SOME ONE-RELATOR HOPFIAN GROUPS

BY

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ABSTRACT. The group presented by

$$(a, t; t^{-1}a't = a^m)$$

is non-Hopfian if $l, m \neq \pm 1$ and $\pi(l) \neq \pi(m)$, where $\pi(l)$ and $\pi(m)$ denote the sets of prime divisors of l and m . By contrast, we prove that if w is a word of the free group $F(a_1, a_2)$ which is not primitive and not a proper power, then the group

$$(a_1, a_2, t; t^{-1}w't = w^m)$$

is Hopfian.

A group is Hopfian if every surjective endomorphism is an automorphism. The best-known example of a non-Hopfian group is probably the group

$$G = (a, t; t^{-1}a^2t = a^3)$$

given by G. Baumslag and D. Solitar in [2]. In fact, Baumslag and Solitar considered all groups

$$G(l, m) = (a, t; t^{-1}a^lt = a^m)$$

and we may summarise their conclusions as follows (incorporating a correction due to S. Meskin [7]):

If $|l| = 1$ or $|m| = 1$ or $|l| = |m|$ then $G(l, m)$ is residually finite and hence Hopfian.

Otherwise $G(l, m)$ is Hopfian if and only if $\pi(l) = \pi(m)$ (where $\pi(l)$, $\pi(m)$ denote the sets of prime divisors of l and m respectively).

These results show the delicacy of the Hopfian property and this is further illustrated by the theorem of G. Baumslag in [1] asserting that the group

$$G(l, m, n) = (a, t; (t^{-1}a^lt a^{-m})^n = 1)$$

is residually finite when $n > 1$ and l and m are coprime. In the same paper, Baumslag also states that if the one-relator group $H(1) = (a_1, a_2, a_3, \dots; r = 1)$, where r is not a proper power, is Hopfian, then so too is the one-relator group

$$H(n) = (a_1, a_2, a_3, \dots; r^n = 1)$$

Hopfian.

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We seek to investigate groups of the form

$$G = (a_1, a_2, \dots, a_n, t; t^{-1}w't = w^m)$$

where w is a word in the free group F on $a_1, a_2, \dots, a_n$ —and w is not a proper power in F . Meskin proved in [7] that if G is residually finite then $|l| = 1$ or $|m| = 1$ or $|l| = |m|$ and G. Baumslag (unpublished) has established the converse of this. In view of Baumslag's result, in determining whether or not G , as given above, is Hopfian we need only consider the situation in which $|l| \neq 1$, $|m| \neq 1$ and $|l| \neq |m|$.

If w is a primitive element of F then our given G is isomorphic to the free product of the Baumslag-Solitar group $G(l, m)$ and a free group of rank $(n - 1)$. In this situation G is Hopfian if and only if $G(l, m)$ is Hopfian. For I. M. S. Dey and H. Neumann proved in [4] that the free product of two finitely generated Hopfian groups is Hopfian—and it is easy to see that a free product is non-Hopfian if one of the factors is non-Hopfian.

Our contribution is the following theorem.

THEOREM. *Let $G = (a_1, a_2, t; t^{-1}w't = w^m)$ where w is a word in a_1, a_2 that is not primitive and not a proper power in the free group $F(a_1, a_2)$. Then G is Hopfian.*

At present we are unable to extend our results beyond the case $n = 2$.

We begin by summarising our notation. We write G for the group

$$(a_1, a_2, \dots, a_n; t^{-1}w't = w^m)$$

where w is a word in $a_1, a_2, \dots, a_n$ that is not primitive and not a proper power in the free group $F = F(a_1, a_2, \dots, a_n)$. Eventually we shall put $n = 2$ but some of our results are valid for arbitrary n . When convenient we shall sometimes write $l[-1] = l$ and $l[1] = m$. As noted previously we may assume $|l| \neq |m|$ —we shall not need to assume $|l| \neq 1$ and $|m| \neq 1$.

We write

$$K = (a_1, a_2, \dots, a_n; w = 1) \quad \text{and} \quad L = K * \langle t \rangle$$

for the free product of K with an infinite cyclic group on t . There is a canonical epimorphism $\theta: G \rightarrow L$ obtained by putting $w = 1$.

We shall rely to a great extent on the theory of HNN-extensions and the corresponding normal form theorem and conjugacy lemma (see Chapter II of C. F. Miller III [8]—where HNN-extensions are called Britton extensions).

LEMMA 1. *G is an HNN-extension of F . In particular F is a subgroup of G .*

PROOF. This is obvious. Q.E.D.

We write $u \sim v$ to mean that u and v are conjugate elements of G . We write $u \sim F$ to mean that there exists $y \in F$ such that $u \sim y$. Given $u \in G$ we

say that u is in *normal form* or is *t -reduced* if u does not contain a subword $t^{-\varepsilon}xt^{\varepsilon}$ with $x \in \langle w^{l[-\varepsilon]} \rangle$.

LEMMA 2. Let $\phi: G \rightarrow G$ be an endomorphism.

(i) Then there is an inner automorphism ψ such that $w\phi\psi \in \langle w \rangle$. If $w\phi \in F$, then ψ corresponds to an element of F .

