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TOPOLOGICAL IRREDUCIBILITY OF NONUNITARY REPRESENTATIONS OF GROUP EXTENSIONS[1]

BY

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ABSTRACT. A functional-analytic approach to the study of the topological irreducibility of certain nonunitary induced representations is set forth. The methods contrast, and in some sense, encompass those first initiated by E. Thieleker in [4], and are amenable to complete irreducibility questions as well. Several sufficient conditions for topological irreducibility are established. A sufficient condition for reducibility is also presented–the latter serving to explain an interesting counterexample due to J. M. G. Fell.

1. **Introduction.** In this paper we establish sufficient conditions for the topological irreducibility of nonunitary representations of semidirect products of the form

$$G = SK, \qquad S \cap K = \{1\},$$

where G is a connected Lie group, S is a connected, closed, normal subgroup of G, and K is a compact subgroup of G. These representations are *induced* from nonunitary representations of closed subgroups of G and are constructed, in the spirit of Mackey [2], as follows. Start with a one-dimensional (nonunitary) character Λ of S and form the stability subgroup

$$K_\Lambda = \left\{k \in K \middle| \Lambda(ksk^{-1}) = \Lambda(s) \text{ for every } s \text{ in } S\right\}$$

of K. Choose an irreducible unitary representation μ of K_Λ. μ acts on a finite dimensional Hilbert space $H(\mu)$. Let $L^2_\mu(K, H(\mu))$ be the Hilbert space of measurable, square integrable, $H(\mu)$-valued functions f on K such that $f(mk) = \mu(m)f(k)$ for (m, k) in $K_\Lambda \times K$. The equation

$$\left(I^{\Lambda,\mu}(sk)f\right)(k_1) = \Lambda\left(k_1 s k_1^{-1}\right)f(k_1 k),$$

$(s, k, k_1) \in S \times K \times K, f \in L^2_\mu(K, H(\mu))$, defines a representation $I^{\Lambda,\mu}$ of G on $L^2_\mu(K, H(\mu))$ where each $I^{\Lambda,\mu}(a)$, $a \in G$, acts as a *bounded* linear operator. If Λ is unitary, in particular, then $I^{\Lambda,\mu}$ is precisely the *unitary* representation of G induced in the classical sense of Mackey [2] by the representation $\Lambda \otimes \mu$ of SK_Λ. By the well-known Mackey theory, moreover, $I^{\Lambda,\mu}$ (for Λ unitary) is

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irreducible; i.e. $I^{\Lambda,\mu}$ is topologically irreducible: $L^2_\mu(K, H(\mu))$ admits no nontrivial $I^{\Lambda,\mu}$ closed invariant subspaces. However for a general nonunitary Λ, $I^{\Lambda,\mu}$ may well be reducible; see §6 for an example due to J. M. G. Fell. Also see Theorem 6.1.

A big step in the study of the representations $I^{\Lambda,\mu}$ for Λ nonunitary has been taken by E. Thieleker in [4]. His approach is infinitesimal in the sense that questions of topological irreducibility are reduced to questions of algebraic irreducibility of the corresponding representations of the universal enveloping algebra of the Lie algebra of G on the space of K-finite vectors. The approach presented here is a global, more direct, one and seems to lead to a simpler analysis of the representations $I^{\Lambda,\mu}$. In particular the methods can also be used to study questions of *complete irreducibility*. On the other hand this paper, in part, owes much to the basic ideas set forth by Thieleker in [4].

Before noting some of the main results proved here it is necessary to introduce additional notation.

Let $L(S)$ denote the Lie algebra of S and let $L(S)^{\mathbf{C}}$ denote the complexification of $L(S)$. Since S is normal and since S, G are connected, $L(S)$ is invariant under the adjoint action of G and, in particular, by restriction we have a representation $\mathrm{Ad}|_K$ of K on $L(S)$ and hence on $L(S)^{\mathbf{C}}$. Since K is compact we can choose (once and for all) an $\mathrm{Ad}|_K$ invariant real inner product $\langle\ ,\ \rangle$ on $L(S)$. $\langle\ ,\ \rangle$ extends to a complex inner product $\langle\ ,\ \rangle$ on $L(S)^{\mathbf{C}}$ such that the action of K on $L(S)^{\mathbf{C}}$ is unitary. We define $d\Lambda$ to be the $\mathbf{C}$ linear map from $L(S)^{\mathbf{C}}$ to $\mathbf{C}$ such that

$$(1.1) \qquad \Lambda(\exp tx) = e^{td\Lambda(x)} \quad \text{for all } x \text{ in } L(S) \text{ and } t \text{ in } R.$$

Identifying the complex dual space of $L(S)^{\mathbf{C}}$ with $L(S)^{\mathbf{C}}$ via $\langle\ ,\ \rangle$ we shall consider $d\Lambda$ as an element of $L(S)^{\mathbf{C}}$. Then Λ is unitary if and only if $d\Lambda \in (-1)^{1/2}L(S)$. Equation (1.1) now reads

$$(1.2) \qquad \Lambda(\exp x) = e^{\langle x, d\Lambda\rangle} \quad \text{for all } x$$

in $L(S)$. Let $-$ denote conjugation of $L(S)^{\mathbf{C}}$ with respect to $L(S)$.

Now we can refer to Theorem 5.9 and Theorem 6.1 *which constitute the key results of this paper*. Theorem 5.9 lists several conditions that insure the topological irreducibility of the $I^{\Lambda,\mu}$. Note that condition (b) is equivalent to the statement that some nonzero complex multiple of $d\Lambda$ lies in $L(S)$. Thus we are restating (in (b)) a result first obtained by Thieleker; see [4, Theorem 4]. Theorem 6.1 shows that under special circumstances the representations $I^{\Lambda,\mu}$ can indeed be reducible.

The irreducibility (or reducibility) properties of the representations $I^{\Lambda,\mu}$ depend largely upon properties of a certain K-invariant subalgebra $A(d\Lambda)$ of $C(K_\Lambda \setminus K)$; see Definition 4.2, Theorem 5.9(j), (k), Theorem 5.10, and

Theorem 6.1. In a forthcoming paper we shall prove the following "double commutant" (or von Neumann density) result. Let $\Delta(I^{\Lambda,\mu})$ denote the linear span of the operators $I^{\Lambda,\mu}(a)$, $a \in G$.

THEOREM 1.3. *If $A(d\Lambda)$ is closed under complex conjugation then the closure of $\Delta(I^{\Lambda,\mu})$ in the strong operator topology coincides with the double commutant $\Delta(I^{\Lambda,\mu})''$ of $\Delta(I^{\Lambda,\mu})$. In particular $I^{\Lambda,\mu}$ is completely irreducible if and only if the commutant $\Delta(I^{\Lambda,\mu})'$ consists of scalar multiples of* 1. *If $d\Lambda$ satisfies* (a), (b) *or* (c) *of Theorem* 5.9 *then $I^{\Lambda,\mu}$ is not only topologically irreducible but is actually completely irreducible.*

2. **Construction of the representations $I^{\Lambda,\mu}$.** In this section G will denote a locally compact group; G need not be a Lie group. We assume that

$$G = SK, \qquad S \cap K = \{1\}, \tag{2.1}$$

where S is a closed, normal subgroup of G, K is a compact subgroup of G, and 1 is the identity element of G. Whenever it is necessary, we shall also assume that G is σ-compact. Let Λ be a continuous homomorphism from S to the nonzero complex numbers $\mathbf{C}^x$; i.e. Λ is a 1-dimensional representation of S. Define

$$K_\Lambda = \left\{k \in K \middle| \Lambda(ksk^{-1}) = \Lambda(s) \text{ for every } s \text{ in } S\right\}. \tag{2.2}$$

