PACKING AND COVERING CONSTANTS FOR CERTAIN FAMILIES OF TREES. II⁽¹⁾

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ABSTRACT. In an earlier paper we considered the problem of determining the packing and covering constants for families of trees whose generating function y satisfied a relation $y = x\phi(y)$ for some power series ϕ in y . In the present paper we consider the problem for some families of trees whose generating functions satisfy a more complicated relation.

1. Introduction. Let T_n denote a tree with n nodes. The distance $d(u, v)$ between nodes u and v of T_n is the number of edges in the unique path joining u and v . (For definitions not given here see [1] or [6].)

A subset $\mathcal{P}$ of nodes of T_n is a k -packing, where k is a fixed positive integer, if $d(u, v) > k$ for every pair of distinct nodes u and v in $\mathcal{P}$. A subset $\mathcal{C}$ of nodes of T_n is a k -covering if for every node u of T_n there exists a node v of $\mathcal{C}$ such that $d(u, v) \leq k$. The k -packing number $P_k(T_n)$ is the number of nodes in any largest k -packing of T_n , and the k -covering number $C_k(T_n)$ is the number of nodes in any smallest k -covering of T_n .

If $\mathcal{F}$ denotes a given family of trees, let $p_k(n)$ and $c_k(n)$ denote the average value of $P_k(T_n)$ and $C_k(T_n)$ over all trees T_n in $\mathcal{F}$ that have n nodes. In [5] (see also [2] and [3]) we determined the asymptotic behaviour of $p_k(n)$ and $c_k(n)$, as $n \rightarrow \infty$, for families of rooted trees whose generating functions $y = y(x)$ satisfied a functional relation of the type $y = x\phi(y)$, subject to some rather mild conditions. Our object here is to consider the usually more difficult case when the generating function does not satisfy a relation of this type.

In particular, we shall investigate rooted unlabelled trees and unrooted unlabelled trees. We shall show that the *packing* and *covering constants*

$$p_1 = \lim_{n \rightarrow \infty} p_1(n)/n \quad \text{and} \quad c_1 = \lim_{n \rightarrow \infty} c_1(n)/n$$

exist in these cases, and we shall obtain estimates for their values. It turns out that p_1 and c_1 have the same values for the family of unrooted unlabelled

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trees as they do for the family of rooted unlabelled trees.

2. Definitions and observations. Let y_n denote the number of trees T_n in $\mathcal{F}$ that have n nodes. We assume that $y_1 = 1$ and that any nontrivial tree in $\mathcal{F}$ may be constructed by joining the roots of a collection of smaller trees from $\mathcal{F}$ to a new node that serves as the root of the resulting tree. We further assume that this construction gives rise to a recurrence relation that expresses y_n in terms of $y_1, \dots, y_{n-1}$ for $n \geq 2$. Finally, we assume that there exists an operator $\gamma\{h_1(x), h_2(x), \dots\}$ defined for any (possibly infinite) sequence of power series $h_1(x), h_2(x), \dots$, such that if

$$y = y(x) = \sum_1^{\infty} y_n x^n,$$

then the recurrence relation for y_n can be expressed in terms of generating functions as

$$y(x) = \gamma\{y(x), y(x^2), \dots\}.$$

For notational convenience we shall write this simply as

$$(2.1) \quad y(x) = \Gamma\{y(x)\}.$$

More generally, if $w(x, z)$ is any power series in the two variables x and z , then we let

$$\Gamma\{w(x, z)\} = \gamma\{w(x, z), w(x^2, z^2), \dots\}$$

where $w(x, z), w(x^2, z^2), \dots$ are interpreted as functions of x for fixed values of z .

