A SUPERPOSITION THEOREM FOR UNBOUNDED CONTINUOUS FUNCTIONS

BY

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ABSTRACT. Let R^n be the n -dimensional Euclidean space. We prove that there are $4n$ real functions φ_q continuous on R^n with the following property: Every real function f , not necessarily bounded, continuous on R^n , can be written $f(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)) + \sum_{q=2n+2}^{4n} h(\varphi_q(x))$, $x \in R^n$, where g, h are 2 real continuous functions of one variable, depending on f .

Let $I = [0, 1]$ be the closed unit interval and let $C(I^n)$, $n = 1, 2, \dots$, be the Banach space of real functions continuous on the cube I^n , with the usual norm. In 1957, Kolmogorov [11] proved the following theorem, giving an elegant solution to the celebrated Hilbert's Problem 13:

For every $n = 1, 2, \dots$ there are $n(2n + 1)$ continuous increasing functions φ_{pq} on I with the following property: Every $f \in C(I^n)$ may be written in the form

$$f(x_1 \cdots x_n) = \sum_{q=1}^{2n+1} g_q \left(\sum_{p=1}^n \varphi_{pq}(x_p) \right), \quad (x_1, \dots, x_n) \in I^n,$$

where the g_q are $2n + 1$ continuous functions of one variable, depending on f .

New research on Hilbert's Problem 13 has been carried out in three main directions:

(a) *Concerning the functions φ_{pq} .* See Fridman [4], [5], Hedberg [6], Henkin [7], Kahane [9], Kaufman [10], Sprecher [14], Vituškin [16].

(b) *Concerning the functions g_q .* See Bassalygo [1], Doss [2], [3], Kahane [8], [9], Lorentz [12], Sternfeld [15].

(c) *Concerning the basic space I^n .* See Ostrand [13]. We quote here Ostrand's theorem for we shall make use of it: Let K be a compact metric space of topological dimension n ; then there are $2n + 1$ real functions φ_q , continuous on K , with the following property: Every real function f continuous on K may be written:

$$f(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in K,$$

where g is a continuous real function of one variable.

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We shall be interested here in a new situation concerning the basic space, namely, instead of I^n , we shall consider the open unit cube I_0^n , or the open unit ball $\mathfrak{B}$ in R^n , or even R^n itself, and the (possibly unbounded) continuous function on I_0^n or $\mathfrak{B}$ or R^n .

We shall prove the following:

THEOREM. *For every fixed n , there are $2n - 1$ functions $\psi_i, i = 1, \dots, 2n - 1$, and $2n + 1$ functions $\varphi_q, q = 1, \dots, 2n + 1$, $4n$ functions in all, continuous on R^n , taking values in the semi-open interval $[0, 1)$, tending to 1 at infinity, with the following property: Every real f , continuous on R^n , may be written:*

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in R^n,$$

where h, g are real functions of one variable continuous on $[0, 1)$.

The same is true if R^n is replaced throughout by I_0^n or by the open unit ball $\mathfrak{B}$ in R^n , or any space homeomorphic to these, and the various forms of the theorem are equivalent. The proof will be carried out for the open unit ball $\mathfrak{B}$.

LEMMA 1. *Let $\mathfrak{R}$ be the closed region bounded by concentric spheres S, S' in R^n , and let $\delta > 0$. Then there are $2n + 1$ sets $C_\delta^q = \bigcup_i C_\delta^q(i)$ satisfying the conditions:*

- (i) *For every $q = 1, \dots, 2n + 1$, the sets $C_\delta^q(i)$ form a finite family of closed disjoint sets of diameter less than δ .*
- (ii) *Every $x \in \mathfrak{R}$ belongs to at least $n + 1$ of the sets $C_\delta^q, q = 1, \dots, 2n + 1$.*

These sets $C_\delta^q(i)$ are the intersection with $\mathfrak{R}$ of the well-known cubes considered by Kolmogorov in his classical paper [11]. There is great freedom in the choice of these sets; in the sequel of this paper we assume that they have been chosen once and for all.