K_Λ is the stability subgroup of K at Λ under the action of K on the 1-dimensional representations of S. K_Λ is a closed, hence compact, subgroup of K and, hence, SK_Λ is a closed subgroup of G. Now choose a continuous, irreducible, unitary representation μ of K_Λ on a Hilbert space $H(\mu)$. Since K_Λ is compact and μ is irreducible, $H(\mu)$ is necessarily finite dimensional. Since K_Λ stabilizes Λ the equation

$$(\Lambda \otimes \mu)(sk) = \Lambda(s)\,\mu(k), \qquad (s, k) \in S \times K_\Lambda, \tag{2.3}$$

defines a continuous, irreducible, nonunitary representation $\Lambda \otimes \mu$ of the closed subgroup SK_Λ on $H(\mu)$. In turn, $\Lambda \otimes \mu$ induces a continuous, nonunitary representation $I^{\Lambda,\mu}$ of G as follows. Let $\mathfrak{L}^2_\mu(K, H(\mu))$ denote the space of measurable functions $f\colon K \to H(\mu)$ such that

(i) $f(mk) = \mu(m)f(k)$ for $(m, k) \in K_\Lambda \times K$,

(ii) $\int_K \|f(k)\|^2\, dk < \infty$,

where dk denotes normalized Haar measure on K. Identifying functions that agree almost everywhere, one obtains a Hilbert space $L^2_\mu(K, H(\mu))$.

PROPOSITION 2.4. *The equation*

$$(T_1(s)f)(k) = \Lambda(ksk^{-1})f(k), \qquad (s, k) \in S \times K, f \in \mathfrak{L}^2_\mu(K, H(\mu)),$$

defines a representation T_1 of S on $L^2_\mu(K, H(\mu))$. T_1 is continuous; i.e. for every ϕ in $L^2_\mu(K, H(\mu))$, the map $s \to T_1(s)\phi$ is continuous from S to $L^2_\mu(K, H(\mu))$. For each s in S, $T_1(s)$ is a bounded operator on $L^2_\mu(K, H(\mu))$.

PROOF. Given f in $\mathfrak{L}^2_\mu(K, H(\mu))$ and s in S, put $M_s = \sup_{k\in K}|\Lambda(ksk^{-1})|^2$ so that

$$\int_K \|(T_1(s)f)(k)\|^2\,dk \leq M_s\int_K \|f(k)\|^2\,dk \tag{2.5}$$

implies $T_1(s)f \in \mathfrak{L}^2(K, H(\mu))$. Moreover $T_1(s)f \in \mathfrak{L}^2_\mu(K, H(\mu))$ and by (2.5) we can define a bounded operator $T_1(s)$ on $L^2_\mu(K, H(\mu))$ such that $T_1(s_1s_2) = T_1(s_1)T_1(s_2)$ for s_1, s_2 in S. Thus we get a representation T_1 of S on $L^2_\mu(K, H(\mu))$. The continuity of T_1 at 1, and hence the continuity of T_1 everywhere, follows from the following easily proved

LEMMA 2.6. *Let V be a neighborhood of* 1 *in S. Then there exists a neighborhood $W = W^1$ of* 1 *in S such that for every k in K, $kWk^{-1} \subset V$.*

Let T_2 be the unitary representation of K on $L^2_\mu(K, H(\mu))$ given by right translation. Then one easily verifies that

$$T_1(ksk^{-1}) = T_2(k)T_1(s)T_2(k^{-1}) \tag{2.7}$$

for all (s, k) in $S \times K$. In other words, because of (2.7) one obtains a continuous representation $I^{\Lambda,\mu}$ of G on $L^2_\mu(K, H(\mu))$ by setting $I^{\Lambda,\mu}(sk) = T_1(s)T_2(k)$ for (s, k) in $S \times K$. Hence $I^{\Lambda,\mu}$ is defined by the equation

$$\begin{aligned}(I^{\Lambda,\mu}(sk)f)(k_1) &= \Lambda(k_1sk_1^{-1})f(k_1k),\\ &(s, k, k_1) \in S \times K \times K, f \in \mathfrak{L}^2_\mu(K, H(\mu)).\end{aligned} \tag{2.8}$$

By (2.5) each $I^{\Lambda,\mu}(sk)$ acts as a *bounded* operator on $L^2_\mu(K, H(\mu))$. We shall call $I^{\Lambda,\mu}$ the (nonunitary) representation of G *induced* by $\Lambda \otimes \mu$ and write

$$I^{\Lambda,\mu} = \operatorname*{ind}_{SK_\Lambda\uparrow G} \Lambda \otimes \mu. \tag{2.9}$$

Indeed when S is abelian and Λ is a unitary character of S, $I^{\Lambda,\mu}$ is the *unitary* representation of G induced by $\Lambda \otimes \mu$ in the classical sense of Mackey; see [2]. Moreover by the Mackey theory one knows that $I^{\Lambda,\mu}$ (for Λ unitary) is irreducible; see [2]. On the other hand given a general nonunitary character Λ of S, $I^{\Lambda,\mu}$ need not be irreducible; see Theorem 6.1. Note that $I^{\Lambda,\mu}|_K$ is unitary. Moreover

$$I^{\Lambda,\mu}|_K = \operatorname*{ind}_{K_\Lambda\uparrow K} \mu. \tag{2.10}$$

3. **Thieleker's density lemma.** The following lemma will play a key role later, when we examine the representations $I^{\Lambda,\mu}$ for irreducibility. In content, it is essentially Lemma 3 of [4], stated in a manner suitable for our purposes. As in §1 we assume that μ is an irreducible unitary representation of K_Λ on a Hilbert space $H(\mu)$. Let $C(K, H(\mu))$ be the space of continuous $H(\mu)$-valued functions on K and let $C_\mu(K, H(\mu))$ be the subspace of $C(K, H(\mu))$ of functions f which satisfy $f(mk) = \mu(m)f(k)$ for $(m, k) \in K_\Lambda \times K$. In

particular for $\mu = 1$ we shall write $C_1(K, H(1) = \mathbf{C}) = C(K_\Lambda \setminus K)$, $L^2_\mu(K, H(1)) = L^2(K_\Lambda \setminus K)$. Note that $C_\mu(K, H(\mu))$ is K invariant; i.e. $C_\mu(K, H(\mu))$ is invariant under *right* translation by K.

