

CYLINDRICITY OF ISOMETRIC IMMERSIONS BETWEEN HYPERBOLIC SPACES

BY

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ABSTRACT. The motivation for this paper was to prove the following analogue of the Euclidean cylinder theorem: any umbilic-free isometric immersion $\eta: H^{n-1} \rightarrow H^n$ between hyperbolic spaces takes the form of a hyperbolic $(n-2)$ -cylinder over a uniquely determined parallelizing curve in H^n . Our approach is through the more general study of isometric immersions generated by one-parameter families of hyperbolic k -planes without focal points. A by-product of this study is a natural extension to curves in H^n of the notion of a parallel family of k -planes along a curve in H^n ; the extension is based on spherical symmetry of variation fields. Existence and uniqueness properties of this extended notion of parallelism are considered.

By a well-known theorem of Hartman and Nirenberg, every isometric immersion $\eta: E^{n-1} \rightarrow E^n$ between Euclidean spaces takes the form of a Euclidean $(n-2)$ -cylinder [2]. That is, $\eta(E^{n-1})$ may be obtained by parallel translation of an $(n-2)$ -plane in E^n along a base curve, where infinitely many choices of base curve are clearly possible. The analogous problem of determining the isometric immersions $\eta: H^{n-1} \rightarrow H^n$ between hyperbolic spaces of curvature -1 was posed by Nomizu in [4]. One may modify this problem by considering only the immersions which are umbilic-free, that is, have nowhere vanishing second fundamental form. Such an immersion is well known to be generated by *hyperbolic $(n-2)$ -planes* (complete, $(n-2)$ -dimensional, totally geodesic submanifolds) in H^n [5]. The umbilic-free isometric immersions of H^{n-1} into H^n were characterized by Ferus in [1]. Ferus' analysis is concerned with the orthogonal trajectories of the generators (for example, see Theorem 3 below), and does not touch on the question of parallelization.

This paper studies isometric immersions $\eta: M \rightarrow H^n$ generated by one-parameter families, without focal points, of hyperbolic k -planes. For such an immersion, the sectional curvatures of M satisfy $K_M < -1$; if $K_M = -1$ then M is isometric to an open subset of H^{k+1} . The original motivation for this study was to prove Theorem 6, which states that any umbilic-free isometric immersion of H^{n-1} into H^n takes the form of a hyperbolic $(n-2)$ -cylinder

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over a uniquely determined parallelizing curve in \bar{H}^n (see further remarks below). Here $\bar{H}^n = H^n \cup H_\infty^n$, where H_∞^n denotes the equivalence classes of asymptotic geodesic rays in H^n . This theorem offers an interesting analogue to the Euclidean cylinder theorem. The possibility of such an analogue was suggested by the classification of developable surfaces in H^3 , Theorem 2, [6].

In §1, for any immersion $\eta: M \rightarrow H^n$ generated by a family of hyperbolic k -planes, the existence of a uniquely determined *striction curve* is obtained (Theorem 1). The striction point on a given k -plane is defined in terms of the *variation field* of the family; this is a vector field defined on and orthogonal to the k -plane, and "tangent to the family". If furthermore $K_M = -1$, it is shown that the striction curve is also the unique parallelizing curve in \bar{H}^n of the family, so that the immersion takes the form of a hyperbolic k -cylinder (Theorem 2).

Here it is necessary to extend to curves in \bar{H}^n , the obvious notion of a *parallel family* of hyperbolic k -planes along a curve in H^n . This is done in a natural way, by proving first that a family $\{H_c\}_{c \in I}$ of k -planes is parallel along a regular curve $\sigma: I \rightarrow H^n$ if and only if for each c , the variation field is covariantly constant on the metric spheres in H_c about $\sigma(c)$ (Proposition 1). Then the extension to curves $\sigma: I \rightarrow \bar{H}^n$ is obtained simply by substituting "horospheres" for "metric spheres" if $\sigma(c) \in H_\infty^n$.