LEMMA 2. *Let $\mathfrak{R}$ be the closed region bounded by the concentric spheres S, S' in R^n , of radius $\alpha, \alpha', \alpha < \alpha'$. There is a decreasing sequence $\varepsilon_m \rightarrow 0$ and there are $2n + 1$ functions φ_q , continuous on $\mathfrak{R}$, such that:*

- (1) $\alpha \leq \varphi_q \leq \alpha'$ in the region $\mathfrak{R}, q = 1, \dots, 2n + 1$,
- (2) $\varphi_q(x) = \alpha$ if and only if $x \in S$,
- (3) $\varphi_q(x) = \alpha'$ if and only if $x \in S'$,
- (4) *for all m , and for all sets $C_{\varepsilon_{m+1}}^q(i)$, not meeting the ε_m -neighborhood of $S \cup S'$, we have*

$$\varphi_q(C_{\varepsilon_{m+1}}^q(i)) \cap \varphi_{q'}(C_{\varepsilon_{m+1}}^{q'}(i')) = \emptyset$$

if either $q \neq q'$ or $q = q', i \neq i'$.

PROOF. Let δ_m be a decreasing sequence tending to 0. Choose two fixed real

functions ϕ_1, ϕ_2 , continuous on $\mathcal{R}$ such that $\phi_1 = \phi_2 = \alpha$ on S , $\phi_1 = \phi_2 = \alpha'$ on S' , $\alpha < \phi_1 < \phi_2 < \alpha'$ between S and S' .

Let A be the set of all $(2n + 1)$ -tuples (φ_q) of functions φ_q , continuous on $\mathcal{R}$ and satisfying the conditions

$$(5) \quad \phi_1 \leq \varphi_q \leq \phi_2.$$

Such functions φ_q satisfy necessarily conditions (1), (2), and (3) of the lemma. With the usual definition of the uniform norm $\|\cdot\|$, A is a complete metric space.

For an integer m , define the subset B_m of A as follows: The element (φ_q) of A belongs to B_m if there exists an integer $l > m$ with the property that for any sets $C_{\delta_l}^q(i)$, $C_{\delta_l}^{q'}(i')$ not meeting the δ_m -neighborhood of $S \cup S'$ we have

$$\varphi_q(C_{\delta_l}^q(i)) \cap \varphi_{q'}(C_{\delta_l}^{q'}(i')) = \emptyset$$

if either $q \neq q'$ or $q = q'$, $i \neq i'$.

We see easily that B_m is open in A .

We shall prove that B_m is dense in A . So let $(\varphi_q^0) \in A$ and let $\varepsilon > 0$ be given. We must show that we can find $(\varphi_q) \in B_m$ such that

$$(6) \quad \|\varphi_q - \varphi_q^0\| < \varepsilon, \quad q = 1, \dots, 2n + 1.$$

On the closed set $\mathcal{R}_m = \mathcal{R} \setminus \delta_m$ -nbhd of $S \cup S'$ we have $\phi_1 < \phi_2$. Hence, there is $\gamma > 0$ such that

$$\phi_1(x) < \phi_2(x) - \gamma, \quad x \in \mathcal{R}_m.$$

Choose l so large that the variation of ϕ_1, ϕ_2 and every φ_q^0 on any set of diameter $< \delta_l$ is less than $\varepsilon/2$ and also less than $\gamma/3$. For any set $C_{\delta_l}^q(i)$ lying in $\mathcal{R}_m$ put $\varphi_q(C_{\delta_l}^q(i)) = k^q(i)$ where the constants $k^q(i)$ are all in the open interval (α, α') , are all different, and

$$(6') \quad |\varphi_q(x) - \varphi_q^0(x)| < \varepsilon, \quad x \in C_{\delta_l}^q(i) \subset \mathcal{R}_m.$$

Moreover, since $\sup_{x \in C} \phi_1(x) < \inf_{x \in C} \phi_2(x) - \gamma/3$, where C stands for $C_{\delta_l}^q(i)$, we may choose the constants $k^q(i)$ such that

$$(5') \quad \sup_{x \in C} \phi_1(x) < k^q(i) < \inf_{x \in C} \phi_2(x)$$

for $C_{\delta_l}^q(i) \subset \mathcal{R}_m$. Next, put $\varphi_q = \alpha$ on S , $\varphi_q = \alpha'$ on S' , and then extend these φ_q , so far defined, to functions φ_q , continuous on $\mathcal{R}$, and satisfying conditions (5) and (6). This proves that B_m is dense in A .