(ii) If $w\phi \in \langle w \rangle$ and $w\phi \neq 1$ then $t\phi = w^{\rho_0}t^{\varepsilon_1}w^{\rho_1} \cdots t^{\varepsilon_r}w^{\rho_r}$ where $\rho_i \in \mathbb{Z}$, $\varepsilon_i = \pm 1$ and $\sum_{i=1}^r \varepsilon_i = 1$.

PROOF. (i) We know that $(w\phi)^l \sim (w\phi)^m$. Since $|l| \neq |m|$ the conjugacy lemma for HNN-extensions shows that $w\phi \sim F$. Without loss of generality we may assume $w\phi \in F$.

We have $(t\phi)^{-1}(w\phi)^l(t\phi) = (w\phi)^m$. Since $l \neq m$, $t\phi \notin F$. Let us write

$$t = x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \cdots x_{r-1} t^{\varepsilon_r} x_r$$

where $\varepsilon_i = \pm 1$ and $x_i \in F$. Then $x_0^{-1}(w\phi)^l x_0 \in \langle w^{l[-\varepsilon_1]} \rangle$. Hence $\langle x_0^{-1}(w\phi)x_0 \rangle \cap \langle w \rangle \neq 1$ giving $x_0^{-1}(w\phi)x_0 \in \langle w \rangle$, since F is free and w is not a proper power. This proves (i).

(ii) Let $w\phi = w^k$ so that $(t\phi)^{-1}w^{kl}(t\phi) = w^{km}$. The lemma follows from the fact that if u is any element of G and $u^{-1}w^p u = w^q$ then $u = w^{\rho_0}t^{\varepsilon_1}w^{\rho_1} \cdots t^{\varepsilon_r}w^{\rho_r}$ and $q = p(m/l)^{\sigma}$ where $\sigma = \sum_{i=1}^r \varepsilon_i$. This fact is established by induction on the number of occurrences of t in the normal form of u and relies heavily on the fact that if $x^{-1}w^r x = w^s$, where $x \in F$, then $r = s$ and $x \in \langle w \rangle$. (Again we do use the fact that w is not a proper power.)

LEMMA 3. Let $u = t^{\varepsilon_1}w^{\rho_1}t^{\varepsilon_2} \cdots w^{\rho_{r-1}}t^{\varepsilon_r}$ be in normal form and suppose that

(i) $u^{-1}w^l u = w^m$,

(ii) $G = \langle a, b, u \rangle$.

Then $u = t$, i.e. $r = 1$ and $\varepsilon_1 = 1$.

PROOF. We distinguish two cases.

Case 1. Suppose $l \nmid m$ and $m \nmid l$. Then $w^l \notin \langle w^m \rangle$ and $w^m \notin \langle w^l \rangle$. Since $u^{-1}w^l u = w^m$ we deduce that $\varepsilon_1 = 1 = \varepsilon_r$.

There must exist equalities of the form

$$(1) \quad t = y_0 u^{\eta_1} y_1 \cdots u^{\eta_s} y_s, \quad y_i \in F, \eta_i = \pm 1,$$

since $G = \langle a, b, u \rangle$. Among all such we consider one with s minimal—we claim that then $s = 1$.

Suppose $s > 1$; then there must exist i such that in reducing $u^{\eta_i} y_{i-1} u^{\eta_i} y_i u^{\eta_{i+1}}$ to normal form the occurrences of t in u^{η_i} are eliminated. For if this were false the normal form of the right-hand side of (1) would contain at least s occurrences of t .

The minimality of s is contradicted immediately if $u^{\eta_i} y_{i-1} u^{\eta_i} \in F$ or

$u^{\eta_i} y_i u^{\eta_{i+1}} \in F$. So part of u^{η_i} is cancelled by $u^{\eta_{i-1}}$ and part by $u^{\eta_{i+1}}$ (strictly only the occurrences of t are cancelled). Since $\varepsilon_1 = \varepsilon_r$ we have $\eta_{i-1} + \eta_i = 0 = \eta_i + \eta_{i+1}$.

Suppose $\eta_{i-1} = 1$; then we may write $u = u_1 u_2$ where $u_2 y_{i-1} u_2^{-1} = x_{i-1} \in F$ and $u_1^{-1} y_i u_1 = x_i \in F$. Now in view of the form of u it is clear that both x_{i-1} and x_i lie in $\langle w \rangle$. In particular $x_{i-1} x_i = x_i x_{i-1}$. Then

$$u y_{i-1} u^{-1} y_i u = u_1 x_{i-1} x_i u_2 = u_1 x_i x_{i-1} u_2 = y_i u y_{i-1}.$$

Again the minimality is contradicted. The argument when $\eta_{i-1} = -1$ is similar.

So $t = y_0 u^{\eta_1} y_1$; as u is in normal form we obtain $\eta_1 = 1$ and $r = 0$ as required.

Case 2. Suppose $l|m$ or $m|l$. Here there are two subcases but they can be treated similarly. So suppose that $m = lm_0$; then $m \nmid l$ as $|l| \neq |m|$.

From the fact that $u^{-1} w^l u = w^m$ we see that $\varepsilon_1 = 1$. However $w^m \in \langle w^l \rangle$ so we must allow for the possibility that $\varepsilon_r = -1$.

If in fact $\varepsilon_r = 1$ we can argue as in Case 1. So suppose $\varepsilon_r = -1$; we consider equations of the form

$$t = z_0 u^{\lambda_1} z_1 \cdots u^{\lambda_s} z_s, \quad z_i \in F, \lambda_i \in \mathbb{Z}, \lambda_i \neq 0.$$

As $G = \langle a, b, u \rangle$ such equations must exist. We again want to consider such an equation with s minimal.