§2 considers foliations of H^n by $(n-1)$ -planes, and contains a criterion for generating such a foliation by parallel translation along a curve in H^n (Theorem 4). §3 studies isometric immersions of H^{k+1} into H^n generated by families of k -planes. For example, it is shown that any parallel family $\{H_c\}$ along a unit-speed curve $\sigma: R \rightarrow H^n$, for which the angle between the family and σ is bounded away from zero, generates an immersion of H^{k+1} into H^n (Theorem 5). Umbilic-free isometric immersions of H^{n-1} into H^n are also discussed in §3.

§4 examines the implications of the extension to curves in \bar{H}^n of the notion of parallel k -plane families. A differentiable manifold-with-boundary structure is placed on \bar{H}^n , namely that induced by the projective model of H^n . Along a regular curve σ lying on H_∞^n , the parallel translation problem has a unique solution for any allowable initial condition (Theorem 8). Of course, unique solutions always exist when σ lies in H^n . Solutions do not generally exist if σ takes both finite and infinite values (the question concerns piecing together solutions of two different problems, because of constraints which operate at H_∞^n). However, if a solution exists, it is unique (Corollary 4).

The method of §4 is to convert the study of hyperbolic k -plane families into the study of Euclidean k -plane families, by using the projective model of H^n . Instead of the parallelizing curve of a hyperbolic family, one may examine the family of focal sets of a Euclidean family (Theorem 7). The classical

approach to focal families (that is, to characteristics and envelopes) seems to be highly computational. Our approach is concise, coordinate-free, and new as far as we know.

Finally, in illustration of the preceding section, §5 examines families of hyperbolic k -planes having parallelizing curves in \overline{H}^n of certain special types.

Our conventions are as follows. On H^n , g or \langle , \rangle denotes the Riemannian metric, ∇ denotes covariant differentiation, and D denotes distance. *Geodesics* are always parametrized by arc-length. In §§1–3, no differentiable structure is assumed for \overline{H}^n . A *curve* is simply a map of an interval unless otherwise described. “Corollary 1.1” refers to part 1 of Corollary 1.

1. **Families of k -planes in H^n .** Let (T_1, \dots, T_k) be a smooth map of an open interval I into the bundle of k -frames on H^n , and H_c be the hyperbolic k -plane tangent to $(T_1, \dots, T_k)(c)$ for each $c \in I$. Then the parametrized k -planes $\{H_c\}_{c \in I}$ will be called a (*one-parameter*) *family* in H^n . The map (T_1, \dots, T_k) will be called a *frame* for the family, and its projection $\alpha = \pi \circ (T_1, \dots, T_k): I \rightarrow H^n$, a *base curve*.

Such a family of hyperbolic k -planes determines a *variation vector field* V_c on each of its members H_c , defined as follows: V_c is the component of $\beta'(c)$ orthogonal to H_c , for any smooth curve β in H^n satisfying $\beta(c) = p$ and $\beta(c + \delta) \in H_{c+\delta}$, $|\delta| < \epsilon$. To see that the orthogonal components for all such curves β agree, one may observe that if any one such β is transverse to H_c at p , then the $H_{c+\delta}$ locally determine a $(k + 1)$ -dimensional submanifold $M(p)$ through p in H^n , and are the level hypersurfaces of a regular function x on $M(p)$. Since by assumption every β satisfies $x \cdot \beta = \text{id}$, then with respect to the H^n -induced metric on $M(p)$, for every β the component of $\beta'(c)$ orthogonal to the level hypersurface H_c is $\|\text{grad } x\|^{-2} \text{grad } x$.

The zeroes of the variation field V_c form the *focal set* at c of the family $\{H_c\}$. The family is *regular* at c if V_c does not vanish identically.