Now the intersection $B = \bigcap_m B_m$ is dense by Baire's theorem and, hence, is nonempty. Choose a fixed $(\varphi_q) \in B$. Then, inductively, there is a subsequence ε_m of δ_m such that for any sets $C_{\varepsilon_{m+1}}^q(i)$, $C_{\varepsilon_{m+1}}^{q'}(i')$ not meeting the ε_m -neighborhood of $S \cup S'$, we have

$$\varphi_q(C_{\varepsilon_{m+1}}^q(i)) \cap \varphi_{q'}(C_{\varepsilon_{m+1}}^{q'}(i')) = \emptyset$$

if either $q \neq q'$ or $q = q'$, $i \neq i'$.

This completes the proof of Lemma 2.

LEMMA 3. Let $\mathcal{R}$ be the closed ring bounded by the two concentric spheres S, S' in R^n , of radius $\alpha, \alpha', \alpha < \alpha'$. Then there are $2n + 1$ functions $\varphi_q, q = 1, \dots, 2n + 1$, continuous on $\mathcal{R}$, taking values in the interval $[\alpha, \alpha']$, such that $\varphi_q(x) = \alpha$ iff $x \in S, \varphi_q(x) = \alpha'$ iff $x \in S', q = 1, \dots, 2n + 1$, with the following property:

To every function F , continuous on $\mathcal{R}$, vanishing on $S \cup S'$, such that $|F(x)| \leq M$ for $x \in \mathcal{R}$, there corresponds a g , continuous on $[\alpha, \alpha']$, such that

$$g(\alpha) = g(\alpha') = 0, \quad |g| \leq \frac{1}{2n+3} M,$$

and

$$(1) \quad \left| F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x)) \right| < \frac{2n+2}{2n+3} M \quad \text{for } x \in \mathcal{R}.$$

PROOF. We may assume $M = 1$. Since $\varepsilon_m \rightarrow 0$ (cf. Lemma 2), we may choose m so large that the oscillation of F , on any set of diameter $< \varepsilon_{m-1}$, is less than $\frac{1}{2} \cdot (2n+3)^{-1}$. Define g as follows:

If $F(x) > 0$ throughout a set $C_{\varepsilon_m}^q(i)$ not meeting the ε_{m-1} -nbhd of $S \cup S'$ put $g(\varphi_q(C_{\varepsilon_m}^q(i))) = (2n+3)^{-1}$. If $F(x) < 0$ throughout such a set, put $g(\varphi_q(C_{\varepsilon_m}^q(i))) = -(2n+3)^{-1}$. Because the closed sets $\varphi_q(C_{\varepsilon_m}^q(i))$ are disjoint, these constructions are consistent. Also, if for some $C_{\varepsilon_m}^q(i)$, the image $g(\varphi_q(C_{\varepsilon_m}^q(i))) = \pm(2n+3)^{-1}$ has been defined, then $\alpha, \alpha' \notin \varphi_q(C_{\varepsilon_m}^q(i))$, since $\varphi_q = \alpha, \alpha'$ only on $S \cup S'$, while $C_{\varepsilon_m}^q(i)$ does not meet $S \cup S'$. Therefore, it is consistent with the above construction to put $g(\alpha) = g(\alpha') = 0$. Finally, extend g to a continuous function on $[\alpha, \alpha']$, still bounded by $(2n+3)^{-1}$.

To prove that relation (1) holds we assume first that $F(x) > (2n+3)^{-1}$. By Lemma 1, there are at least $n+1$ sets $C_{\varepsilon_m}^q(i)$ containing x . By our choice of m , no such set can meet the ε_{m-1} -nbhd of $S \cup S'$ [$F = 0$ on $S \cup S'$ while $F > \frac{1}{2} \cdot (2n+3)^{-1}$ on such a set]. Therefore

$$(2) \quad F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x)) \leq 1 - \frac{n+1}{2n+3} + \frac{n}{2n+3} = \frac{2n+2}{2n+3}$$

since at least $n+1$ of the terms $g(\varphi_q(x))$ are equal to $(2n+3)^{-1}$. Also, the left side of (2) is larger than $(2n+3)^{-1} - (2n+1)/(2n+3)$, hence larger than $-(2n+2)/(2n+3)$, so that (1) is verified in this case.

The case $F(x) < -(2n+3)^{-1}$ is treated similarly.

Finally, if $|F(x)| \leq (2n+3)^{-1}$, the expression on the left side of (2) has absolute value not exceeding $(2n+2)/(2n+3)$ so that (1) holds also in this case.

The proof of Lemma 3 is now complete.