We claim that in this event

(a) if $i > 1$ and $\lambda_i > 0$, then $z_{i-1} \notin \langle w^l \rangle$;

(b) if $i < s$ and $\lambda_i < 0$, then $z_i \notin \langle w^l \rangle$.

We note that for any $\lambda > 0$ and any $j \in \mathbb{Z}$, $u^{-\lambda} w^j u^\lambda = w^{jlm_0}$. Hence, if (a) is violated then for some $i > 1$

$$u^{\lambda_{i-1}} z_{i-1} u^\lambda = u^{\lambda_{i-1} + \lambda} z'_{i-1}, \quad \text{where } z'_{i-1} = u^{-\lambda} z_i u^\lambda.$$

Clearly the minimality of s is contradicted. Similarly if (b) is violated, then, for some $i < s$,

$$u^{\lambda} z_i u^{\lambda_{i+1}} = z'_i u^{\lambda + \lambda_{i+1}}$$

which again is contradictory.

Now write $u = u_1 u_2 u_1^{-1}$ where u_2 is cyclically t -reduced. Since $\varepsilon_1 = 1$ and $\varepsilon_r = -1$, $u_1 \notin F$. Also, since $u^{-1} w^l u = w^m$, $\sum_{i=1}^r \varepsilon_i = 1$ so that $u_2 \notin F$. Our equation becomes

$$t = z_0 u_1 u_2^{\lambda_1} u_1^{-1} z_1 \cdots u_1 u_2^{\lambda_s} u_1^{-1} z_s.$$

Suppose $s > 1$; there cannot exist i such that $u_1 u_2^{\lambda_i} u_1^{-1} z_i u_1 u_2^{\lambda_{i+1}} u_1^{-1} \in F$. It follows that there exists i such that in reducing

$$u_1 u_2^{\lambda_i} u_1^{-1} z_{i-1} u_1 u_2^{\lambda_i} u_1^{-1} z_i u_1 u_2^{\lambda_{i+1}} u_1^{-1}$$

to normal form, the occurrences of t in $u_1 u_2^\lambda u_1^{-1}$ are eliminated partly from the left and partly from the right. This means that $z_{i-1} \in \langle w' \rangle$ and $z_i \in \langle w' \rangle$. The former implies that $\lambda_i < 0$ and the latter that $\lambda_i > 0$.

We conclude that $t = z_0 u_1 u_2^\lambda u_1^{-1} z_1$ which is impossible since the right-hand side is in normal form and has at least three occurrences of t . Q.E.D.

COROLLARY 4. *Let $\phi: G \rightarrow G$ be a surjective endomorphism such that $F\phi \subseteq F$ and $w\phi \in \langle w \rangle$, $w\phi \neq 1$. Then $t\phi = w^{\rho_0} t w^{\rho_1}$.*

PROOF. By Lemma 2, $t\phi = w^{\rho_0} t^{\epsilon_1} w^{\rho_1} \cdots t^{\epsilon_r} w^{\rho_r}$. Let $u = t^{\epsilon_1} w^{\rho_1} \cdots w^{\rho_r - \epsilon_r}$; since $G = \langle a\phi, b\phi, t\phi \rangle$ and $F\phi \subseteq F$ we obtain $G = \langle a, b, u \rangle$. By Lemma 3, $u = t$. Q.E.D.

PROPOSITION 5. *Let $G = (a_1, a_2, \dots, a_n, t; t^{-1} w' t = w^m)$ where w is not primitive and not a proper power in the free group $F = F(a_1, a_2, \dots, a_n)$. If ϕ is a surjective endomorphism such that $F\phi \subseteq F$, then ϕ is an automorphism.*

PROOF. Certainly $w\phi \in F$ and, using Lemma 2, we may assume $w\phi \in \langle w \rangle$. More explicitly this asserts that for some $k \in \mathbb{Z}$

$$w(a_1\phi, a_2\phi, \dots, a_n\phi) = w^k$$

and this equality holds in F . Suppose $k \neq \pm 1$; by a theorem of G. Baumslag and A. Steinberg in [3], the rank of the group $\langle a_1\phi, a_2\phi, \dots, a_n\phi, w \rangle$ is at most $(n-1)$. Then certainly the rank of $\langle a_1\phi, a_2\phi, \dots, a_n\phi \rangle$ is at most $(n-1)$ and so G can be generated by at most n elements. This means that

$$L = (a_1, a_2, \dots, a_n; w = 1) * \langle t \rangle$$

can be generated by at most n elements. By Gruško's theorem

$$K = (a_1, a_2, \dots, a_n; w = 1)$$

is generated by at most $(n-1)$ elements. By a theorem of W. Magnus (Corollary 5.14.2 of [5]), K is a free group. By Whitehead's theorem (Theorem N.3 of Chapter 3 of [5]), w is primitive in F . Thus $w\phi = w^{\pm 1}$.