The results in [5] can be used to determine recursively the number of trees T_n in $\mathcal{F}$ for which $P_k(T_n) = i$ or for which $C_k(T_n) = i$, for $1 \leq i \leq n$. However, because of the general difficulty in establishing the existence and finding the explicit values of the packing and covering constants

$$p_k = \lim_{n \rightarrow \infty} p_k(n)/n \quad \text{and} \quad c_k = \lim_{n \rightarrow \infty} c_k(n)/n$$

when the operator Γ cannot be expressed as $\Gamma\{y(x)\} = x\phi(y(x))$, we shall restrict our attention to the constants p_1 and c_1 . We shall use the fact (see [4]) that $P_{2k}(T_n) = C_k(T_n)$ for any tree T_n in determining c_1 , so we shall actually be considering 1-packings and 2-packings.

Let p_{ni} and c_{ni} denote the fraction of trees T_n in $\mathcal{F}$ for which $P_1(T_n) = i$ and $P_2(T_n) = C_1(T_n) = i$, respectively, for $1 \leq i \leq n$. If

$$P = P(x, z) = \sum_{n,i} p_{ni} y_n z^i x^n$$

and

$$C = C(x, z) = \sum_{n,i} c_{ni} y_n z^i x^n,$$

then clearly $P(x, 1) = C(x, 1) = y(x)$. Furthermore, if

$$p(x) = \sum_n p_1(n) y_n x^n \quad \text{and} \quad c(x) = \sum_n c_1(n) y_n x^n,$$

where $p_1(n)$ and $c_1(n)$ denote the expected packing and covering numbers defined in §1, then

$$(2.2) \quad p(x) = (\partial/\partial z)(P(x, z))_{z=1}$$

and

$$(2.3) \quad c(x) = (\partial/\partial z)(C(x, z))_{z=1}.$$

Thus if we can determine the generating functions $P(x, z)$ and $C(x, z)$ then, in principle, we can determine the expected packing and covering numbers $p_1(n)$ and $c_1(n)$. In order to obtain defining relations for $P(x, z)$ and $C(x, z)$, we need to introduce some auxiliary functions.

3. Preliminary results. We say that a rooted tree T_n with root r is a *type* (k, j) tree, where $0 \leq j \leq k$, if

$$\max\{d(r, \mathcal{P}) : \mathcal{P} \text{ is a largest } k\text{-packing of } T\} = j.$$

Thus, for example, a type $(k, 0)$ tree is one in which the root belongs to every largest k -packing. Let the generating functions $F_0 = F_0(x, z)$ and $F_1 = F_1(x, z)$ be defined with respect to the type $(1, 0)$ and type $(1, 1)$ trees in $\mathcal{T}$ in the same way that $P(x, y)$ was defined with respect to all the trees in $\mathcal{T}$; thus, for example, the coefficient of $z^i x^n$ in F_0 is the number of type $(1, 0)$ trees T_n in $\mathcal{T}$ for which $P_1(T_n) = i$. And let the generating functions $G_0 = G_0(x, z)$, $G_1 = G_1(x, z)$, and $G_2 = G_2(x, z)$ be similarly defined with respect to the trees of type $(2, 0)$, $(2, 1)$, and $(2, 2)$ in $\mathcal{T}$. Then,

$$(3.1) \quad F_0 + F_1 = P$$

and

$$(3.2) \quad G_0 + G_1 + G_2 = C.$$

Furthermore, if $f_i = f_i(x) = F_i(x, 1)$ and $g_j = g_j(x) = G_j(x, 1)$, then

$$(3.3) \quad f_0(x) + f_1(x) = g_0(x) + g_1(x) + g_2(x) = y(x).$$

Trees of type (k, j) were characterized in [5] and it was shown that the characterization yielded a defining relation for the generating functions P and C in terms of the defining relation for $y(x)$. In the present context these results may be expressed as follows when $k = 1$ and $k = 2$.