LEMMA 4. Let $\mathcal{R}$ be the closed ring bounded by the two concentric spheres S, S' , in R^n , of radius $\alpha, \alpha', \alpha < \alpha'$; then there are $2n + 1$ functions $\varphi_q, q = 1, \dots, 2n + 1$, continuous on $\mathcal{R}$, taking values in the interval $[\alpha, \alpha']$, such that $\varphi_q(x) = \alpha$ for $x \in S, \varphi_q(x) = \alpha'$ for $x \in S'$ with the following property:

To every function F , continuous on $\mathcal{R}$ and vanishing on $S \cup S'$, there corresponds a real function g , defined and continuous on the interval $[\alpha, \alpha']$, such that

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in \mathcal{R}.$$

Observe that for such a g we necessarily have $g(\alpha) = g(\alpha') = 0$

This is deduced from Lemma 3, using the very familiar Kolmogorov technique; see [11].

LEMMA 5. Let α_m be an increasing sequence of positive numbers tending to 1, and let S_m be the spheres in R^n of center 0, and radius α_m . Then there are $2n - 1$ functions $\psi_i, i = 1, \dots, 2n - 1$, continuous on the open unit ball $\mathcal{B}$ in R^n of center 0, taking values in $[0, 1]$, with the following property: To every f continuous on $\mathcal{B}$ there corresponds a function h , continuous on $[0, 1]$, such that

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in S_{2m}, m = 1, 2, \dots$$

PROOF. By Ostrand's theorem [13], since $\dim S_{2m} = n - 1$, there are $2n - 1$ functions $\psi_i^m, i = 1, \dots, 2n - 1$, continuous on S_{2m} , taking values in the interval $[\alpha_{2m}, \alpha_{2m+1}]$ with the property that every f continuous on S_{2m} may be written:

$$f(x) = \sum_{i=1}^{2n-1} h_m(\psi_i^m(x)) \quad \text{for } x \in S_{2m}$$

where h_m is continuous on $[\alpha_{2m}, \alpha_{2m+1}]$.

Let ψ_i be a continuous function on the open unit ball $\mathcal{B}$ such that $\psi_i(x) = \psi_i^m(x)$ for $x \in S_{2m}$, and taking values in $[0, 1]$.

If now h is a continuous function on $[0, 1]$ such that $h(y) = h_m(y)$ for $y \in [\alpha_{2m}, \alpha_{2m+1}]$, then

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) \quad \text{for } x \in S_{2m}, m = 1, 2, \dots$$

This proves Lemma 5.

THEOREM There are $4n$ functions $\psi_i, \varphi_q, i = 1, \dots, 2n - 1, q = 1, \dots, 2n + 1$, continuous on the open unit ball $\mathcal{B}$ in R^n , taking values in the semi-open interval $[0, 1]$, with the following property:

To every real function f , continuous on the open ball $\mathfrak{B}$, there correspond two real functions h, g , continuous on $[0, 1)$ such that

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in \mathfrak{B}.$$

PROOF. By Lemma 5, starting with any increasing sequence $\alpha_0 = 0, \alpha_1, \dots, \alpha_m, \dots$ of real numbers tending to 1 and with the spheres S_m of center 0 and radius α_m , we have $2n-1$ functions $\psi_i, i = 1, \dots, 2n-1$, continuous on $\mathfrak{B}$, taking values in $[0, 1)$ such that to every f continuous on $\mathfrak{B}$ there corresponds a real function h , continuous on $[0, 1)$, such that

$$(1) \quad f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in S_{2m}, m = 0, 1, 2, \dots$$

Next, by Lemma 4, if $\mathfrak{R}_m$ is the closed ring bounded by the two spheres S_{2m}, S_{2m+2} , then there are $2n+1$ functions $\varphi_q^m, q = 1, \dots, 2n+1$, continuous on $\mathfrak{R}_m$, taking values in the interval $[\alpha_{2m}, \alpha_{2m+2}]$ such that $\varphi_q^m(x) = \alpha_{2m}$ for $x \in S_{2m}$, $\varphi_q^m(x) = \alpha_{2m+2}$ for $x \in S_{2m+2}$ with the property that to every F continuous on $\mathfrak{R}_m$ and vanishing on $S_{2m} \cup S_{2m+2}$, there corresponds a real function g_m , defined and continuous on $[\alpha_{2m}, \alpha_{2m+2}]$, such that $g_m(\alpha_{2m}) = g_m(\alpha_{2m+2}) = 0$ and such that

$$(2) \quad F(x) = \sum_{q=1}^{2n+1} g_m(\varphi_q^m(x)), \quad x \in \mathfrak{R}_m.$$

Let $\varphi_q(x) = \varphi_q^m(x)$ for $x \in \mathfrak{R}_m, m = 0, 1, \dots, q = 1, \dots, 2n+1$. These functions φ_q are continuous on the open unit ball $\mathfrak{B}$ and take values in the semi-open interval $[0, 1)$.