LEMMA 3.1 (THIELEKER'S DENSITY LEMMA). *Let $F \neq 0$ be a K invariant subspace of $C_\mu(K, H(\mu))$ and let A be a K invariant subspace of $C(K_\Lambda \setminus K)$. If A is dense in $L^2(K_\Lambda \setminus K)$, then AF (the subspace of $C_\mu(K, H(\mu))$ generated by all pointwise products ϕf, $\phi \in A, f \in F$) is dense in $L^2_\mu(K, H(\mu))$.*

For the sake of completeness we will sketch a proof of Lemma 3.1 in this section. First it is necessary to show

LEMMA 3.2. *Let $0 \neq W \subset L^2_\mu(K, H(\mu))$ be a closed K invariant subspace. Then W contains a nonzero element of $C_\mu(K, H(\mu))$.*

PROOF. Let $U = I^{\Lambda,\mu}|_K$ so that by (2.8) U is the unitary representation of K on $L^2_\mu(K, H(\mu))$ given by right translation. Given ϕ in $C(K)$ we can form the Fourier transform of ϕ at U; i.e. form the bounded operator

$$\hat{\phi}(U) = \int_K \phi(k)U(k)\,dk \quad \text{on } L^2_\mu(K, H(\mu)).$$

Let $f \in W, f \neq 0$. For each ϕ in $C(K)$

$$\langle \hat{\phi}(U)f, f \rangle = \int_K \phi(k)\langle U(k)f, f \rangle\,dk.$$

Hence if $\hat{\phi}(U)f = 0$ for every ϕ in $C(K)$, $\langle U(k)f, f\rangle = 0$ for every k in $K \Rightarrow f = 0$. In other words there is some ϕ in $C(K)$ such that $\hat{\phi}(U)f \neq 0$. On the other hand, since W is U invariant and since U is unitary, W is $\hat{\phi}(U)$ invariant; i.e. $\hat{\phi}(U)f \in W$. Moreover $\hat{\phi}(U)f \in C_\mu(K, H(\mu))$. In fact, by Fubini's theorem, one sees that $\hat{\phi}(U)f$ is the unique element of $C_\mu(K, H(\mu))$ such that

$$\begin{aligned}\left\langle \left(\hat{\phi}(U)f\right)(k), \alpha \right\rangle &= \int_K \phi(k_1)\langle f(kk_1), \alpha\rangle\,dk_1 \\ &= \int_K \phi(k^{-1}k_1)\langle f(k_1), \alpha\rangle\,dk_1 \quad \text{for } (k, \alpha)\end{aligned}$$

in $K \times H(\mu)$. The last integral implies the continuity of $\hat{\phi}(U)f$.

The proof of Lemma 3.1, which we now sketch, is the same, with one exception, as that of Lemma 3 in [4], given by Thieleker. First note that AF is K invariant since A, F are K invariant. Since K acts unitarily, it follows that the orthogonal complement $(AF)^\perp$ of AF in $L^2_\mu(K, H(\mu))$ is also K invariant. Assume that $(AF)^\perp \neq 0$. Noting that $(AF)^\perp$ is a closed subspace of $L^2_\mu(K, H(\mu))$, we deduce by Lemma 3.2 that $(AF)^\perp$ contains a nonzero continuous function f. If $\gamma \in F$ is arbitrary define $\gamma_f \in C(K_\Lambda \setminus K)$ by $\gamma_f(k) = \langle f(k), \gamma(k)\rangle$; at this point we use the continuity and the unitarity of μ. For every ϕ in A

$$\int_K \bar{\phi}(k)\gamma_f(k)\,dk = \int_K \langle f(k), (\phi\gamma)(k)\rangle\,dk = 0$$

since $\phi\gamma \in AF, f \in (AF)^\perp$. By hypothesis, A is dense in $L^2(K_\Lambda \setminus K)$. Therefore for almost all k in K, $\gamma_f(k) = 0$. But γ_f is continuous so $\gamma_f(k) = 0$ for all k in K. Now let k, k_1 in K be arbitrary. Since $0 \neq F$ is K invariant we can choose $\gamma \in F$ such that $\gamma(1) \neq 0$. If $R(k^{-1}k_1)\gamma$ denotes right translation of γ by $k^{-1}k_1$, then $R(k^{-1}k_1)\gamma \in F$ and we have just seen that for all l in K, $(R(k^{-1}k_1)\gamma)_f(l) = 0$. Taking $l = k$ we get

$$\begin{aligned} 0 &= \big(R(k^{-1}k_1)\gamma\big)_f(k) = \big\langle f(k), \big(R(k^{-1}k_1)\gamma\big)(k)\big\rangle \\ &= \langle f(k), \gamma(k_1)\rangle \quad \text{for } (k, k_1) \text{ in } K \times K. \end{aligned} \tag{3.3}$$

Let $H_0 = \{\mu(m)\gamma(1) | m \in K_\Lambda\} = \{\gamma(m) | m \in K_\Lambda\}$. Since μ is irreducible and since $\gamma(1) \neq 0$, the closed subspace of $H(\mu)$ generated by H_0 coincides with $H(\mu)$. Thus by (3.3), $\langle f(k), \alpha\rangle = 0$ for every (α, k) in $H(\mu) \times K$ so $f = 0$. This contradiction shows that $(AF)^\perp = 0$ so that AF is dense in $L^2_\mu(K, H(\mu))$.

4. **The algebra** $A(d\Lambda)$. *From this point on* we shall assume that G is a *connected Lie group* and that S is also connected; see (2.1). To any 1-dimensional representation Λ of S we shall associate a complex K invariant subalgebra $A(d\Lambda)$ of $C(K_\Lambda \setminus K)$ having the following properties:

(i) $A(d\Lambda)$ contains the constant functions on K.

(ii) $A(d\Lambda)$ separates the points of $K_\Lambda \setminus K$.

(iii) If $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$ then $I^{\Lambda,\mu}$ is topologically irreducible.

Here $d\Lambda$ is the differential of Λ given by (1.2). In particular, by the Stone-Weierstrass theorem, a sufficient condition that $I^{\Lambda,\mu}$ be topologically irreducible is that $A(d\Lambda)$ be closed under complex conjugation.

As in §1 let $\langle\ ,\ \rangle$ be a K invariant complex inner product on $L(S)^{\mathbf{C}}$. Given $z, x \in L(S)^{\mathbf{C}}$ we define $\phi_{z,x} \in C(K)$:

$$\phi_{z,x}(k)e^{\langle \mathrm{Ad}(k)x, z\rangle} \quad \text{for } k \in K. \tag{4.1}$$

DEFINITION 4.2. Given $z \in L(S)^{\mathbf{C}}$, $A(z)$ is the complex subspace of $C(K)$ generated by the functions $\phi_{z,x}$ in (4.1) where x varies over $L(S)^{\mathbf{C}}$.

By (4.1) $\phi_{z,x_1} \cdot \phi_{z,x_2} = \phi_{z,x_1+x_2}$ for z, x_1, x_2 in $L(S)^{\mathbf{C}}$. Hence $A(d\Lambda)$ is a *subalgebra* of $C(K)$ (under pointwise multiplication). That we actually have $A(z) \subset C(K_\Lambda \setminus K)$ follows from

PROPOSITION 4.3. $K_\Lambda = \{k \in K | \mathrm{Ad}(k)\,d\Lambda = d\Lambda\}$.

The proof of Proposition 4.3, which uses the fact that S is connected, is easy.