LEMMA 1. *If $y = \Gamma\{y\}$, then*

$$F_0 = z\Gamma\{F_1\}, \quad F_1 = \Gamma\{P\} - \Gamma\{F_1\},$$

and

$$(3.4) \quad P = \Gamma\{P\} + (z - 1)\Gamma\{F_1\}.$$

LEMMA 2. If $y = \Gamma\{y\}$, then

$$G_0 = z\Gamma\{G_2\},$$

$$G_1 = z\Gamma\{G_0z^{-1} + G_1 + G_2\} - z\Gamma\{G_1 + G_2\},$$

$$G_2 = \Gamma\{G_1 + G_2\} - \Gamma\{G_2\},$$

and

$$(3.5) \quad C = z\Gamma\{G_0z^{-1} + G_1 + G_2\} + (1 - z)G_2.$$

Notice that (3.4) and (3.5) both reduce to the relation $y = \Gamma\{y\}$ when $z = 1$. These lemmas and (2.2) and (2.3) imply the following relations for $p(x)$ and $c(x)$.

COROLLARY 1. If $y = \Gamma\{y\}$, then

$$p(x) = (\partial/\partial z)(\Gamma\{P\})_{z=1} + f_0(x).$$

COROLLARY 2. If $y = \Gamma\{y\}$, then

$$c(x) = (\partial/\partial z)(\Gamma\{G_0z^{-1} + G_1 + G_2\})_{z=1} + g_0(x) + g_1(x).$$

4. Rooted unlabelled trees. Let $\mathcal{F}$ denote the family of (nonisomorphic) rooted unlabelled trees. It is well known (see, e.g., [1, p. 52]) that the generating function $y(x)$ for trees in $\mathcal{F}$ satisfies the relation

$$(4.1) \quad y(x) = x \exp \sum_1^{\infty} y(x^k)/k.$$

Thus the family $\mathcal{F}$ satisfies (2.1) with

$$(4.2) \quad \Gamma\{y(x)\} = x \exp \sum_1^{\infty} y(x^k)/k.$$

We observe, for later use, that if $w(x, z)$ denotes any power series in x and z , then

$$(4.3) \quad \frac{\partial}{\partial z} \Gamma\{w(x, z)\} = \Gamma\{w(x, z)\} \sum_1^{\infty} \frac{\partial w(x^k, z^k)}{\partial z} z^{k-1}.$$

The following results are straightforward consequences of Lemmas 1 and 2, Corollaries 1 and 2, and relations (4.2) and (4.3).

THEOREM 1. If $\mathcal{F}$ denotes the family of rooted unlabelled trees then

$$(4.4) \quad p(x) = y(x) \sum_1^{\infty} p(x^k) + f_0(x),$$

where

$$(4.5) \quad f_0(x) = y(x) - f_1(x) = x \exp \sum_1^{\infty} f_1(x^k)/k.$$

THEOREM 2. *If $\mathcal{F}$ denotes the family of rooted unlabelled trees, then*

$$(4.6) \quad c(x) = y(x) \sum_1^{\infty} \{c(x^k) - g_0(x^k)\} + g_0(x) + g_1(x),$$

where

$$(4.7) \quad g_0(x) = x \exp \sum_1^{\infty} g_2(x^k)/k,$$

$$(4.8) \quad g_1(x) = y(x) - g_0(x) \exp \sum_1^{\infty} g_1(x^k)/k,$$

and

$$(4.9) \quad g_2(x) = y(x) - g_0(x) - g_1(x).$$

We shall use the following lemma (see [5] for the proof of a similar result) in estimating the packing and covering constants for the family $\mathcal{F}$ of rooted unlabelled trees.