Now by the same argument as above, the group $\langle a_1\phi, a_2\phi, \dots, a_n\phi \rangle$ is free of rank n and thus ϕ is injective on F . Suppose $v\phi = 1$ and $v \neq 1$. Then certainly $v \notin F$. By the normal form theorem for HNN-extensions v must contain a subword $t^{-\epsilon} z t^\epsilon$ where $z \in \langle w'^{l-\epsilon} \rangle$. By Corollary 4, $t\phi = w^{\rho_0} t w^{\rho_1}$ and so v must contain a subword $t^{-\epsilon} y t^\epsilon$ such that $y\phi = z = w^{kl[l-\epsilon]}$. Now we know that $w\phi = w^\eta$, $\eta = \pm 1$ and hence $(w^{\eta kl[l-\epsilon]})\phi = y\phi$. As ϕ is injective on F , $y \in \langle w'^{l-\epsilon} \rangle$. Inductively, this means that $v \in F$ which is impossible. Q.E.D.

We have on the face of it proved a little more than was necessary to obtain Proposition 5. It would suffice to prove that $t\phi$ was of the form $w^{\rho_0} t^{\epsilon_1} x_1 \cdots x_{r-1} t^{\epsilon_r} w^{\rho_r}$ with $\epsilon_1 = 1 = \epsilon_r$. But much of Lemma 3 is needed for

this and we have obtained additional information about automorphisms of G .

We now turn to the situation where we must impose the additional hypothesis that $n = 2$. This is done in order to deal with the situation where $F\phi \not\subseteq F$ (even allowing for inner automorphisms). We shall show that, in these circumstances, ϕ cannot be surjective.

Given $u_1, u_2, u_3 \notin F$ we call (u_1, u_2, u_3) a *reduction triple* if in the process of reducing $u_1 u_2 u_3$ to normal form, the occurrences of t in u_2 are eliminated.

LEMMA 6. Let $u_i \notin F$, $i = 1, 2, \dots, s$, suppose that $u_1 u_2 \cdots u_s \sim F$. Then, for some i , (u_{i-1}, u_i, u_{i+1}) is a reduction triple—including the possibility that $i = 1$ or $i = s$ in which case $i - 1$ and $i + 1$ are to be interpreted modulo s .

PROOF. If the conclusion were false the cyclically t -reduced form of $u_1 u_2 \cdots u_s$ would not be free of occurrences of t . Q.E.D.

We begin with the situation in which $a_1 \phi \notin F$ and $a_2 \phi \in F$. Firstly we require a rather technical lemma.

LEMMA 7. Let $u \notin F$ and $y \in F$ be such that $(u^\varepsilon, y^r u^\eta, y^s u^v)$, where $r, s \in \mathbb{Z}$ and $\varepsilon, \eta, v = \pm 1$ is a reduction triple. Then the following possibilities occur:

- (i) (a) $\varepsilon + \eta = 0$ and $y\theta = 1$ or
(b) $\eta + v = 0$ and $y\theta = 1$,
- (ii) (a) $\varepsilon = \eta$, and $u^\varepsilon \theta = (y\theta)^{-r}$ or
(b) $\eta = v$, and $u^\eta \theta = (y\theta)^{-s}$,
- (iii) $\varepsilon = \eta = v$ and $u^\varepsilon = u_0 x u_0^{-1} y^{-r}$, for some $x \in F$ and some u_0 .

(Recall that $\theta: G \rightarrow L$ is the canonical epimorphism described at the outset.)

PROOF. It is clear that either $u^\varepsilon y^r u^\eta$ or $u^\eta y^s u^v$ is not in normal form.

Suppose the former occurs; if $\varepsilon + \eta = 0$ then $y^r \sim \langle w \rangle$ (in F). This gives $(y\theta)^r = 1$. Since w is not a proper power, K is torsion free (see [5]). Thus $y\theta = 1$ and (i)(a) holds.

Let $\varepsilon = \eta = 1$; if in fact $u y^r u \in F$ then analysis of the reduction of the $u y^r u$ to normal form gives rise to a system of equations in the following way. Let $u = x_0 t^{\kappa_1} x_1 \cdots t^{\kappa_p} x_p$. Then

$$\begin{aligned} x_p y^r x_0 &= w^{i_1/l[\kappa_p]}, & \kappa_p + \kappa_1 &= 0, \\ x_{p-1} w^{i_2/l[-\kappa_p]} x_1 &= w^{i_{p-1}/l[\kappa_{p-1}]}, & \kappa_{p-1} + \kappa_2 &= 0, \\ &\vdots \\ x_1 w^{i_2/l[-\kappa_2]} x_{p-1} &= w^{i_1/l[\kappa_1]}, & \kappa_1 + \kappa_p &= 0, \end{aligned}$$

where the group equations hold in F .

Now clearly $2(\kappa_1 + \cdots + \kappa_p) = 0$ and so $p = 2q$, $q \in \mathbb{Z}$. Then among the

group equations there occurs

$$x_q w^{i_q+1/l-\kappa_{q+1}} w_q = w^{i_q/l\kappa_q}.$$

Clearly $(x_q \theta)^2 = 1$ and so $x_q \theta = 1$, since K is torsion free.

Let $u_0 = x_0 t^{\kappa_1} x_1 \cdots x_{q-1} t^{\kappa_q}$. The above equations imply that

$$x_q t^{\kappa_{q+1}} \cdots t^{\kappa_p} x_p = x_q w^{-i_q+1/l-\kappa_{q+1}} u_0^{-1} y^{-r}$$

and hence that $u = u_0 \hat{x}_q u_0^{-1} y^{-r}$, where $\hat{x}_q = x_q w^{-i_q+1/l-\kappa_{q+1}}$. But clearly $\hat{x}_q \theta = 1$ and so $u \theta = (y \theta)^{-r}$. The argument is similar when $\varepsilon = -1$ and we have (ii)(a).