The algebra $A(z)$ is, in a sense, a global analogue of the algebra, introduced by Thieleker, which is generated by the functions $k \to \langle \mathrm{Ad}(k)x, z\rangle$, $k \in K$, where z is fixed and x varies, $x, z \in L(S)^{\mathbf{C}}$. From our "global" point of view the use of the exponential function in (4.1) is natural. This will become clearer when direct use of equation (2.8) is made and when z is taken to be the element $d\Lambda$ of $L(S)^{\mathbf{C}}$. As we have seen, because of exponentiation, the subspace $A(z)$ is automatically a subalgebra. $A(z)$ clearly contains the function 1 on K and is K invariant; i.e. $A(z)$ is invariant under *right* translation by K. In particular for $z = d\Lambda$ we have

PROPOSITION 4.4. *The K invariant subalgebra $A(d\Lambda)$ of $C(K_\Lambda \setminus K)$ contains the constant functions on K and separates the points of $K_\Lambda \setminus K$.*

PROOF. Suppose $k_1, k_2 \in K$ such that $\phi(k_1) = \phi(k_2)$ for every ϕ in $A(d\Lambda)$. We must show that $k_2k_1^{-1} \in K_\Lambda$. Let $s \in S$ be arbitrary. Since S is connected $s = \prod_j \exp x_j$, $x_j \in L(S)$. Then

$$\begin{aligned}
\Lambda\left(k_2k_1^{-1}s(k_2k_1^{-1})^{-1}\right) &= \prod_j \Lambda\left(k_2k_1^{-1}\exp x_j(k_2k_1)^{-1}\right)\\
&= \prod_j \Lambda\left(\exp \operatorname{Ad}(k_2k_1^{-1})x_j\right)\\
&= \prod_j \phi_{d\Lambda,\operatorname{Ad}(k_1^{-1})x_j}(k_2) \quad \text{(by (1.1) and (4.1))}\\
&= \prod_j \phi_{d\Lambda,\operatorname{Ad}(k_1^{-1})x_j}(k_1) \quad \text{(since } \phi(k_1) = \phi(k_2) \text{ for every } \phi \text{ in } A(d\Lambda))\\
&= \prod_j e^{\langle x_j, d\Lambda\rangle} = \prod_j \Lambda(\exp x_j) = \Lambda(s) \quad \text{(again by (1.1) and (4.1))},
\end{aligned}$$

which shows that $k_2k_1^{-1} \in K_\Lambda$, since s was arbitrary.

The following theorem is of fundamental importance. Its application in conjunction with the density lemma (Lemma 3.1) will give us statement (iii) on page 74.

THEOREM 4.5. *Let $W \subset L^2_\mu(K, H(\mu))$ be a closed $I^{\Lambda,\mu}|_S$ invariant subspace. Then W is also $A(d\Lambda)$ invariant; i.e. if $\phi \in A(d\Lambda)$ and $f \in W$, then $\phi f \in W$.*

PROOF. The heart of the proof involves justifying differentiation under the integral sign and the use of Lebesgue's theorem on dominated convergence. Note that since $A(d\Lambda) \subset C(K_\Lambda \setminus K)$, $A(d\Lambda)W$ is indeed contained in $L^2_\mu(K, H(\mu))$. Take $f \in W$, f fixed once and for all. Suppose $x \in L(S)$. Then by (1.1), (2.8), and (4.1)

$$I^{\Lambda,\mu}(\exp x)f = \phi_{d\Lambda,x}f; \tag{4.6}$$

i.e. since W is $I^{\Lambda,\mu}|_S$ invariant

$$\phi_{d\Lambda,x}f \in W \quad \text{for every } x \text{ in } L(S). \tag{4.7}$$

However we must show that (4.7) holds for every x in $L(S)^{\mathbf{C}}$. Again let

$x \in L(S)$ be arbitrary but fixed. Let $\phi \in C(K)$ be defined by

$$\phi(k) = \langle \mathrm{Ad}(k)x, d\Lambda \rangle. \tag{4.8}$$

Then for every (m, k) in $K_\Lambda \times K$, $\phi(mk) = \langle \mathrm{Ad}(m)\mathrm{Ad}(k)x, d\Lambda \rangle = \langle \mathrm{Ad}(k)x, \mathrm{Ad}(m^{-1})d\Lambda \rangle = \langle \mathrm{Ad}(k)x, d\Lambda \rangle$ (by Proposition 4.3) $= \phi(k)$; i.e. $\phi \in C(K_\Lambda \setminus K) \Rightarrow \phi f \in L^2_\mu(K, H(\mu))$. We claim that $\langle \phi f, \Psi \rangle = 0$ for every Ψ in $W^\perp$. To prove this, choose a neighborhood N of 0 in R whose closure $\bar{N}$ is compact and define $F: K \times N \to \mathbf{C}$ by

$$F(k, t) = \left\langle e^{t\langle \mathrm{Ad}(k)x, d\Lambda \rangle} f(k), \Psi(k) \right\rangle. \tag{4.9}$$

Then

(a) $k \to F(k, t) \in L^1(K)$ for every t in N, by Hölder's inequality,

(b) $\partial F(k, t)/\partial t = e^{t\langle \mathrm{Ad}(k)x, d\Lambda \rangle} \langle \mathrm{Ad}(k)x, d\Lambda \rangle \langle f(k), \Psi(k) \rangle$ for every (k, t) in $K \times N$, and

(c) $|\partial F(k, t)/\partial t| < M \|f(k)\| \, \|\Psi(k)\|$ for every (k, t) in $K \times N$, where $k \to M\|f(k)\| \, \|\Psi(k)\| \in L^1(K)$, by Hölder's inequality, and

$$|e^{s\langle \mathrm{Ad}(k)x, d\Lambda \rangle} \langle \mathrm{Ad}(k)x, d\Lambda \rangle| < M$$

for every (k, s) in $K \times \bar{N}$.

In other words, because of (a), (b), (c) we can say the following. Put $G(t) = \int_K F(k, t)\, dk$, $t \in N$. Then $G'(t)$ exists on N, $k \to \partial F(k, t)/\partial t \in L^1(K)$ for every t in N and $G'(t) = \int_K \partial F(k, t)/\partial t\, dt$, $t \in N$. In particular,

$$\begin{aligned} G'(0) &= \int_K \frac{\partial F}{\partial t}(k, 0)\, dt = \int_K \left\langle \langle \mathrm{Ad}(k)x, d\Lambda \rangle f(k), \Psi(k) \right\rangle dk \quad \text{(by (b))} \\ &= \langle \phi f, \Psi \rangle \quad \text{(by (4.8))}; \end{aligned}$$

i.e.

$$\begin{aligned} \langle \phi f, \Psi \rangle &= \lim_{t \to 0} \frac{1}{t} \left[G(t) - G(0) \right] \\ &= \lim_{t \to 0} \frac{1}{t} \int_K \left[F(k, t) - F(k, 0) \right] dk \\ &= \lim_{t \to 0} \frac{1}{t} \int_K \left[\left\langle e^{t\langle \mathrm{Ad}(k)x, d\Lambda \rangle} f(k), \Psi(k) \right\rangle - \left\langle f(k), \Psi(k) \right\rangle \right] dk \\ &\qquad\qquad \text{(by (4.9))} \\ &= \lim_{t \to 0} \frac{1}{t} \left[\langle \phi_{d\Lambda, tx} f, \Psi \rangle - \langle f, \Psi \rangle \right] = 0 \end{aligned}$$

by (4.7) since $\Psi \in W^\perp$. Thus we have shown that $\phi f \in (W^\perp)^\perp$. But $(W^\perp)^\perp = W$ since W is closed; i.e. $\langle \mathrm{Ad}(\cdot)x, d\Lambda \rangle f \in W$ for every x in $L(S) \Rightarrow$

$$\langle \mathrm{Ad}(\cdot)x, d\Lambda \rangle f \in W \quad \text{for every } x \in L(S)^{\mathbf{C}}.$$

