TOPOLOGICAL ENTROPY AT AN Ω -EXPLOSION

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ABSTRACT. In this paper an example is given of a C^2 map g from the circle onto itself, which permits an Ω -explosion. It is shown that topological entropy (considered as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at g .

1. Introduction. Let g denote any C^2 mapping of the circle onto itself which satisfies the following properties (see Figure 1):

FIGURE 1

- (1) g has an expanding fixed point e and a contracting fixed point c , and these are the only fixed points of g .
- (2) g preserves orientation at e and c .
- (3) g has nondegenerate singularities t and s , and these are the only singularities of g .
- (4) The points $e, t, s, g(s),$ and c are distinct and in order on the circle in the counterclockwise direction.
- (5) g is one-to-one on each of the intervals $(e, t), (t, s),$ and (s, e) . Here we

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use the notation (a, b) to denote the open arc from a counterclockwise to b .

$$(6) \ g(t) = e.$$

These properties imply that $\Omega(g) = \{e, c\}$, where $\Omega(g)$ denotes the nonwandering set (see [2], [5], or [7] for definition).

It is easy to see that g permits an Ω -explosion. By this we mean that for any neighborhood N of g in $C^2(S^1, S^1)$, there is a map $f \in N$ with $\Omega(f)$ infinite. See Proposition 9 in §4 for a proof.

Let ent denote topological entropy (see §2 for the definition). Our main result is the following:

THEOREM A. *The map $\text{ent}: C^2(S^1, S^1) \rightarrow \mathbb{R}$ is continuous at g .*

Theorem A implies that for any bifurcation through g , at the map g there is no sudden jump in the amount of action.

To prove Theorem A we first, in §3, obtain an upper bound for entropy (Theorem 6). Then, in §4, we construct a sequence (f_n) of maps with $\text{ent}(f_n) \rightarrow 0$ as $n \rightarrow \infty$. Finally, in §5, we prove Theorem A by using Theorem 6 to show that for arbitrarily large n if f is close enough to g in $C^2(S^1, S^1)$, then $\text{ent}(f) < \text{ent}(f_n)$.

2. Preliminary definitions and results. We first review the definition of topological entropy given in [1]. Let X be a compact space and $f: X \rightarrow X$ a continuous map. For any two open covers $\mathcal{A}$ and $\mathcal{B}$ of X , let $\mathcal{A} \vee \mathcal{B}$ denote $\{A \cap B: A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$, and let $f^{-1}(\mathcal{A})$ denote $\{f^{-1}(A): A \in \mathcal{A}\}$. Let $M_n(f, \mathcal{A})$ denote the minimum cardinality of a subcover of X of

$$(*) \quad \mathcal{A} \vee f^{-1}(\mathcal{A}) \vee \dots \vee f^{-(n+1)}(\mathcal{A}).$$

We set

$$\text{ent}(f, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(M_n(f, \mathcal{A}))),$$

where $\ln$ denotes the natural logarithm. It is easy to see that this limit exists and is finite (see [1]). Finally, we define the topological entropy of f by

$$\text{ent}(f) = \sup(\text{ent}(f, \mathcal{A}))$$

where the supremum is taken over all open covers $\mathcal{A}$ of X . If X is a metric space it suffices to consider any sequence of open covers whose diameter approaches zero (see [1]). By the diameter of an open cover $\mathcal{A}$ we mean the supremum of the diameters of the open sets in $\mathcal{A}$.

We now state some basic facts about topological entropy which will be used later. In each of the four propositions, f is a continuous map of a compact space X into itself. Proposition 2 follows immediately from the definition, and Proposition 4 follows from Proposition 3.

PROPOSITION 1 (SEE [5]). *If X is a metric space, $\text{ent}(f) = \text{ent}(f|_{\Omega(f)})$.*

PROPOSITION 2. If X is finite, $\text{ent}(f) = 0$.

PROPOSITION 3 (SEE [1]). If X_1 and X_2 are closed subsets of X , with $X_1 \cup X_2 = X$ and $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then

$$\text{ent}(f) = \max\{\text{ent}(f|_{X_1}), \text{ent}(f|_{X_2})\}.$$

PROPOSITION 4. If K is a closed subset of X , with $f(K) \subset K$, then $\text{ent}(f|_K) \leq \text{ent}(f)$.

We will assume the reader is familiar with the following terminology (see [2] or [7]); nonwandering set, expanding fixed point, contracting fixed point, and stable manifold of a contracting fixed point (denoted $W^s(c)$). For any point $x \in W^s(c)$, we will use the notation $\text{sism}(x)$ to denote the component of $W^s(c)$ which contains x .