LEMMA 3. *Let $A(x) = \sum_0^{\infty} a_n x^n$, $B(x) = \sum_0^{\infty} b_n x^n$, and $A(x)B(x) = \sum_0^{\infty} c_n x^n$ and suppose there exists a positive constant ρ such that $B(\rho) \neq 0$ and $b_n = O(n^{-3/2}\rho^{-n})$ as $n \rightarrow \infty$.*

(i) *If $a_n = O(n^{-3/2}\rho^{-n})$, then $c_n = O(n^{-3/2}\rho^{-n})$ as $n \rightarrow \infty$.*

(ii) *If $a_n \sim cn^{-1/2}\rho^{-n}$, where $c > 0$, then $c_n \sim B(\rho)a_n$ as $n \rightarrow \infty$.*

Otter [7] (see also [1, pp. 209–213]) showed that the generating function $y(x)$ of the family $\mathcal{F}$ of rooted unlabelled trees has radius of convergence $\rho = .3383 \dots$ and that about the point $x = \rho$ it has an expansion of the type

$$(4.10) \quad y(x) = 1 - b(\rho - x)^{1/2} + b_2(\rho - x)^1 + b_3(\rho - x)^{3/2} + \dots,$$

where $b = 2.6811 \dots$. From this he deduced that

$$(4.11) \quad y_n \sim (b/2\sqrt{\pi})\rho^{-n+1/2}n^{-3/2}$$

as $n \rightarrow \infty$.

We remark, for later use, that it follows from (4.1) that

$$(4.12) \quad xy'(x)(1 - y(x)) = y(x) \cdot Q(x)$$

where

$$(4.13) \quad Q(x) = 1 + \sum_2^{\infty} y'(x^k)x^k;$$

relations (4.10) and (4.12) imply (see [1, pp. 212, 213]) that

$$(4.14) \quad Q(\rho) = \frac{1}{2}b^2\rho = 1.21599 \dots$$

5. The packing constant for rooted unlabelled trees.

THEOREM 3. *If $\mathcal{F}$ denotes the family of rooted, unlabelled trees, then there exists a constant p_1 , where $.603 < p_1 < .609$, such that $p_1(n)/n \rightarrow p_1$ as $n \rightarrow \infty$.*

PROOF. For notational convenience we shall let $\mathcal{C}_n(f)$ denote the coefficient of x^n in any power series $f(x)$.

Equation (4.4) can be rewritten as

$$(5.1) \quad p(x) = (y/(1-y))H(x) + f_0(x)$$

where

$$H(x) = \sum_2^{\infty} p(x^k) + f_0(x);$$

and, in view of (4.12), this can be rewritten as

$$(5.2) \quad p(x) = xy'(x)H(x)/Q(x) + f_0(x).$$

Since $0 \leq \mathcal{C}_n(f_0) \leq y_n$, equation (4.11) implies that $\mathcal{C}_n(f_0) = O(n^{-3/2}\rho^{-n})$. It follows from (4.11) and the definition of $p(x)$ that $\sum_2^{\infty} p(x^k)$ has radius of convergence $\rho^{1/2} > \rho = .3383 \dots$. Hence, $\mathcal{C}_n(H) = O(n^{-3/2}\rho^{-n})$.

Since $Q(x)$ is analytic for $|x| < \rho^{1/2}$, it follows from (4.14) that $Q(x) \neq 0$ for $|x| < \rho + 2\epsilon$ for some $\epsilon > 0$; thus $1/Q(x)$ is analytic for $|x| < \rho + 2\epsilon$ and, consequently, $\mathcal{C}_n(1/Q) = O((\rho + \epsilon)^{-n}) = O(n^{-3/2}\rho^{-n})$. These facts imply that $\mathcal{C}_n(H/Q) = O(n^{-3/2}\rho^{-n})$ by Lemma 3(i).

Moreover, $\mathcal{C}_n(xy') \sim cn^{-1/2}\rho^{-n}$, where $c = b\rho^{1/2}/2\sqrt{\pi}$, by (4.11). Therefore, we may apply Lemma 3(ii) to (5.2), and conclude that

$$p_1(n)y_n \sim ny_n H(\rho)/Q(\rho),$$

or that

$$p_1(n)/n \rightarrow p_1 = H(\rho)/Q(\rho)$$

as $n \rightarrow \infty$. Since $Q(\rho)$ is known, by (4.12), it remains to estimate $H(\rho)$.

