfy the inductive hypotheses. Defining $\underline{B}$: $H \times H \times V \to G$ by

$$\underline{B}(x,y; s) = f_i(x, s)f_i(y, s)f_i(xy, s)^{-1}f_i(\exp A_i(x, y, s), s),$$

we have $\underline{B}$ analytic and vanishing to order i in the V variables, i.e.

$$\exp^{-1}\underline{B}(x,y; s) = B'(x,y; s) + B''(x,y; s),$$

where B' and B'' are again $\mathbf{R}$-analytic, B' is homogeneous of degree $i + 1$ in the V variables, and B'' is of order at least $i + 2$ in the V variables. Let $B(x,y; s)$ denote the class of $B'(x,y; s)$ in g/h.

LEMMA 4. *$B(x,y; s)$ satisfies the cocycle relation*

$$x \circ B(y, z; s) - B(xy, z; s) + B(x, yz; s) - B(x,y; s) = 0.$$

PROOF. We have

$$\underline{B}(x,y; s)f_i(\exp A_i(x, y, s), s)^{-1} = f_i(x, s)f_i(y, s)f_i(xy, s)^{-1}.$$

So

$$\begin{aligned}\underline{B}(x,y; s)&f_i(\exp A_i(x, y, s), s)^{-1}\underline{B}(xy, z; s)f_i(\exp A_i(xy, z, s), s)^{-1}\\ &= f_i(x, s)f_i(y, s)f_i(z, s)f_i(xyz, s)^{-1}\\ &= \left\{f_i(x, s)\left(\underline{B}(y, z; s)f_i(\exp A_i(y, z, s), s)^{-1}\right)f_i(x, s)^{-1}\right\}\\ &\quad\times \underline{B}(x, yz; s)f_i(\exp A_i(x, yz, s), s)^{-1}.\end{aligned}$$

Now $\exp^{-1}\underline{B}(*, *; s) = O(s^{i+1})$, $A_i(*, *, s) = O(s)$, $\exp^{-1}f_i(\exp A_i(*, *, s), s) = O(s)$, and $f_i(x, s) = \exp F_i(x, s)x$, where $F_i(x, s) = O(s)$. So the commutator of $\underline{B}(*, *; s)$ with either $\exp F_i(x, s)$ or $f_i(\exp A_i(*, *, s), s)$ is of order $O(s^{i+2})$. Thus we may rewrite the above equation as

$$\begin{aligned}\underline{B}(x,y; s)&\underline{B}(xy, z; s)\underline{B}(x, yz; s)^{-1}\left\{x\underline{B}(y, z; s)x^{-1}\right\}^{-1}\\ &= \left\{f_i(x, s)f_i(\exp A_i(y, z, s), s)^{-1}f_i(x, s)^{-1}\right\}f_i(\exp A_i(x, yz, s), s)^{-1}\\ &\quad\times f_i(\exp A_i(xy, z, s), s)f_i(\exp A_i(x, y, s), s)\exp O(s^{i+2}).\end{aligned}$$

By repeated application of the "group closure of f_i to ith order" (point (5) of the inductive hypotheses), the smallness of $A_i(\exp r, \exp t, s) = O(rts)$, and the fact that the bracketed expression on the right-hand side of the equation is the conjugate of $f_i(\exp A_i(y, z, s), s)^{-1}$ by $f_i(x, s)$, we get

$$\text{right-hand side} = f_i(w(x, y, z, s), s)\exp O(s^{i+2}),$$

with $w\colon H \times H \times H \times V \to H$ an analytic function, $\exp^{-1} w(x, y, z, s) = O(s)$. So

$$\begin{aligned}\underline{B}(x, y; s)\underline{B}(xy, z; s)\underline{B}(x, yz; s)^{-1}\{x\underline{B}(y, z; s)x^{-1}\}^{-1}\\ = f_i(w(x, y, z, s), s)\exp O(s^{i+2}).\end{aligned}$$

Taking $\exp^{-1}$ of both sides,

$$(*)\quad\begin{aligned}-\{x \circ B'(y, z; s) - B'(xy, z; s) + B'(x, yz; s) - B'(x, y; s)\}\\ = \exp^{-1} f_i(w(x, y, z, s), s) + O(s^{i+2}).\end{aligned}$$

Then $\exp^{-1} w(x, y, z, s) = O(s^{i+1})$ and $F_i(w(x, y, z, s), s) = O(s^{i+2})$. Finally, breaking our last equation $(*)$ into h and $h^{\perp}$ components, we obtain the desired result. Q.E.D. LEMMA 4.