3. An upper bound for entropy. The proof of Theorem 6 (which modifies a theorem of [2]) uses the following lemma from [2] (see [2, §3, Lemma 5]).

LEMMA 5. Let $f \in C^0(S^1, S^1)$. Let $K_1, \dots, K_n$ be proper closed intervals of S^1 , such that for each $i = 1, \dots, n-1$, $f|_{K_i}$ is a homeomorphism and $f(K_i) \subset K_{i+1}$. Let $\mathcal{Q}$ be a covering of $K_1 \cup \dots \cup K_n$ by finitely many open intervals, such that $\forall A \in \mathcal{Q}$ and $i = 1, \dots, n$, $A \cap K_i$ is an interval (or empty). Let $\text{card}(\mathcal{Q}) = k$ (card denotes cardinality). Then there is a subset $\mathcal{B}$ of $(*)$ which covers K_1 , with $\text{card}(\mathcal{B}) \leq n \cdot k$.

In Theorems 6 and A we will use the following definition. Let $\mathcal{C}$ be a finite collection of closed intervals on S^1 , $\mathcal{C} = \{I(1), \dots, I(p)\}$, and let $f \in C^0(S^1, S^1)$. We denote by $K_n(f, \mathcal{C})$ the number of distinct nonempty sets of the form

$$I(j_1) \cap f^{-1}(I(j_2)) \cap \dots \cap f^{-(n+1)}(I(j_n)),$$

where $j_i \in \{1, \dots, p\}$ for $i = 1, \dots, n$.

THEOREM 6. Let $f \in C^1(S^1, S^1)$, and let $\mathcal{C} = \{I(1), \dots, I(p)\}$ be a finite collection of proper closed intervals on S^1 . Let $W = S^1$ or $W = S^1 - (O_1 \cup O_2 \cup \dots \cup O_m)$ where O_i is a component of the stable manifold of a contracting periodic point c_i for $i = 1, \dots, m$. Suppose the following conditions hold.

- (1) $I(1) \cup \dots \cup I(p) = W$.
- (2) For $j = 1, \dots, p$, f maps $I(j)$ homeomorphically onto its image.
- (3) For any $i = 1, \dots, p$ and $j = 1, \dots, p$, $f(I(i)) \cap I(j)$ is an interval.

Then

$$\text{ent}(f) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{C}))).$$

PROOF. Let δ denote the minimum length of the intervals $S^1 - I(j)$ where $j = 1, \dots, p$. Let $\mathcal{Q}$ be any finite cover of $I(1) \cup \dots \cup I(p)$ by open

intervals with the diameter of $\mathcal{Q}$ less than δ . Let $k = \text{card}(\mathcal{Q})$.

Let

$$I(j_1, j_2, \dots, j_n) = I(j_1) \cap f^{-1}(I(j_2)) \cap \dots \cap f^{(-n+1)}(I(j_n))$$

where $j_i \in \{1, 2, \dots, p\} \quad \forall i = 1, \dots, n$. Then each nonempty $I(j_1, j_2, \dots, j_n)$ is a closed interval and f maps $I(j_1, j_2, \dots, j_n)$ homeomorphically into $I(j_2, \dots, j_n)$. For any fixed $I(j_1, j_2, \dots, j_n)$, by Lemma 5 (with $K_i = I(j_1, \dots, j_n)$), there is a subset of $(*)$ which covers $I(j_1, j_2, \dots, j_n)$ of cardinality at most $n \cdot k$.

Let $X = S^1 - (\cup_{i=1}^m W^s(c_i))$. Then X is a compact set with $f(X) \subset X$ and $f^{-1}(X) \subset X$.

Let $\mathcal{Q}(X)$ be the open cover of X defined by $\mathcal{Q}(X) = \{A \cap X : A \in \mathcal{Q}\}$. Then for any $I(j_1, j_2, \dots, j_n)$, the minimal number of open sets of

$$\mathcal{Q}(X) \vee f^{-1}(\mathcal{Q}(X)) \vee \dots \vee f^{(-n+1)}(\mathcal{Q}(X))$$

needed to cover $X \cap I(j_1, j_2, \dots, j_n)$ is equal to the minimal number of open sets of $(*)$ needed to cover $X \cap I(j_1, j_2, \dots, j_n)$. Also X is contained in the union of all the $I(j_1, j_2, \dots, j_n)$. Hence

$$M_n(f|X, \mathcal{Q}(X)) \leq (K_n(f, \mathcal{Q})) \cdot n \cdot k.$$

This implies that

$$\text{ent}(f|X, \mathcal{Q}(X)) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{Q}))).$$

Since the diameter of $\mathcal{Q}(X)$ may be taken to be arbitrarily small, we have

$$\text{ent}(f|X) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{Q}))).$$

But X contains all nonwandering points of f except for the finite set $\{c_1, c_2, \dots, c_m\}$. Thus, using Propositions 1-4,

$$\text{ent}(f) = \text{ent}(f|\Omega(f)) = \text{ent}(f|X) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{Q}))). \quad \text{Q.E.D.}$$

4. Construction of the sequence f_n . Let f_n be any map in $C^2(S^1, S^1)$ which satisfies properties (1)-(4) of g in §1 and the following:

(5') There are points $l \in (e, t)$ and $k \in (t, s)$ with $f(l) = f(k) = e$.