Put now

$$(3) \quad F(x) = f(x) - \sum_{i=1}^{2n-1} h(\psi_i(x)), \quad x \in \mathfrak{B}.$$

Then F is continuous on $\mathfrak{B}$, and, by (1), $F(x) = 0$ for $x \in S_{2m}, m = 0, 1, \dots$. Put $g(x) = g_m(x)$ for $x \in [\alpha_{2m}, \alpha_{2m+2}]$. Since $g(\alpha_{2m}) = g(\alpha_{2m+2}) = 0$, this function is unambiguously defined and is continuous on $[0, 1)$. We have, by (2),

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in \mathfrak{R}_m, m = 0, 1, 2, \dots,$$

that is $F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), x \in \mathfrak{B}$, Finally, by (3),

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \quad x \in \mathfrak{B},$$

and the theorem is proved.

REFERENCES

1. L. A. Bassalygo, *On the representation of continuous functions of two variables by means of continuous functions of one variable*, Vestnik Moskov Univ. Ser. I Mat. Meh. **21** (1966), no. 1, 58–73. (Russian) MR 32 #7684.
2. R. Doss, *On the representaton of continuous functions of two variables by means of addition and continuous functions of one variable*, Colloq. Math. **10** (1963), 249–259. MR 27 #5882.
3. ———, *Representation of continuous functions of several variables*, Amer. J. Math. **98** (1976), 375–383.
4. B. L. Fridman, *Improvement in the smoothness of functions in the Kolmogorov superposition theorem*, Dokl. Akad. Nauk SSSR **177** (1967), 1019–1022 = Soviet Math. Dokl. **8** (1967), 1550–1553. MR 37 #663.
5. ———, *A nowhere dense space of linear superpositions of functions of several variables*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 814–846 = Math. USSR Izv. **6** (1972), 807–838. MR 47 #6969.
6. T. Hedberg, Appendix II in H. S. Shapiro's *Topics in approximation theory*, Lecture Notes in Math., Vol. 187, Springer-Verlag, Berlin and New York, 1971.
7. G. M. Henkin, *Linear superpositions of continuously differentiable functions*, Dokl. Akad. Nauk SSSR **157** (1964), 288–290 = Soviet Math. Dokl. **5** (1964) = 948–950. MR 29 #3596.
8. J. P. Kahane, *Sur les rearrangements de fonctions de la classe A*, Studia Math. **31** (1968), 287–293. MR 39 #6007.
9. ———, *Sur le théorème de superposition de Kolmogorov*, J. Approximation Theory **13** (1975), 229–234.
10. R. Kaufman, *Linear superposition of smooth functions*, Proc. Amer. Math. Soc. **46** (1974), 360–362. MR 50 #4861.
11. A. N. Kolmogorov, *On the representation of continuous functions of several variables by superposition of continuous functions of one variable and addition*, Dokl. Akad. Nauk SSSR **114** (1957), 953–956; English transl., Amer. Math. Soc. Transl. (2) **28** (1963), 55–59. MR 22 #2669; 27 #3760.
12. G. G. Lorentz, *Approximations of functions*, Holt, Rinehart and Winston, New York, 1966. MR 35 # 4642.
13. P. A. Ostrand, *Dimension of metric spaces and Hilbert's problem 13*, Bull. Amer. Math. Soc. **71** (1965), 619–622. MR 31 #1654.
14. D. A. Sprecher, *On the structure of continuous functions of several variables*, Trans. Amer. Math. Soc. **115** (1965), 340–355. MR 35 #1737.
15. J. Sternfeld, *Dimension theory and superpositions of continuous functions*, Israel J. Math. **20** (1975), 300–320.
16. A. G. Vituškin, *Proof of the existence of analytic functions of several complex variables which are not representable by linear superpositions of continuously differentiable functions of fewer variables*, Dokl. Akad. Nauk SSSR **156** (1964), 1258–1261 = Soviet Math. Dokl. **5** (1964), 793–796. MR 29 #2427.

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