So suppose that $u^{\varepsilon} v^r u^{\eta} \notin F$. Then $u^{\eta} y^s u^{\nu}$ is not in normal form; if $\eta + \nu = 0$ we have (i)(b). So assume $\varepsilon = \eta = \nu$. If $u^{\eta} y^s u^{\nu} \in F$ we obtain (ii)(b). Otherwise, taking $\varepsilon = 1$, we obtain two systems of equations, viz.

$$\begin{aligned} x_p y^r x_0 &= w^{i_p/l\kappa_p}, & \kappa_p + \kappa_1 &= 0, \\ &\vdots \\ x_c w^{i_c+1/l-\kappa_{c+1}} x_{p-c} &= w^{i_c/l\kappa_c}, & \kappa_c + \kappa_{p+1-c} &= 0, \end{aligned}$$

from reduction in $uy^r u$, and

$$\begin{aligned} x_p y^s x_0 &= w^{k_p/l\kappa_p}, & \kappa_p + \kappa_1 &= 0, \\ &\vdots \\ x_{p+2-c} w^{k_{p+2-c}+1/l\kappa_{p+2-c}} x_{c-2} &= w^{k_{p+2-c}/l\kappa_{p+2-c}}, & \kappa_{p+2-c} + \kappa_{c-1} &= 0, \end{aligned}$$

from reduction in $uy^s u$.

As before $2(\kappa_1 + \cdots + \kappa_p) = 0$ and $p = 2q$. It is easy to see that $c < q + 1$ or $c = q + 1$ or $p + 2 - c < q + 1$. If $c < q + 1$ we obtain $u \theta = (y \theta)^{-r}$ and if $p + 2 - c < q + 1$ we obtain $u \theta = (y \theta)^{-s}$, arguing as above.

Let $c = q + 1$; then we still have $u = u_0 \hat{x}_q u_0^{-1} y^{-r}$ by substituting appropriately. We cannot, however, conclude that $\hat{x}_q \theta = 1$; but we have (iii). A similar argument is given when $\varepsilon = -1$.

LEMMA 8. *Let $u \notin F$, $y \in F$ be such that there is a word $z(a_1, a_2)$ such that $z(u, y) \sim F$. Then for any $g \in G$, $\langle u, y, g \rangle \neq G$.*

PROOF. Clearly we may assume that $z(a_1, a_2)$ is cyclically reduced. Rather less obviously we may assume that $z(a_1, a_2)$ is not a proper power. For suppose $z(a_1, a_2) = z_0(a_1, a_2)^p$. Then $z_0(u, y)^p \sim F$; we claim $z_0(u, y) \sim F$. Suppose not; then $z_0(u, y)$ can be expressed as $v_1 v_2 v_1^{-1}$ where v_2 is cyclically t -reduced, $v_2 \notin F$. But then $z_0(u, y)^p = v_1 v_2^p v_1^{-1}$ and $v_2^p \sim F$.

Let us write $z(a_1, a_2) = a_1^{\varepsilon_1} a_2^{r_1} a_1^{\varepsilon_2} a_2^{r_2} \cdots a_1^{\varepsilon_f} a_2^{r_f}$ where $\varepsilon_i = \pm 1$, $r_f \neq 0$, and $r_i \neq 0$ if $\varepsilon_i + \varepsilon_{i+1} = 0$. Then we have

$$u^{\varepsilon_j} y^r u^{\varepsilon_j} y^{r_2} \cdots u^{\varepsilon_j} y^{r_f} \sim F.$$

By Lemma 5, there is reduction triple $(u^{\varepsilon_j-1}, y^{r_j-u^{\varepsilon_j}}, y^{\varepsilon_j} u^{\varepsilon_j+1})$.

We claim that either $y\theta = 1$ or $u\theta \in \langle y\theta \rangle$. This is immediate if either (i) or (ii) of Lemma 7 occurs. So suppose that $\varepsilon_{j-1} = \varepsilon_j = \varepsilon_{j+1} = 1$ and $u = u_0 x u_0^{-1} y^{-r}$ where $r = r_{j-1}$. (The argument when $\varepsilon_{j-1} = \varepsilon_j = \varepsilon_{j+1} = -1$ is virtually identical to what follows.) Substituting in $z(u, y)$ we obtain a word of the form

$$y^{s_1} u_0 x^{\varepsilon_1} u_0^{-1} y^{s_1} \cdots u_0 x^{\varepsilon_f} u_0^{-1} y^{s_f}$$

where $s_i = r_i - \frac{1}{2}(\varepsilon_i + \varepsilon_{i+1})r$, $i = 1, 2, \dots, f$, $s_0 = 0$ or r according as $\varepsilon_1 = 1$ or -1 , and $s_f = r_f - r$ or r_f according as $\varepsilon_f = 1$ or -1 .