Let

$$f_0(x) = \sum_1^{\infty} \alpha_n x^n \quad \text{and} \quad f_1(x) = \sum_1^{\infty} \beta_n x^n.$$

Relations (4.1) and (4.5) imply that

$$y(x) = x \prod_1^{\infty} (1 - x^j)^{-y_j} \quad \text{and} \quad f_0(x) = x \prod_1^{\infty} (1 - x^j)^{-\beta_j},$$

by a standard argument (see, e.g., [1, p. 54]). These relations and the fact that $y = f_0 + f_1$ may be combined to show that

$$f_0(x)e^{f_0(x)} = xe^{y(x)} \prod_1^{\infty} \{(1 - x^j)e^{x^j}\}^{-\beta_j} = y(x) \prod_1^{\infty} \{(1 - x^j)e^{x^j}\}^{\alpha_j}.$$

We observe that $y(\rho) = 1$, by (4.10), and that $\alpha_1 = \alpha_3 = 1$. Since $(1 - t)e^t < 1$ for $0 < t < 1$, it follows that

$$f_0(\rho)e^{f_0(\rho)} < (1 - \rho)(1 - \rho^3)e^{\rho + \rho^3} < .9275.$$

Furthermore, $\beta_2 = \beta_3 = 1$, so

$$f_0(\rho)e^{f_0(\rho)} > \rho e(1 - \rho^2)^{-1}(1 - \rho^3)^{-1}e^{-\rho^2 - \rho^3} > .9268.$$

These inequalities imply that

$$(5.3) \quad .540 < f_0(\rho) < .541.$$

Now, $p_1(1) = p_1(2) = 1$, $p_1(3) = 2$, and $p_1(n) \leq n - 1$ for $n \geq 2$; hence,

$$x + x^2 + 4x^3 < p(x) \leq xy'(x) - y(x) + x$$

for $x > 0$. The lower bound for $p(x)$ implies that

$$\sum_2^\infty p(\rho^k) > \rho^2(1 - \rho)^{-1} + \rho^4(1 - \rho^2)^{-1} + 4\rho^6(1 - \rho^3)^{-1} > .193.$$

And, since $y(x) > x + x^2 + 2x^3$, the upper bound implies that

$$\sum_2^\infty p(\rho^k) \leq Q(\rho) - 1 - \rho^4(1 - \rho^2)^{-1} - 2\rho^6(1 - \rho^3)^{-1} < .199,$$

appealing to (4.14). These estimates and (5.3) yield the inequality

$$.733 < H(\rho) < .740.$$

When we combine this with (4.14) we find that

$$.603 < p_1 = H(\rho)/Q(\rho) < .609,$$

as required.

We remark that it can be shown that approximately 35% of the trees T_n in $\mathcal{F}$ are type (1, 0) trees and approximately 65% are type (1, 1) trees, when n is large.

6. The covering constant for rooted unlabelled trees.

THEOREM 4. *If $\mathcal{F}$ denotes the family of rooted unlabelled trees, then there exists a constant c_1 , where $.332 < c_1 < .335$, such that $c_1(n)/n \rightarrow c$, as $n \rightarrow \infty$.*

PROOF. If we let

$$R(x) = g_1(x) + \sum_2^\infty \{c(x^k) - g_0(x^k)\}$$

and appeal to (4.12) and (4.13) again, we find that equation (4.6) can be rewritten as

$$(6.1) \quad c(x) = xy'(x) \cdot (R(x)/Q(x)) + g_0(x) + g_1(x).$$

The same argument that led to (5.2) shows that $c_1(n)/n \rightarrow c_1 = R(\rho)/Q(\rho)$,

as $n \rightarrow \infty$, so it remains to estimate $R(\rho)$.