Next let $z \in L(S)^{\mathbf{C}}$, $\Psi \in W^\perp$. Choose M such that $|\langle \mathrm{Ad}(k)z, d\Lambda \rangle| < M$ for every k in K and choose $M_1 \ni \sum_{p=0}^n M^P/P! < M_1$ for $n = 1, 2, 3, \ldots$. Put

$$f_n(k) = \sum_{p=0}^{n} \frac{\langle \mathrm{Ad}(k)z, d\Lambda\rangle^p}{p!} \langle f(k), \Psi(k)\rangle,$$

$k \in K$. Then each $f_n \in L^1(K)$, by Hölder's inequality, and

$$|f_n(k)| \leq \|f(k)\| \, \|\Psi(k)\| M_1$$

for every n, k. Thus by the Lebesgue dominated convergence theorem

$$\begin{aligned}\left\langle e^{\langle \mathrm{Ad}(\cdot)z, d\Lambda\rangle} f, \Psi \right\rangle &= \int_K \lim_{n\to\infty} f_n(k)\, dk \\ &= \lim_{n\to\infty} \int_K f_n(k)\, dk = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} \left\langle \langle \mathrm{Ad}(\cdot)z, d\Lambda\rangle^p f, \Psi \right\rangle = 0\end{aligned}$$

since $\langle \mathrm{Ad}(\cdot)z, d\Lambda\rangle^p f \in W$ by (4.10) (and induction on p) and since $\Psi \in W^\perp$; i.e. $e^{\langle \mathrm{Ad}(\cdot)z, d\Lambda\rangle} f \in (W^\perp)^\perp = W$ for every z in $L(S)^{\mathbf{C}}$ implies $\phi f \in W$ for every ϕ in $A(d\Lambda)$.

We can now state the main result of this section.

THEOREM 4.11. *If $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$ then the nonunitary induced representation $I^{\Lambda,\mu}$ of G is topologically irreducible*; *see* (2.8) *and Definition* 4.2.

PROOF. Let $0 \neq W \subset L^2_\mu(K, H(\mu))$ be a closed $I^{\Lambda,\mu}$ invariant subspace. In particular W is $I^{\Lambda,\mu}|_K$ invariant; i.e. W is invariant under right translation by K. By Lemma 3.2 there is a continuous function f in $C_\mu(K, H(\mu))$ with $f \in W, f \neq 0$. Let $F = R(K)f =$ subspace of $C_\mu(K, H(\mu))$ spanned by the right K translates of f. Assuming that $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$ we can apply Lemma 3.1 to conclude that $A(d\Lambda)F$ is dense in $L^2_\mu(K, H(\mu))$; i.e.

(4.12) $$\overline{A(d\Lambda)F} \text{ (closure in } L^2) = L^2_\mu(k, H(\mu)).$$

Also W is $I^{\Lambda,\mu}|_S$ invariant. Thus if $\phi \in A(d\Lambda)$ and $k \in K$, $\phi I^{\Lambda,\mu}(k)f \in W$ by Theorem 4.5 since $I^{\Lambda,\mu}(k)f \in W$. In other words, $A(d\Lambda)F \subset W$ so because $W \subset L^2_\mu(K, H(\mu))$ is closed we have $L^2_\mu(K, H(\mu)) = \overline{A(d\Lambda)F}$ (by (4.12)) $\subset \overline{W} = W$, which proves irreducibility.

5. **Sufficient conditions for irreducibility.** If $A(d\Lambda)$ is uniformly dense in $C(K_\Lambda \setminus K)$ then $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$ since $C(K_\Lambda \setminus K)$ is dense in $L^2(K_\Lambda \setminus K)$ and since K is compact. Hence by Proposition 4.4, the Stone-Weierstrass theorem, and Theorem 4.11 we have

THEOREM 5.1. *If $A(d\Lambda)$ is closed under complex conjugation then $I^{\Lambda,\mu}$ is topologically irreducible.*

Now we shall describe some general conditions on $d\Lambda$ which will guarantee the closure of $A(d\Lambda)$ under complex conjugation. First note that the K invariant inner product $\langle\ ,\ \rangle$ on $L(S)^{\mathbf{C}}$ satisfies $\overline{\langle z_1, z_2\rangle} = \langle \bar{z}_1, \bar{z}_2\rangle$ for z_1, z_2 in $L(S)^{\mathbf{C}}$, where $-$ also denotes complex conjugation on $L(S)^{\mathbf{C}}$ with respect to

$L(S)$. Since $\mathrm{Ad}|_K$ leaves $L(S)$ invariant it follows that

$$\bar{\phi}_{d\Lambda,z}(k) = e^{\langle \mathrm{Ad}(k)\bar{z}, \overline{d\Lambda}\rangle} \tag{5.2}$$

for $(z, k) \in L(S)^{\mathbf{C}} \times K$; see (4.1). Recall that Λ is unitary if and only if $d\Lambda \in (-1)^{1/2}L(S)$. From (5.2) it is clear that $\overline{A(d\Lambda)} \subset A(d\Lambda)$ for Λ unitary so that Theorem 5.1 implies $I^{\Lambda,\mu}$ is irreducible if, in particular, Λ is unitary.

THEOREM 5.3. *If $k \to \langle \mathrm{Ad}(k)_\Lambda, d\Lambda\rangle$ is a real-valued function on K then $\overline{A(d\Lambda)} \subset A(d\Lambda)$.*

PROOF. Let $K\overline{d\Lambda}$ be the subspace of $L(S)^{\mathbf{C}}$ spanned by $\{\mathrm{Ad}(k)\overline{d\Lambda} | k \in K\}$. Thus we can write $L(S)^{\mathbf{C}} = K\overline{d\Lambda} \oplus (K\overline{d\Lambda})^{\perp}$. If $z \in L(S)^{\mathbf{C}}$ is arbitrary, $\bar{z} = \sum_j c_j \mathrm{Ad}(k_j)\,\overline{d\Lambda} + z_1$ where $z_1 \in (K\overline{d\Lambda})^{\perp}$, $k_j \in K$ and $c_j \in \mathbf{C}$. Then for every k in K

$$\langle \mathrm{Ad}(k)\bar{z}, \overline{d\Lambda}\rangle = \sum_j c_j \langle \mathrm{Ad}(kk_j)\,\overline{d\Lambda}, \overline{d\Lambda}\rangle$$

since $\langle \mathrm{Ad}(k)z_1, \overline{d\Lambda}\rangle = \langle z_1, \mathrm{Ad}(k^{-1})\overline{d\Lambda}\rangle = 0$. By (5.2)

$$\begin{aligned}\bar{\phi}_{d\Lambda,z}(k) &= \prod_j \exp\{c_j \langle \mathrm{Ad}(kk_j)\,\overline{d\Lambda}, \overline{d\Lambda}\rangle\} \\ &= \prod_j \exp\{c_j\, \overline{\langle \mathrm{Ad}(kk_j)d\Lambda, d\Lambda\rangle}\} = \prod_j \exp\{c_j \langle \mathrm{Ad}(kk_j)\, d\Lambda, d\Lambda\rangle\}\end{aligned}$$

since $k_1 \to \langle \mathrm{Ad}(k_1)d\Lambda, d\Lambda\rangle$ is real-valued by hypothesis. Thus

$$\begin{aligned}\bar{\phi}_{d\Lambda,z}(k) &= \prod_j \exp\{\langle \mathrm{Ad}(k)c_j \mathrm{Ad}(k_j)d\Lambda, d\Lambda\rangle\} \\ &= \prod_j \phi_{d\Lambda, c_j \mathrm{Ad}(k_j)d\Lambda}(k)\end{aligned}$$

for every k in K, which shows that $\bar{\phi}_{d\Lambda,z} \in A(d\Lambda)$ for every z in $L(S)^{\mathbf{C}}$. Since $A(d\Lambda)$ is spanned by the functions $\phi_{d\Lambda,z}$, $z \in L(S)^{\mathbf{C}}$, Theorem 5.3 follows.