We firstly consider the possibility that $s_1 = s_2 = \cdots = s_{f-1} = 0$. Then $r_i = \frac{1}{2}(\varepsilon_i + \varepsilon_{i+1})r$, $i = 1, 2, \dots, (f-1)$. This means $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_f$, for otherwise there exists i such that $\varepsilon_i + \varepsilon_{i+1} = 0$ and, hence, $r_i = 0$. Thus $r_1 = \cdots = r_{f-1} = \pm r$. Assume $\varepsilon_1 = 1$ so that $r_1 = r$. Then $z(u, y) = u_0 x^f u_0^{-1} y^{f-r}$. Since $z(a_1, a_2)$ is not a proper power, $r_f \neq r$. The fact that $y_0 x^f u_0^{-1} y^{f-r} \sim F$ means that either $x^f \sim \langle w \rangle$ or $y^{f-r} \sim \langle w \rangle$ in F . Hence $u\theta = 1$ or $v\theta = 1$. If $\varepsilon_1 = -1$ and $r_1 = -r$, then $z(u, y) = y^r u_0 x^{-f} u_0^{-1} y^r$. Since $z(a_1, a_2)$ is not a proper power, $r_f \neq -r$. As above we obtain $u\theta = 1$ or $y\theta = 1$.

Now we may suppose that some $s_i \neq 0$, $i = 1, 2, \dots, f-1$. This means that $z(u, y)$ is equal after free cancellation to a word of the form

$$y^{s_1} u_0 x^{\lambda_1} u_0^{-1} y^{t_1} \cdots y^{t_{f-1}} u_0 x^{\lambda_f} u_0^{-1} y^{s_f}$$

where $\lambda_i, t_i \in \mathbb{Z}$, $q \geq 2$, $t_i \neq 0$, $i = 1, 2, \dots, f$, λ_1 has the same sign as ε_1 , and λ_q the same sign as ε_f . Note that the initial and terminal powers of y are unchanged. Since $z(u, y) \sim F$, this word is not cyclically reduced. If $u_0 x^{\lambda_1} u_0^{-1}$ or $u_0^{-1} y^{t_1} u_0$, for some i , is not in normal form we obtain respectively $x\theta = 1$ or $y\theta = 1$. The claim follows in this case. If the above word is in normal form then it follows that $u_0 y^{s_1+s_0} u_0^{-1}$ is not in normal form. Then of course $y\theta = 1$ except when $s_f + s_0 = 0$ (this cannot immediately be ruled out). If $s_f + s_0 = 0$, then $u_0^{-1} x^{\lambda_q+\lambda_1} u_0$ is not in normal form. If $\lambda_q + \lambda_1 \neq 0$, this gives $x\theta = 1$ and we are all right. So assume $\lambda_q + \lambda_1 = 0$. Then of course ε_1 and ε_f are of opposite sign. If $\varepsilon_1 = 1$ and $\varepsilon_f = -1$, then $s_0 = 0$ and $s_f = r_f$ so that $r_f = 0$. This is contradictory. If $\varepsilon_1 = -1$ and $\varepsilon_f = 1$, then $s_0 = r$ and $s_f = r_f - r$ so again $r_f = 0$ and we have a contradiction.

We can now show that $\langle u, y, g \rangle \neq G$. If $\langle u, y, g \rangle = G$, then $\langle u\theta, y\theta, g\theta \rangle = L$. This means L is generated by at most two elements. By our usual argument, this contradicts the fact that w is not primitive.

PROPOSITION 9. *Let $\phi: G \rightarrow G$ be an endomorphism such that $a\phi \notin F$ and $b\phi \in F$. Then ϕ is not surjective.*

PROOF. If ϕ is surjective then $G = \langle a\phi, b\phi, t\phi \rangle$. Since $w\phi \sim \langle w \rangle$ by Lemma 2 and $w\phi = w(a\phi, b\phi)$, Lemma 8 yields a contradiction.

For any $u \in G$, let $l_t(u)$ denote the number of occurrences of t in u .

PROPOSITION 10. Let $u, v \in G, u, v \notin F$ be such that

(1) there exists a word $z(a_1, a_2)$ such that $z(u, v) \sim F$,

(2) there exists $g \in G$ such that $G = \langle u, v, g \rangle$.

Then there exist $x, y \in F$ and $u_0 \in G$ such that $u = u_0xu_0^{-1}$ and $v = u_0yu_0^{-1}$.

PROOF. We proceed by induction on $l_t(u) + l_t(v)$. We therefore assume the proposition false and that among all pairs u and v satisfying the hypotheses, but not the conclusion, we have chosen a pair with $l_t(u) + l_t(v)$ minimal.

Suppose $u \sim F$, say $u = u_0xu_0^{-1}, x \in F$. Then we have

$$u_0x^mu_0^{-1}v^{n_1} \cdots u_0x^mu_0^{-1}v^{n_r} \sim F$$

where $z(a_1, a_2) = a_1^{m_1}a_2^{n_1} \cdots a_1^{m_r}a_2^{n_r}$. Let $\hat{v} = u_0^{-1}vu_0$ and $\hat{g} = u_0^{-1}gu_0$. Then $G = \langle x, \hat{v}, \hat{g} \rangle$ and $z(x, \hat{v}) \sim F$. By Lemma 8, $\hat{v} \in F$, i.e. $v = u_0yu_0^{-1}, y \in F$. Our minimality assumption is contradicted.

The argument when we assume $v \sim F$ is similar.