We may view B as an element of $\mathcal{C}^{\omega}(H \times H, g/h) \otimes P^{i+1}(V)$, where $P^{i+1}(V)$ denotes the space of $(i + 1)$st-degree homogeneous polynomials in $\{s_i\}$ (a basis of V). By the foregoing lemma, and by the assumption $H^2(H, g/h) = 0$, we have an analytic function $C \in \mathcal{C}^{\omega}(H, g/h) \otimes P^{i+1}(V)$ such that $B = \delta^1 C$, that is,

$$B(x, y; s) = x \circ C(y; s) - C(xy; s) + C(x; s).$$

Let F_{i+1} be defined by $F_{i+1}(x, s) = F_i(x, s) - p_{h^{\perp}}C(x; s)$, with $p_{h^{\perp}}$ as defined above. Let $f_{i+1}(x, s) = \exp F_{i+1}(x, s)x$. Defining $D\colon H \times H \times V \to g$ by

$$\begin{aligned}D(x, y, s) = \exp^{-1}\{f_{i+1}(x, s)f_{i+1}(y, s)f_{i+1}(xy, s)^{-1}\\ \times f_{i+1}(\exp A_i(x, y, s), s)\},\end{aligned}$$

we find, by our construction, that $D(x, y, s) = D'(x, y, s) + D''(x, y, s)$, where $D' \in \mathcal{C}^{\omega}(H \times H, h) \otimes P^{i+1}(V)$ and $D''(x, y, s) = O(s^{i+2})$. Let $A_{i+1}(x, y, s) = A_i(x, y, s) - D'(x, y, s)$.

It is straightforward to verify that $F_{i+1}, f_{i+1}, A_{i+1}$ satisfy the inductive hypotheses.

In this manner we obtain formal power series $\hat{F} \in \mathcal{C}^{\omega}(H, h^{\perp})[[s_i]]$ and $\hat{A} \in \mathcal{C}^{\omega}(H \times H, h)[[s_i]]$; defining $\hat{f}(x, s) = \exp \hat{F}(x, s)x$, we have (from inductive hypothesis (5)) that $\{\hat{f}(x, s) | x \in H\}$ is formally closed under group multiplication, that is

$$\hat{f}(x, s)\hat{f}(y, s)\hat{f}(xy, s)^{-1}\hat{f}(\exp \hat{A}(x, y, s), s) = e$$

formally, with (by hypothesis (4))

$$\hat{A}(x, y, 0) = \hat{A}(x, e, s) = \hat{A}(e, y, s) = 0.$$

From $\hat{F}$ we may obtain a formal power series map from V into Hom$(h, h^{\perp}) \times (h^{\perp})^p$, with the image of s being denoted $(\hat{a}_{0s}, \langle \hat{a}_{is} \rangle)$, by setting $\hat{a}_{is} = \hat{F}(x_i, s)$ and $\hat{a}_{0s}(z) = \lim_{t \to 0}(\hat{F}(\exp tz, s))/t$. By the formal group closure, we get that $(\hat{a}_{0s}, \langle \hat{a}_{is} \rangle)$ formally satisfies the relations R_i; thus we have a formal map from V to S_H whose derivative (i.e. first-order term) is nonsingular at 0. Now we use Artin's Lemma [1], which assures us, in this situation, of an analytic map α from S (a neighborhood of 0 in V) to S_H which agrees with our formal power series map to first order. In particular, S_H contains a manifold of the dimension of Ker δ^1, in fact an analytic nonsingular image of a neighborhood of 0 in Ker δ^1. By Lemma 2, S_H is also contained in an analytic manifold of dimension equal to that of Ker δ^1. Namely, Ker δ^1 is the intersection of the tangent planes to the equations $\overline{R_i}$; choose just enough such equations $\overline{R_i'}$ to exactly determine Ker δ^1, and then the variety determined by the analogous R_i' will be a manifold, containing S_H, with tangent plane equal to Ker δ^1. Therefore S_H itself must be a manifold, of the right dimension. In fact, the derivative of α at 0 reflects the identity of $S \subset$ Ker δ^1 with the tangent plane to S_H, as may be verified.

Applying Lemma 3 to the map $\alpha: S \to S_H$ we obtain the desired deformation $\mathbf{H}$, parametrized by S. (Alternatively, we could have built a deformation $\mathbf{H}_H$, parametrized by S_H, by applying Lemma 3 to the identity map on S_H.) The conclusions of the theorem are now straightforward. Q.E.D. THEOREM 2.

REMARK. It was necessary, in this section, to restrict our attention to the case $K = H \subset G$, since the analogous theorem for $K \subset H \subset G$ is false, as the following example shows.

EXAMPLE. Let G be the universal cover of E^2, namely the semidirect product of $\mathbf{R}$ with $\mathbf{R}^2$, given by $(\theta, x)(\theta', x') = (\theta + \theta', R_{\theta'}x + x')$, with $\theta, \theta' \in \mathbf{R}$, $x, x' \in \mathbf{R}^2$. Let $H = \mathbf{R}$, embedded as $\{(\theta, 0) | \theta \in \mathbf{R}\}$, and let $K = \mathbf{Z}$, embedded as $\{(2\pi m, 0) | m \in \mathbf{Z}\}$. Then $H^2(H, g/h) = H^2(K, g/h) = 0$, while $H^1(K, g/h) \cong g/h \neq 0$. We can calculate that for any nearby fiber H_s of a deformation of H, $K \subset H_s$, and so to first-order K does not move in the g/h direction. This contradicts our notion of what a universality theorem would say in the $K \subset H \subset G$ case, so we conclude there can be no such theorem.

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