(6') g is one-to-one on each of the intervals (e, l) , (l, t) , (t, s) , and (s, e) .

(7') $f_n(t) \in \text{sism}(c^{-n})$ where c^{-n} is defined as follows. Let $c^0 = c$. Then for $i = 1, \dots, n$ let c^{-i} denote the unique inverse image (under f_n) of c^{-i+1} in (e, l) . Recall that $\text{sism}(c^{-n})$ denotes the component of $W^s(c)$ which contains c^{-n} .

The map f_3 is pictured in Figure 2. Let H_1, H_2, H_3, H_4, H_5 be the disjoint

FIGURE 2

closed intervals which form the complement of

$$\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \text{slsm}(c^{-2}) \cup \text{slsm}(c^{-3}) \cup \text{slsm}(t).$$

Note that $\text{slsm}(c) = (k, e)$. We can define a 5×5 matrix A_3 by $A_3(i, j) = 1$ if $f_3(H_i) \cap H_j \neq \emptyset$ and $A_3(i, j) = 0$ otherwise. Note that $f_3(H_i) \cap H_j \neq \emptyset$ implies $f_3(H_i) \supset H_j$. It is easy to see from Figure 2 that

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can define a matrix A_n analogously, and A_n is the $(n+2) \times (n+2)$ matrix

$$A_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the missing rows have ones on the super diagonal and zeros elsewhere.

The following proposition follows from Theorem D of [2].

PROPOSITION 7. $\text{ent}(f_n) = \ln(\lambda_n)$ where λ_n denotes the largest eigenvalue of A_n .

THEOREM 8. $\text{ent}(f_n) \rightarrow 0$ as $n \rightarrow \infty$.

PROOF. A straightforward calculation shows that for $n \geq 3$ the characteristic polynomial of A_n is $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Now λ_n is the largest root of p_n , and it is easy to see that $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$. The theorem now follows from Proposition 7. Q.E.D.

PROPOSITION 9. For any neighborhood N of g in $C^2(S^1, S^1)$, there is a map in N with positive entropy (and hence by Propositions 1 and 2 infinite nonwandering set).

PROOF. Let N be any neighborhood of g in $C^2(S^1, S^1)$. There is (for large enough n) a map $h \in N$ satisfying properties (1)–(4) and (5')–(7') of the map f_n in the sequence defined above. Hence $\text{ent}(h) = \ln(\lambda_n)$ where λ_n is the largest root of $p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)$. Clearly $\lambda_n > 1$, so $\ln(\lambda_n) > 0$. Q.E.D.

5. Proof of Theorem A.

THEOREM A. The map $\text{ent}: C^2(S^1, S^1) \rightarrow \mathbb{R}$ is continuous at g .

PROOF. Let $\varepsilon > 0$. Choose N large enough that $\text{ent}(f_N) < \varepsilon$ where f_N is the N th term of the sequence defined in §4. Choose $\delta > 0$ such that if $d(g, f) < \delta$, where d denotes a metric on $C^2(S^1, S^1)$, then the following hold.

(1) f has an expanding fixed point $e(f)$ and a contracting fixed point $c(f)$ and these are the only fixed points of f .

(2) f preserves orientation at $e(f)$ and $c(f)$.

(3) f has nondegenerate singularities $t(f)$ and $s(f)$ and these are the only singularities of f .

(4) The points $e(f)$, $t(f)$, $s(f)$, $f(s(f))$, and $c(f)$ are distinct and in order on the circle in the counterclockwise direction.

(5) Either (5A) holds or (5B) and (5C) hold.

(5A) $f(t(f)) \in [c(f), e(f)]$ and f is one-to-one on each of the intervals $(e(f), t(f))$, $(t(f), s(f))$ and $(s(f), e(f))$.

(5B) There are points $l(f) \in (e(f), t(f))$ and $k(f) \in (t(f), s(f))$ with $f(l(f)) = f(k(f)) = e(f)$.