So suppose $u \not\sim F$ and $v \not\sim F$. Let $u = u_1u_2u_1^{-1}$ and $v = v_1v_2v_1^{-1}$ be in normal form with u_2 and v_2 cyclically t -reduced. Let $\hat{u} = v_1^{-1}u_1u_2u_1^{-1}v_1$ and $\hat{g} = v_1^{-1}gv_1$. Then $G = \langle \hat{u}, v_1, \hat{g} \rangle$ and $z(\hat{u}, v_2) \sim F$. If $\hat{u}$ (equivalently $u_1^{-1}v_1$) is not in normal form then the induction hypothesis gives $v_2 \sim F$ which is contradictory. So we may assume $\hat{u}$ is in normal form.

Case (i). Let $\hat{u}$ be cyclically t -reduced. Then, in practice, we have $u_1 = v_1 = 1$, i.e. $u = \hat{u}$ and $v = v_2$.

So we have

$$u^mu^{n_1} \cdots u^mv^{n_r} \sim F$$

where $z(a_1, a_2) = a_1^{m_1}a_2^{n_1} \cdots a_1^{m_r}a_2^{n_r}$ and u and v are cyclically t -reduced. By Lemma 6 we can find a reduction triple. The following are the only possibilities:

(1) $(u^\epsilon, u^\epsilon, v^\eta)$,

(2) $(u^\epsilon, v^\eta, v^\eta)$,

(3) $(u^\epsilon, v^\eta, u^\nu)$,

(4) $(v^\epsilon, u^\eta, v^\nu)$,

where $\epsilon, \eta, \nu = \pm 1$.

Suppose (1) occurs; then we can write $v^\eta = v_3v_4$ where $u^\epsilon v_3 \in F$. Thus $l_t(u^\epsilon v) < l_t(v)$. Now $G = \langle u, v, g \rangle$; also we can regard $z(u, v)$ as $z(u, u^{-\epsilon}(u^\epsilon v))$ if $\eta = 1$ and as $z(u, (vu^{-\epsilon})u^\epsilon)$ if $\eta = -1$. In either case we can construct a word $z^*(a_1, a_2)$ such that $z^*(u, u^\epsilon v^\eta) \sim F$. By the induction hypothesis $u \sim F$ which is contradictory. Clearly (2) is similar to (1).

Suppose (3) occurs. If $\epsilon + \nu = 0$ then either $u^\epsilon v^\eta$ or $v^\eta u^\nu$ is t -reduced—since

v is cyclically t -reduced. Suppose $v^\eta u^\eta$ is t -reduced. Then we can write $u^\epsilon = u_5 u_6$ where $u_6 v^\eta \in F$. Then $l_i(u^\epsilon v^\eta) < l_i(u)$ and $G = \langle u^\epsilon v^\eta, v, g \rangle$. Also we can transform $z(u, v)$ into $z^*(u^\epsilon v^\eta, v)$ and deduce that $v \sim F$ by the induction hypothesis. The alternative case is similar and so too is (4).

Case (ii). Suppose that $\hat{u}$ is not cyclically t -reduced. Write $u_3 = v_1^{-1} u_1$; thus $\hat{u} = u_3 u_2 u_3^{-1}$. We have

$$u_3 u_2^m u_3^{-1} v_2^{\eta_1} \cdots u_3 u_2^m u_3^{-1} v_2^{\eta_r} \sim F$$

and Lemma 6 ensures the existence of a reduction triple. The possibilities are:

- (1) $(u_2^\epsilon, u_3^{-1}, v_2^\eta)$,
- (2) $(v_2^\eta, v_2^\eta, u_3)$,
- (3) $(u_3^{-1}, v_2^\eta, u_3)$,
- (4) $(u_3^{-1}, v_2^\eta, v_2^\eta)$,
- (5) $(v_2^\eta, u_3, u_2^\epsilon)$.

We examine (1) in detail. Here we must have $v_2^\eta = v_3 v_4$ where $u_3^{-1} v_3 \in F$; write $x = u_3^{-1} v_3$. Then $z(x^{-1} u_4 x, (v_4 v_3)^\eta) \sim F$ and $G = \langle x^{-1} u_4 x, (v_4 v_3)^\eta, v_3^{-1} g v_3 \rangle$. Since $l_i(x^{-1} u_4 x) < l_i(\hat{u})$ we obtain $(v_4 v_3)^\eta \sim F$ and hence $v_2 \sim F$.

The remaining cases are dealt with in a similar manner and the proof is complete. Q.E.D.

We come finally to the proof of our main theorem.

PROOF OF THE THEOREM. Let ϕ be a surjective endomorphism. By Proposition 9, either $\langle a_1 \phi, a_2 \phi \rangle \subseteq F$ or $a_1 \phi \notin F$ and $a_2 \phi \notin F$. If $\langle a_1 \phi, a_2 \phi \rangle \subseteq F$ then by Proposition 5, ϕ is an automorphism. So let $a_1 \phi \notin F$ and $a_2 \phi \notin F$. By Lemma 2 $w \phi \sim \langle w \rangle$. We may apply Proposition 10 with $z = w$, $u = a_1 \phi$, $v = a_2 \phi$. Thus there exists an inner automorphism ψ such that $\langle a_1 \phi \psi, a_2 \phi \psi \rangle \subseteq F$. By Proposition 5, $\phi \psi$ is an automorphism and hence ϕ is an automorphism. Q.E.D.

Our results enable us to say something about the automorphism group of G . Let $S = \{ \mu \in \text{Aut } F : w\mu = w^\epsilon, \epsilon = \pm 1 \}$. S is the group of all automorphisms μ of F such that $\langle w\mu \rangle = \langle w \rangle$.