PROPOSITION 5.4. *The function $k \to \langle \mathrm{Ad}(k)d\Lambda, d\Lambda\rangle$ is real-valued on K if any of the following conditions holds*:

(i) *$\langle d\Lambda, x\rangle\overline{\langle d\Lambda, y\rangle}$ is real for every x, y in $L(S)$; note $\overline{\langle d\Lambda, y\rangle} = \langle \overline{d\Lambda}, y\rangle$.*

(ii) *$K_\Lambda k K_\Lambda = K_\Lambda k^{-1} K_\Lambda$ for every k in K.*

(iii) *$d\Lambda = c\overline{d\Lambda}$ for some $c \in \mathbf{C}^x$.*

(iv) *Every closed K invariant subspace of $L^2(K_\Lambda \setminus K)$ is spanned by real functions; here K acts on $L^2(K_\Lambda \setminus K)$ by left translation.*

(v) *$f(k) = f(k^{-1})$ for every k in K for any $f \in C(K)$ which satisfies $f(m_1 k m_2) = f(k)$ for every k in K, m_1, m_2 in K_Λ.*

If the adjoint representation of K on $L(S)^{\mathbf{C}}$ is irreducible, then $k \to \langle \mathrm{Ad}(k)d\Lambda, d\Lambda\rangle$ is real-valued on K if and only if (i) *holds.*

PROOF. Choose a real orthonormal basis $\{e_j\}$ of $L(S)$. Then for each k in K

the matrix coefficients $\langle \mathrm{Ad}(k)e_j, e_i\rangle$ are real numbers. On the other hand $\{e_j\}$ is a complex orthonormal basis of $L(S)^{\mathbf{C}}$ so that $d\Lambda = \Sigma_j \langle d\Lambda, e_j\rangle e_j$ implies

$$\langle \mathrm{Ad}(k)\, d\Lambda, d\Lambda\rangle = \sum_{i,j} \langle \mathrm{Ad}(k)e_j, e_i\rangle \langle d\Lambda, e_j\rangle \overline{\langle d\Lambda, e_i\rangle}\,.$$

Hence $\langle \mathrm{Ad}(\cdot) d\Lambda, d\Lambda\rangle$ is real-valued on K if (i) holds. Conversely suppose $\langle \mathrm{Ad}(\cdot) d\Lambda, d\Lambda\rangle$ is real-valued on K and, in addition, the adjoint representation of K on $L(S)^{\mathbf{C}}$ is irreducible. Then by the Schur orthogonality relations, for every x, y in $L(S)$,

$$\langle d\Lambda, x\rangle \overline{\langle d\Lambda, y\rangle} = \dim_{\mathbf{C}} L(S)^{\mathbf{C}} \int_K \langle \mathrm{Ad}(k) d\Lambda, d\Lambda\rangle \langle \mathrm{Ad}(k)x, y\rangle\, dk$$

is real. Suppose (ii) holds. Then given k in K there exist m_1, m_2 in K_Λ such that $k = m_1 k^{-1} m_2$. Then

$$\begin{aligned}\langle \mathrm{Ad}(k)\, d\Lambda, d\Lambda\rangle &= \big\langle \mathrm{Ad}(m_1)\mathrm{Ad}(k^{-1})\mathrm{Ad}(m_2) d\Lambda, d\Lambda\big\rangle \\ &= \big\langle \mathrm{Ad}(k^{-1})\, d\Lambda, d\Lambda\big\rangle \quad \text{(see Proposition 4.3)} \\ &= \langle d\Lambda, \mathrm{Ad}(k)\, d\Lambda\rangle = \overline{\langle \mathrm{Ad}(k)\, d\Lambda, d\Lambda\rangle}\end{aligned}$$

which shows that $\langle \mathrm{Ad}(k) d\Lambda, d\Lambda\rangle$ is real for every k in K. Clearly (iii) implies (ii). Now (ii), (iv), (v) are equivalent by [3]. Q.E.D.

In the special case that (K, K_Λ) is a *symmetric pair* of compact type the double coset condition in (ii) of Proposition 5.4 can be replaced by a simple Weyl group condition. Thus suppose:

(5.5) K is connected, the center of K is discrete, σ is an involutive automorphism of K which fixes K_Λ pointwise, and K_Λ is open in the fixed point set of σ.

Let $\mathfrak{k}$, $\mathfrak{k}_\Lambda$ denote the Lie algebras of K, K_Λ, let σ_* denote the differential of σ and define $p = \{x \in \mathfrak{k} \mid \sigma_* x = -x\}$. Then $\mathfrak{k} = \mathfrak{k}_\Lambda \oplus p$ is a Cartan decomposition of $\mathfrak{k}$. Let $\mathfrak{a} \subset p$ be a maximal abelian subspace and let $M' = \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} \subset \mathfrak{a}\}$.

PROPOSITION 5.6. *Let* (K, K_Λ) *be subject to* (5.5). *Then the following are equivalent*

(i) *There is an m in M′ such that* $\mathrm{Ad}(m)|_{\mathfrak{a}} = -1$.

(ii) $K_\Lambda k K_\Lambda = K_\Lambda k^{-1} K_\Lambda$ *for every k in K.*

(i), (ii) *are equivalent to conditions* (iv), (v) *in Proposition* 5.4.

For a proof, which is valid even in a more general context, see [3].

COROLLARY 5.7. *If the symmetric space* $K_\Lambda \setminus K$ *is rank* 1 (*i.e.* $\dim \mathfrak{a} = 1$) *then* $k \to \langle \mathrm{Ad}(k) d\Lambda, d\Lambda\rangle$ *is real-valued on K.*

PROOF. Condition (i) of Proposition 5.6 holds so that Proposition 5.4 applies.

Another special case is the following.

PROPOSITION 5.8. *Suppose $z(K)$ is the center of K and $\overline{d\Lambda}$ lies in the $z(K)$ cyclic subspace of $L(S)^{\mathbf{C}}$ generated by $d\Lambda$. Then $A(d\Lambda)$ is closed under complex conjugation.*

PROOF. By hypothesis, $\overline{d\Lambda} = \sum_j c_j \mathrm{Ad}(k_j) d\Lambda$ where $k_j \in z(K)$, $c_j \in \mathbf{C}$. Then for every k in K, and z in $L(S)^{\mathbf{C}}$

$$\langle \mathrm{Ad}(k)\bar{z}, \overline{d\Lambda} \rangle = \sum \bar{c}_j \langle \mathrm{Ad}(k_j^{-1}) \mathrm{Ad}(k)\bar{z}, d\Lambda \rangle$$
$$= \Big\langle \mathrm{Ad}(k) \sum_j \bar{c}_j \mathrm{Ad}(k_j^{-1})\bar{z}, d\Lambda \Big\rangle$$

so by (5.2), $\bar{\phi}_{d\Lambda, z} \in A(d\Lambda)$.