Also f is one-to-one on each of the intervals $(e(f), l(f))$, $(l(f), t(f))$, $(t(f), s(f))$, and $(s(f), e(f))$.

(5C) $f(t(f)) \in (e(f), c^{-N}(f))$ where $c^{-N}(f)$ is defined as follows. Let

$c^0(f) = c(f)$. Then for $i = 1, \dots, N$ let $c^{-i}(f)$ denote the unique inverse image (under f) of $c^{-i+1}(f)$ in $(e(f), l(f))$.

Let $f \in C^2(S^1, S^1)$ with $d(g, f) < \delta$. We will show that $\text{ent}(f) < \varepsilon$. If property (5A) above holds, it follows that $\Omega(f) = \{e(f), c(f)\}$, and $\text{ent}(f) = 0$. Hence we may assume that (5B) and (5C) hold.

We define a collection of proper closed intervals $\mathcal{C}(f) = \{I_1, \dots, I_{N+2}\}$ as follows. Let $I_1, \dots, I_N$ be the components of the complement in $[e(f), c^{-1}(f)]$ of

$$\text{slsm}(c^{-1}(f)) \cup \dots \cup \text{slsm}(c^{-N}(f)).$$

Let $I_{N+1} = [l(f), t(f)]$ and $I_{N+2} = [t(f), k(f)]$. Then if W is the complement in S^1 of

$$\text{slsm}(c(f)) \cup \text{slsm}(c^{-1}(f)) \cup \dots \cup \text{slsm}(c^{-N}(f))$$

we have $I_1 \cup \dots \cup I_{N+2} = W$. Hence by Theorem 6,

$$\text{ent}(f) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{C}(f)))).$$

Let $H_1, \dots, H_{N+2}$ be the components of the complement in S^1 of the following set (defined with respect to f_N):

$$\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \dots \cup \text{slsm}(c^{-N}) \cup \text{slsm}(t).$$

Let $h = f_N$ and $\mathcal{D}(h) = \{H_1, \dots, H_{N+2}\}$.

It will be helpful for the reader to see Figure 2, in which $N = 3$ and H_1, H_2, H_3, H_4 , and H_5 are as indicated. In the case $N = 3$ one may also use Figure 2 for a picture of the intervals I_1, I_2, I_3, I_4 , and I_5 . To do this, of course, we must replace e, t, c , etc., by $e(f), t(f), c(f)$, etc. Then in the modified figure, I_1, I_2 , and I_3 are intervals corresponding to H_1, H_2 , and H_3 , while $I_4 = [l(f), t(f)]$ and $I_5 = [t(f), k(f)]$.

We may assume that the H_i are numbered as in Figure 2, and the I_i are numbered analogously. We claim that for each positive integer n , $K_n(f, \mathcal{C}(f)) \leq K_n(h, \mathcal{D}(h))$. To prove this claim, suppose that

$$I(j_1) \cap f^{-1}(I(j_2)) \cap \dots \cup f^{(-n+1)}(I(j_n)) \neq \emptyset.$$

Then $f(I(j_i)) \cap I(j_{i+1}) \neq \emptyset$ for $i = 1, \dots, n-1$. By construction, whenever $f(I(i)) \cap I(k) \neq \emptyset$, $A_n(i, k) = 1$ where A_n is the matrix defined in §4. Hence $h(H(j_i)) \supset H(j_{i+1})$ for $i = 1, \dots, n-1$. This implies that

$$H(j_1) \cap h^{-1}(H(j_2)) \cap \dots \cap h^{(-n+1)}(H(j_n)) \neq \emptyset.$$

This proves our claim that for each positive integer n , $K_n(f, \mathcal{C}(f)) \leq K_n(h, \mathcal{D}(h))$.

Let X be the complement in S^1 of the stable manifold of c (with respect to $h = f_N$). Let $\mathcal{D}(X) = \{H_1 \cap X, \dots, H_{N+2} \cap X\}$. Then $\mathcal{D}(X)$ is an open

cover of X , and for each positive integer n (since the H_i are pairwise disjoint),

$$K_n(h, \mathfrak{D}(h)) = M_n(h|X, \mathfrak{D}(X)).$$

We have for each positive integer n ,

$$K_n(f, \mathcal{C}(f)) \leq K_n(h, \mathfrak{D}(h)) = M_n(h|X, \mathfrak{D}(X)).$$

Also,

$$\text{ent}(f) \leq \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(K_n(f, \mathcal{C}(f)))).$$

Hence

$$\text{ent}(f) \leq \text{ent}(h|X, \mathfrak{D}(X)) \leq \text{ent}(h) < \varepsilon. \quad \text{Q.E.D.}$$

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