LEMMA A. Let $\mu \in S$. Then the mapping $\mu^*: G \rightarrow G$ defined by

$$\begin{aligned} x &\rightarrow x\mu, & x &\in F, \\ \mu^*: & \\ t &\rightarrow t \end{aligned}$$

is an automorphism of G .

PROOF. It is easy to see that $(t^{-1} w t) \mu^* = (w^m) \mu^*$. Since μ^* is clearly surjective it is an automorphism. Q.E.D.

LEMMA B. S is embedded in $\text{Aut } G$.

PROOF. The map sending $\mu \rightarrow \mu^*$ as above is easily seen to be an embedding. Q.E.D.

LEMMA C. Let $\phi \in \text{Aut } G$ be such that $F\phi \subseteq F$. Then there exists $x \in F$ such that

- (i) $x^{-1}(w\phi)x = w^\varepsilon, \quad \varepsilon = \pm 1,$
- (ii) $x^{-1}(t\phi)x = tw^p.$

PROOF. This is proved by an easy adaptation of the proof of Proposition 5. Q.E.D.

LEMMA D. Let $\phi \in \text{Aut } G$ be such that $F\phi \subseteq F$ and $w\phi = w^\varepsilon, \varepsilon = \pm 1$. Then $F\phi = F$.

PROOF. Suppose not; then, say, $a_1 \notin F\phi$. Since ϕ is surjective and $t\phi = w^p t w^p$, it is clear that $a_1 \in \langle F\phi, w, t \rangle$. We claim that $a_1 \in \langle F\phi, w \rangle$.

Let $a_1 = y_0 t^{\eta_1} y_1 \cdots t^{\eta_s} y_s$, where $y_i \in \langle F\phi, w \rangle, i = 0, 1, 2, \dots, s$. If $s = 0$, then there is nothing to prove. If $s > 0$, then the right-hand side is not t -reduced. So for some $i, t^{\eta_i} y_i t^{\eta_{i+1}} \in \langle w^{l-1} \rangle$. By induction, $a_1 \in \langle F\phi, w \rangle$.

However, $w = w^\varepsilon \phi, \varepsilon = \pm 1$, and thus $w \in F\phi$. This is contradictory. Q.E.D.

Write $\text{Inn } G$ for the group of inner automorphisms of G .

PROPOSITION E. Let $\phi \in \text{Aut } G$. Then there exists $\psi \in \text{Inn } G$ such that

- (i) $F\phi\psi = F$,
- (ii) $w\phi\psi = w^\varepsilon, \varepsilon = \pm 1$,
- (iii) $t\phi\psi = -w^p$, some $p \in \mathbb{Z}$.

PROOF. This is immediate from Proposition 9 and the Lemmas just above. Q.E.D.

We can now describe $\text{Aut } G$.

Let S^* be the copy of S embedded in $\text{Aut } G$. Let $\sigma: G \rightarrow G$ be the automorphism of G defined by

$$\begin{aligned} \sigma^*: \quad & x \rightarrow x, \quad x \in F, \\ & t \rightarrow tw. \end{aligned}$$

Then $\text{Aut } G = \langle S^*, \sigma, \text{Inn } G \rangle$. For let $\phi \in \text{Aut } G$; let ψ be such that (i), (ii) and (iii) of Proposition E are satisfied. Let μ be the restriction to F of $\phi\psi$. Then $\phi\psi = \mu^* \sigma^p$.

We note the following facts:

- (a) $\langle \sigma \rangle$ is infinite cyclic.
- (b) $\langle S^*, \sigma \rangle = S^* \times \langle \sigma \rangle$, the direct product of S^* and $\langle \sigma \rangle$.
- (c) $\text{Inn } G \cong G$ since G has trivial centre.

(d) $\langle S^*, \sigma \rangle \cap \text{Inn } G = \langle \beta^l \rangle$ where β is the inner automorphism of G corresponding to w .

To see (d), let $\phi \in \langle S^*, \sigma \rangle \cap \text{Inn } G$. If $\phi = \mu^* \sigma^p$, then for some $v \in G$ we have $v^{-1}tv = tw^p$ and also $v^{-1}wv = w^\varepsilon$, $\varepsilon = \pm 1$. If $|l|, |m| \neq 1$, the latter immediately yields $v \in \langle w \rangle$ and so $\phi \in \langle \beta \rangle$. Then the first of the two equalities yields $v \in \langle w^l \rangle$. Since $\beta^l = (\gamma^*)^l \sigma^{l-m}$ where γ is the inner automorphism of F corresponding to w , we have the desired conclusion.

Suppose that $l = 1$; then the equation $v^{-1}wv = w^\varepsilon$ yields only the fact v is of the form $w^{q_0} t^{q_1} w^{q_1} \cdots t^{q_s} w^{q_s}$ where $\sum_{i=1}^s q_i \geq 0$, $i = 1, 2, \dots, s$ (and possibly $s = 0$). However, a simple induction argument shows that this is inconsistent with $v^{-1}tv = tw^p$ unless $s = 0$. Then we can argue as before.

In conclusion we remark that our description of $\text{Aut } G$ is only a relative one since it depends on knowledge of S . J. McCool has proved in [6] that S is always finitely presented. Thus we know that $\text{Aut } G$ is always finitely generated. It remains to be seen whether or not $\text{Aut } G$ is also finitely related.

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