The main results of this paper can now be stated. Again we assume that in (2.1) G is a connected Lie group and S is also connected. $L(S)$ is the Lie algebra of S, $d\Lambda \in L(S)^{\mathbf{C}}$ is given by (1.2), K_Λ is given by (2.2) (or Proposition 4.3), the bar – denotes complex conjugation on $\mathbf{C}$ or on $L(S)^{\mathbf{C}}$ with respect to $L(S)$, and $\langle\ ,\ \rangle$ is a K invariant complex inner product on $L(S)^{\mathbf{C}}$ as in §1.

THEOREM 5.9. *The nonunitary induced representation $I^{\Lambda,\mu} = \mathrm{ind}_{SK_\Lambda \uparrow G} \Lambda \otimes \mu$ (see (2.8)) is topologically irreducible if any one of the following conditions holds.*

(a) *$\langle d\Lambda, x\rangle \overline{\langle d\Lambda, y\rangle}$ is real for every x, y in $L(S)$; note $\overline{\langle d\Lambda, y\rangle} = \langle \overline{d\Lambda}, y\rangle$ for every y in $L(S)$.*

(b) *$d\Lambda = c\overline{d\Lambda}$ for some nonzero c in $\mathbf{C}$ (Thieleker).*

(c) *The function $k \to \langle \mathrm{Ad}(k) d\Lambda, d\Lambda\rangle$ is real-valued on K.*

(d) *$K_\Lambda k K_\Lambda = K_\Lambda k^{-1} K_\Lambda$ for every k in K.*

(e) *Every closed K invariant subspace of $L^2(K \setminus K_\Lambda)$, where K acts on $L^2(K \setminus K_\Lambda)$ by left translation, is spanned by real functions.*

(f) *$f(k) = f(k^{-1})$ for every k in K for any continuous function f on K which satisfies $f(m_1 k m_2) = f(k)$ for every k in K, m_1, m_2 in K_Λ.*

(g) *$\overline{d\Lambda}$ lies in the $z(K)$ cyclic subspace of $L(S)^{\mathbf{C}}$ generated by $d\Lambda$ where $z(K)$ is the center of K.*

(h) *(K, K_Λ) is a symmetric pair of compact type (see (5.5)) and (i) in Proposition 5.6 holds.*

(i) *$K_\Lambda \setminus K$ is a rank 1 symmetric space.*

(j) *The algebea $A(d\Lambda)$ (see Definition 4.2) is dense in $L^2(K_\Lambda \setminus K)$.*

(k) *$A(d\Lambda)$ is closed under complex conjugation.*

For various redundancies and equivalences among conditions (a) through (k) see Theorem 5.3, Proposition 5.4, Proposition 5.6, Corollary 5.7, and Proposition 5.8. The proof of Theorem 5.9 follows from Theorem 5.1 and the other results of this section.

THEOREM 5.10. *Suppose $\mu = 1$ is the trivial representation of K_Λ on $H(\mu) = \mathbf{C}$. Then $I^{\Lambda,1}$ is irreducible if and only if $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$.*

PROOF. The point is that $A(d\Lambda)$ is an $I^{\Lambda,1}$ invariant subspace of $L^2(K_\Lambda \setminus K)$. In fact if $(x, z) \in L(S) \times L(S)^{\mathbf{C}}$ is arbitrary then by (4.1) and (4.6)

$$I^{\Lambda,1}(\exp x)\phi_{d\Lambda,z} = \phi_{d\Lambda,x} \cdot \phi_{d\Lambda,z} = \phi_{d\Lambda,x+z}.$$

It follows then that $A(d\Lambda)$ is $I^{\Lambda,1}|_S$ invariant since S is connected. On the other hand $I^{\Lambda,1}|_K$ is right translation on $L^2(K_\Lambda \setminus K)$ and $A(d\Lambda)$ is always invariant under right translation. Hence $A(d\Lambda)$ and hence $\overline{A(d\Lambda)}$ (closure in $L^2(K_\Lambda \setminus K)$) are $I^{\Lambda,1}$ invariant. If $I^{\Lambda,1}$ is topologically irreducible then we must have $\overline{A(d\Lambda)} = L^2(K_\Lambda \setminus K)$. Conversely if $A(d\Lambda)$ is dense in $L^2(K_\Lambda \setminus K)$ then $I^{\Lambda,1}$ is topologically irreducible by Theorem 5.9(j).

The proof shows

COROLLARY 5.11. *If $\mu = 1$ is the trivial representation of K_Λ on $\mathbf{C}$ then $A(d\Lambda)$ is an $I^{\Lambda,1}$ invariant subspace of $L^2(K_\Lambda \setminus K)$.*

As an example consider the Euclidean group of motions $E_n = R^n. SO(n)$ of R^n, where $SO(n)$ acts linearly on R^n. If $\Lambda \neq 1$, $K_\Lambda = SO(n-1)$ and for $n > 3$, (d) of Theorem 5.9 holds (or (i)). Hence *the induced representations $I^{\Lambda,\mu}$ of E_n are always topologically irreducible for $n > 3$.* For $n = 2$ this is not the case; see p. 82.

6. A reducibility theorem. The nonunitary induced representations $L^{\Lambda,\mu} = \mathrm{ind}_{SK_\Lambda \uparrow G}\Lambda \otimes \mu$, contrary to the unitary case, can indeed be *reducible.* Theorem 6.1 of this section shows that when K is abelian, when $\mu = 1$, and when $d\Lambda$ is a "weight vector" of $\mathrm{Ad}|_K$, then $I^{\Lambda,1}$ is reducible. The latter hypotheses are satisfied, for example, when G is taken to be the two-dimensional Euclidean group of motions of the plane E_2 and Λ is the representaiton of $S = R^2$ whose infinitesimal character $d\Lambda$ is $[1/(-1)^{1/2}] \in \mathbf{C}^2 = L(S)^{\mathbf{C}}$; see page 82. This example is due to J. M. G. Fell and was the motivation for our formulation of Theorem 6.1. A similar example is given on page 83. On the other hand if $n > 3$ we have seen (page 81 that the induced representations $I^{\Lambda,\mu}$ of E_n are always topologically irreducible.

The following theorem is more or less known.

SCHOLIUM. *Let K be a compact abelian group and let $\hat{K}$ denote the dual group of unitary characters of K. Then K is connected if and only if $\hat{K}$ is torsion free (i.e. $\hat{K}$ has no elements of finite order other than the identity).*

We shall only need that *connectedness* implies no torsion. In fact when K is a compact, connected, abelian Lie group–the case which we shall consider–K is a torus and $\hat{K} = \mathbf{Z}^{\dim K}$ where $\mathbf{Z}$ is the ring of integers. Thus $\hat{K}$ is clearly torsion free.

THEOREM 6.1. *Suppose K is abelian and suppose* $\mathrm{Ad}|_K$ *leaves* $\mathbf{C}d\Lambda$ *invariant; i.e. there is a unitary character χ of K such that* $\mathrm{Ad}(k)d\Lambda = \chi(k)d\Lambda$ *for every k in K. Assume that $\chi \neq 1$. Then the nonunitary induced representation $I^{\Lambda,1}$ is reducible. In fact $\overline{A(d\Lambda)}$ (closure in $L^2(K_\Lambda \setminus K)$) is a nonzero, proper, closed $I^{\Lambda,1}$ invariant subspace; see Definition* 4.2 *and the paragraph which precedes the statement of Theorem* 5.9.