Since $c_1(1) = 1$ and $1 < c_1(n) \leq \frac{1}{2}n$ when $n \geq 2$, it is not difficult to see that

$$\begin{aligned} x^2 + 2x^3 &< y(x) - g_0(x) \leq c(x) - g_0(x) < c(x) - x \\ &< x^2 + 2x^3 + \frac{1}{2} \sum_4^{\infty} n y_n x^n < \frac{1}{2} x y'(x) - \frac{1}{2} x - x^3 \end{aligned}$$

for $x > 0$. From this we may conclude, by an argument similar to the one used in §5, that

$$(6.2) \quad .0179 < \sum_2^{\infty} \{c(\rho^k) - g_0(\rho^k)\} < .020.$$

We now turn to the problem of estimating $g_1(\rho)$.

We observe that

$$\begin{aligned} (6.3) \quad \sum_2^{\infty} \{g_1(\rho^k) + g_2(\rho^k)\}/k &= \sum_2^{\infty} y(\rho^k)/k - \sum_2^{\infty} g_0(\rho^k)/k \\ &< \ln(1/\rho e) - \sum_2^{\infty} \rho^k/k = \ln(1/\rho e) + \ln(1 - \rho) + \rho < .0092, \end{aligned}$$

where we have appealed to (4.1) and the fact that $g_0(x) > x$. Since $g_1(x) > x^2 + x^3$, it follows that

$$(6.4) \quad \sum_2^{\infty} g_1(\rho^k)/k > -\ln(1 - \rho^2) - \ln(1 - \rho^3) - \rho^2 - \rho^3 > .0078.$$

Equations (4.7)–(4.9) imply that

$$(6.5) \quad g_0(x) = (y(x) - g_1(x)) \exp \left\{ - \sum_1^{\infty} g_1(x^k)/k \right\}$$

and

$$(6.6) \quad y(x) - g_1(x) = x \exp \left(y(x) - g_0(x) + \sum_2^{\infty} \{g_1(x^k) + g_2(x^k)\}/k \right).$$

If we let $z = 1 - g_1(\rho)$ and apply inequalities (6.3) and (6.4) to (6.5) we find that

$$ze^{z-1.0092} < g_0(\rho) < ze^{z-1.0078}.$$

When we apply (6.3) and (6.4) to (6.6) we obtain the inequality

$$\rho e^{1.0078} < ze^{g_0(\rho)} < \rho e^{1.0092}.$$

These last two inequalities imply that $.6130 < z < .6140$, or that

$$.3860 < g_1(\rho) < .3870.$$

This last inequality combined with (4.14) and (6.2) yields the estimate

$$.332 < c_1 = R(\rho)/Q(\rho) < .335,$$

as required.

We remark that it can be shown that approximately 15%, 37%, and 48% of the trees T_n in $\mathcal{F}$ are of type (2, 0), (2, 1), and (2, 2), respectively, when n is large.

7. Unrooted unlabelled trees. Let $\mathcal{G}$ denote the family of (nonisomorphic) unrooted unlabelled trees. In what follows, functions such as $P(x, z)$, $C(x, z)$, $y(x)$, etc. will denote the same functions as were defined earlier with respect to the family $\mathcal{F}$ of rooted unlabelled trees; functions such as $\bar{P}(x, z)$, $\bar{C}(x, z)$, $\bar{y}(x)$, etc. will denote the corresponding functions defined with respect to the family $\mathcal{G}$.

Otter [7] (see also [1, p. 57]) has shown that $\bar{y}(x)$ can be expressed in terms of $y(x)$ by the relation

$$(7.1) \quad \bar{y}(x) = y(x) + \frac{1}{2}(y^2(x) - y(x^2))$$

and from this and (4.10) he deduced that

$$(7.2) \quad \bar{y}_n \sim (b^3/4\sqrt{\pi})\rho^{-n+3/2}n^{-5/2}$$

as $n \rightarrow \infty$, when $\bar{y}_n$ denotes the number of unrooted unlabelled trees with n nodes, and b and ρ are the constants defined in §4. It turns out that the generating functions $\bar{p}(x)$ and $\bar{c}(x)$ for the expected packing and covering numbers of trees in $\mathcal{G}$ can also be expressed in terms of the corresponding generating functions $p(x)$ and $c(x)$ for trees in the family $\mathcal{F}$.

THEOREM 5. *If $\mathcal{G}$ denotes the family of unrooted unlabelled trees, then*

$$(7.3) \quad \bar{p}(x) = p(x)(1 - y(x)) + p(x^2) + \frac{1}{2}(f_0^2(x) - f_0(x^2)),$$

and

$$(7.4) \quad \begin{aligned} \bar{c}(x) &= c(x)(1 - y(x)) + c(x^2) \\ &+ \frac{1}{2}(g_0^2(x) - g_0(x^2) + 2g_0(x)g_1(x)). \end{aligned}$$

PROOF. Otter's dissimilarity characteristic theorem (see [7] or [1, p. 57]) states, in effect, that the number of unrooted unlabelled trees with n nodes equals the number of rooted unlabelled trees with n nodes minus the number of trees with n nodes that can be formed by joining the roots of two smaller nonisomorphic rooted unlabelled trees. If D denotes the tree formed by joining the roots of two rooted trees A and B , then it is not difficult to see that $p_1(D) = p_1(A) + p_1(B)$ unless A and B are both type (1, 0) trees in which case $p_1(D) = p_1(A) + p_1(B) - 1$.

If we express Otter's dissimilarity characteristic theorem in terms of generating functions we find that

$$(7.5) \quad \bar{P}(x, z) = P(x, z) - \frac{1}{2} \left\{ (P^2(x, z) - P(x^2, z^2)) - (1 - z^{-1})(F_0^2(x, z) - F_0(x^2, z^2)) \right\}.$$

The factor $\frac{1}{2}$ is present because the order of the subtrees A and B is immaterial; the term involving $(1 - z^{-1})$ corrects the exponent of z in the exceptional case when both A and B are type $(1, 0)$ trees; and the terms $-P(x^2, z^2)$ and $-F_0(x^2, z^2)$ exclude the possibility that $A = B$. (Notice that (7.5) reduces to (7.1) when $z = 1$.) Equation (7.3) follows upon differentiating both sides of equation (7.5) with respect to z and then setting $z = 1$.

The derivation of (7.4) is quite similar. The main difference is that $c_1(D) = c_1(A) + c_1(B)$ unless A and B are both type $(2, 0)$ trees, or one of them is a type $(2, 0)$ tree and the other is a type $(2, 1)$ tree; in these two cases $c_1(D) = c_1(A) + c_1(B) - 1$. (Notice that the latter case automatically excludes the possibility that $A = B$.) It is not difficult to see that this implies that

$$\bar{C}(x, z) = C(x, z) - \frac{1}{2} \left\{ (C^2(x, z) - C(x^2, z^2)) - (1 - z^{-1}) \cdot (G_0^2(x, z) - G_0(x^2, z^2) + 2G_0(x, z)G_1(x, z)) \right\}.$$

Equation (7.4) now follows upon differentiating both sides of this equation with respect to z and then setting $z = 1$.

THEOREM 6. *If $\mathcal{G}$ denotes the family of unrooted unlabelled trees, then $\bar{p}_1(n)/n \rightarrow p_1$ and $\bar{c}_1(n)/n \rightarrow c_1$, when p_1 and c_1 denote the packing and covering constants of the family $\mathcal{G}$ of rooted unlabelled trees.*

PROOF. For notational convenience we shall let $\phi_1, \phi_2, \dots$ denote power series in which the coefficient of x^n can be shown to be $O(\rho^{-n}n^{-3/2})$.