Note that if $\chi = 1$, then $I^{\Lambda,1}$ is topologically irreducible since in fact $K = K_\Lambda$ by Proposition 4.3; i.e. $I^{\Lambda,1} = \Lambda$. Note also that $\chi \in C(K_\Lambda \setminus K)$ since for every (m, k) in $K_\Lambda \times K$,

$$\chi(mk)d\Lambda = \mathrm{Ad}(m)\mathrm{Ad}(k)\, d\Lambda = \chi(k)\mathrm{Ad}(m)\, d\Lambda = \chi(k)\, d\Lambda.$$

PROOF. Suppose $f \in L^2(K_\Lambda \setminus K)$ and $z \in L(S)^{\mathbf{C}}$ are arbitrary. By Hölder's inequality $L^2(K) \subset L^1(K)$ so by (4.1) and by Lebesgue's theorem on dominated convergence

$$\begin{aligned}\int_K \phi_{d\Lambda,z}(k)\bar{f}(k)\, dk &= \int_K e^{\langle \mathrm{Ad}(k)z, d\Lambda\rangle}\bar{f}(k)\, dk\\ &= \int_K e^{\langle z, \chi(k^{-1})d\Lambda\rangle}\bar{f}(k)\, dk = \int_K e^{\langle z, d\Lambda\rangle\chi(k)}\bar{f}(k)\, dk\\ &= \sum_{n=0}^{\infty} \frac{\langle z, d\Lambda\rangle^n}{n!}\int_K \chi^n(k)\bar{f}(k)\, dk.\end{aligned} \tag{6.2}$$

Now take $f = \bar{\chi} \in L^2(K_\Lambda \setminus K)$; see above note. Then

$$\int_K \chi^n(k)\bar{f}(k)\, dk = \int_K \chi^{n+1}(k)\, dk = 0 \tag{6.3}$$

for $n = 0, 1, 2, 3, \ldots$, by the orthogonality relations. The point is that K is connected (K is a continuous image of G) so by the Scholium, or the remarks following it, $\hat{K}$ is torsion free; i.e. $\chi^{n+1} \neq 1$ since $\chi \neq 1$ and hence the orthogonality relations do apply. Thus (6.2) and (6.3) imply that for $f = \bar{\chi}$

$$\int_K \phi_{d\Lambda,z}(k)\bar{f}(k)\, dk = 0 \quad \text{for every } z$$

in $L(S)^{\mathbf{C}}$. In other words $\bar{\chi} \in A(d\Lambda)^\perp$ and since $\bar{\chi} \neq 0$ this shows that $A(d\Lambda)$ cannot be dense in $L^2(K_\Lambda \setminus K)$. Theorem 6.1 is now a consequence of Theorem 5.10 and Corollary 5.11. Q.E.D.

Now we shall consider Fell's example. Let $G = E_2$ be the Euclidean group of motions of the plane. Here $S = R^2$, $K = SO(2)$ is abelian, and $L(S)^{\mathbf{C}}$ may be identified with $\mathbf{C}^2$. Let Λ be the representation of S whose infinitesimal character is $d\Lambda = [1/(-1)^{1/2}] \in \mathbf{C}^2$. If

$$\sigma(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \in SO(2),$$

then $\mathrm{Ad}(\sigma(\theta))\, d\Lambda = e^{i\theta}\, d\Lambda$ for every θ in R. Therefore by Theorem 6.1 the corresponding nonunitary induced representation $I^{\Lambda,1}$ of E_2 is reducible.

PROPOSITION 6.4. *Let Λ be any nonunitary character of R^2. In particular since $\Lambda \neq 1$, $K_\Lambda = \{1\}$. The corresponding induced representation $I^{\Lambda,1}$ of $E_2 = R^2$. $SO(2)$ is topologically irreducible if and only if the infinitesimal character $d\Lambda$ is an* $\mathrm{Ad}|_{SO(2)}$ *cyclic vector.*

PROOF. If $d\Lambda$ is cyclic then the irreducibility assertion is a consequence of Theorem 5.9(g). Conversely let $SO(2)\, d\Lambda$ denote the $\mathrm{Ad}|_{SO(2)}$ cyclic subspace of $L(R^2)^{\mathbf{C}}$ generated by $d\Lambda$. Then since $L(R^2)^{\mathbf{C}}$ has dimension 2, $SO(2)\, d\Lambda = \mathbf{C}\, d\Lambda$ if $d\Lambda$ is not cyclic. Then $L^{\Lambda,1}$ is reducible by Theorem 6.1.

As another example we consider $K = SO(2)$ as a group of automorphisms of the *Heisenberg group* S. Typical elements of S, K are written, respectively,

$$[x, y, z] = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where x, y, z, θ are real numbers. The corresponding semidirect product G, for which S ia a normal factor, is called the *Oscillator group* and consists of 4×4 real matrices of the form

$$\begin{bmatrix} 1 & x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta & z \\ 0 & \cos\theta & \sin\theta & y \\ 0 & -\sin\theta & \cos\theta & -x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly the Lie algebra $L(S)$ of S consists of matrices of the form

$$[a, b, c] = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where a, b, c are real numbers. An easy calculation shows that

(6.5)

$$\mathrm{Ad}(\sigma(\theta))[a, b, c]$$

$$= \begin{bmatrix} 0 & a\cos\theta + b\sin\theta & -a\sin\theta + b\cos\theta & c \\ 0 & 0 & 0 & -a\sin\theta - b\cos\theta \\ 0 & 0 & 0 & -a\cos\theta - b\sin\theta \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $[a, b, c] \in L(S)^{\mathbf{C}}$; i.e. for $a, b, c \in \mathbf{C}$; here θ is any real number. Any character of S has the form $\Lambda_{\alpha,\beta}$ where $\Lambda_{\alpha,\beta}([x, y, z]) = e^{\alpha x + \beta y}$, $(\alpha, \beta) \in \mathbf{C}^2$, and $[x, y, z] \in S$. $d\Lambda_{\alpha,\beta}$ in $L(S)^{\mathbf{C}}$ is given by $d\Lambda_{\alpha,\beta} = [\bar{\alpha}, \bar{\beta}, 0]$. If $\Lambda_{\alpha,\beta} \neq 1$, $K_{\Lambda_{\alpha,\beta}} = \{1\}$. In particular if $d\Lambda_{\alpha,\beta} = [1, (-1)^{1/2}, 0]$ (i.e. $\alpha = 1$, $\beta =$

$-(-1)^{1/2})$ then by (6.5), $\mathrm{Ad}(\sigma(\theta))\, d\Lambda = \exp\{(-1\theta)^{1/2}\}\, d\Lambda$ for every real number θ and by Theorem 6.1 the corresponding induced representation $I^{\Lambda_{1,-(-1)^{1/2}},1}$ of G is reducible. On the other hand, for $(\alpha, \beta) \in \mathbf{C}^2$ arbitrary

$$\langle \mathrm{Ad}(\sigma(\theta))\, d\Lambda_{\alpha,\beta}, d\Lambda_{\alpha,\beta} \rangle = (|\alpha|^2 + |\beta|^2)\cos\theta + (\bar{\beta}\alpha - \bar{\alpha}\beta)\sin\theta$$

by (6.5). This is real if $\alpha\bar{\beta}$ is real. Hence by Theorem 5.9(c) *the corresponding induced representation* $I^{\Lambda_{\alpha,\beta},1}$ *of* G *is topologically irreducible whenever* $\alpha\bar{\beta}$ *is a real number*.

As we indicated in §1 (see Theorem 1.3) the complete irreducibility properties of the representations $I^{\Lambda,\mu}$ will be investigated in a forthcoming paper.

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