Equations (7.3) and (5.1) imply that

$$(7.6) \quad \begin{aligned} \bar{p}(x) = y(x)(H(x) - f_0(x)) + f_0(x) + \frac{1}{2}f_0^2(x) \\ + p(x^2) - \frac{1}{2}f_0(x^2). \end{aligned}$$

It follows from (4.5) that

$$f'_0 = f_0(y' - f'_0) + \phi_1.$$

If we differentiate both sides of equation (7.6) and apply this relation for f'_0 , we find that

$$\begin{aligned} \bar{p}' &= y'(H - f_0) + f'_0 + f_0f'_0 + \phi_2 \\ &= y'(H - f_0) + f_0(y' - f'_0) + f_0f'_0 + \phi_3 = y'H + \phi_3. \end{aligned}$$

It follows from Lemma 3(ii), therefore, that

$$n\bar{p}_1(n)\bar{y}_n \sim ny_n H(\rho),$$

or that

$$\bar{p}_1(n)/n \sim y_n H(\rho)/n\bar{y}_n \sim 2H(\rho)/b^2\rho = H(\rho)/Q(\rho) = p_1,$$

as $n \rightarrow \infty$, by (7.2), (4.11), (4.14), and Theorem 3. This completes the proof of the first part of the theorem.

Equations (7.4), (4.12), and (6.1) imply that

$$\begin{aligned} \bar{c}(x) &= y(x)R(x) + (1 - y(x))(g_0(x) + g_1(x)) \\ &\quad + \frac{1}{2}g_0^2(x) + g_0(x) \cdot g_1(x) + c(x^2) - \frac{1}{2}g_0(x^2). \end{aligned}$$

When we differentiate this and use the fact that $y = g_0 + g_1 + g_2$, we obtain

$$\bar{c}' = y'R + y'(1 - y) + g'_1(y - g_1) - g'_2(1 - g_2) + \phi_4.$$

From (4.12) and (4.13) we conclude that $y'(1 - y) = \phi_5$; hence,

$$(7.7) \quad \bar{c}' = y'R + g'_1(y - g_1) - g'_2(1 - g_2) + \phi_6.$$

It follows from equations (4.7) and (4.8) and Lemma 3(i) that

$$g'_0 = g_0g'_2 + \phi_7 \quad \text{and} \quad y' - g'_1 = (y - g_1)(g'_1 + g'_2) + \phi_8.$$

These two equations and the fact that $y = g_0 + g_1 + g_2$ imply that

$$(7.8) \quad g'_1(y - g_1) = g'_2(1 - g_2) + \phi_9.$$

Consequently,

$$\bar{c}' = y'R + \phi_{10},$$

by (7.7) and (7.8). Hence,

$$n\bar{c}_1(n)\bar{y}_n \sim ny_n R(\rho), \quad \text{or} \quad \bar{c}_1(n)/n \rightarrow c_1 = R(\rho)/Q(\rho)$$

as $n \rightarrow \infty$, by the same argument as before.

We remark that it is not true that $p_1(n) = \bar{p}_1(n)$ and $c_1(n) = \bar{c}_1(n)$ for all n . For example, $p_1(5) = 29/9$, $\bar{p}_1(5) = 10/3$, $c_1(5) = 16/9$ and $\bar{c}_1(5) = 5/3$.

REFERENCES

1. F. Harary and E. Palmer, *Graphical enumeration*, Academic Press, New York, 1973. MR 50 #9682.
2. A. Meir and J. W. Moon, *The expected node-independence number of random trees*, Nederl. Akad. Wetensch. Proc. Ser. A 76 = Indag. Math. 35 (1973), 335–341. MR 49 #10607.
3. ———, *The expected node-independence number of various types of trees*, Recent Advances in Graph Theory, Academia Praha, 1975, pp. 351–363.

4. _____, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975), 225–233.
5. _____, *Packing and covering constants for certain families of trees. I*, J. Graph Theory (to appear).
6. J. W. Moon, *Counting labelled trees*, Canad. Math. Monos., No. 1, Canad. Math. Congress, Montreal, Que., 1970. MR 43 #98.
7. R. Otter, *The number of trees*, Ann. of Math. (2) **49** (1948), 583–599. MR 10, 53.

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