Given a frame for the family, define $\eta: I \times \mathbb{R}^k \rightarrow H^n$ by

$$(1) \quad \eta(c, c_1, \dots, c_k) = \exp \sum_{i=1}^k c_i T_i(c).$$

Denote by x, x_i and X, X_i respectively the coordinate functions and vector fields on $I \times \mathbb{R}^k$. Then $\eta_* X$ at $x = c$ is an H^n -Jacobi field on H_c (that is, Jacobi along every geodesic in H_c). Since by definition, V_c coincides with the component of $\eta_* X$ orthogonal to H_c , it follows from the form of Jacobi fields in H^n that V_c is Jacobi. Therefore the length function

$$(2) \quad l_c = \|V_c\|$$

on H_c (carrying the H^n -induced metric) is convex, and strictly convex where nonvanishing. Since H_c is isometric to H^k , the following lemma applies:

LEMMA 1. For any convex function $f: H^k \rightarrow \mathbf{R}$, the limit of f along a geodesic ray in a fixed asymptote class is independent of the choice of ray.

PROOF. By convexity, $f \circ \gamma$ has a limit in $[-\infty, \infty]$ along any geodesic ray γ . For $j = 1, 2$, suppose $\lim f \circ \gamma_j = n_j$, where $n_1 < n_2$ and γ_1 and γ_2 are distinct asymptotic rays. Choose sequences $\{p_{1i}\}$ and $\{p_{2i}\}$ diverging to infinity along γ_1 and γ_2 respectively, and a point q in the hyperbolic 2-plane containing γ_1 and γ_2 , such that p_{2i} lies on the geodesic segment δ_i joining p_{1i} and q . Then $D(p_{1i}, p_{2i})$ is bounded while $D(p_{2i}, q)$ becomes arbitrarily large. Since $f \circ \delta_i$ is convex, $n_1 < n_2$ would contradict finiteness of $f(q)$. Therefore $n_1 = n_2$, as required.

Now let the variation function $\bar{l}_c: \bar{H}_c \rightarrow [0, \infty]$ be the extension, whose existence is guaranteed by Lemma 1, of the length function given by (2).

LEMMA 2. If a family $\{H_c\}$ of k -planes in H^n has no focal points at c , then the variation function at c has a single minimum point, which is the limit as $\delta \rightarrow 0$ of the point of \bar{H}_c closest to $H_{c+\delta}$.

PROOF. It is a consequence of strict convexity that l_c has at most one critical point and \bar{l}_c has exactly one minimum point. (If \bar{l}_c is finite at a point $p \in H_{c_\infty}$, then l_c must decrease on all geodesic rays to p .)

For $p \in H_c$, let $\beta_p(c + \delta)$ be the point of $H_{c+\delta}$ nearest p . Then it is not hard to see that $\beta_p = \exp_p kN$ for smooth function k and vector-valued function N , where $k(c) = 0$ and $N(c) \perp H_c$. Since $\beta'_p(c) = k'(c)N(c)$, which is orthogonal to H_c , then $V_{cp} = \beta'_p(c)$. It follows that

$$(3) \quad l_c(p) = \lim_{\delta \rightarrow 0^+} [D(p, H_{c+\delta})/\delta].$$

The function $D(\cdot, H_{c+\delta})$ on H_c is strictly convex, and therefore extends to \bar{H}_c with exactly one minimum point p_δ . From (3) and the fact that convergence of convex functions on Riemannian manifolds is uniform on compact subsets, we may conclude easily that p_δ converges to the minimum point of \bar{l}_c , as was claimed. This completes the proof of Lemma 2.

Next suppose $\{H_c\}_{c \in I}$ is a family without focal points. Let $M = \{(c, p): c \in I, p \in H_c\}$, and define $\eta: M \rightarrow H^n$ by $\eta(c, p) = p$. A frame (T_1, \dots, T_k) for the family determines an obvious coordinatization of M by $I \times \mathbf{R}^k$, for which η is expressed by (1); and thus determines a differentiable structure on M , with respect to which η is an immersion. This structure is independent of choice of frame. Letting M carry the η -induced Riemannian metric, we say that the isometric immersion $\eta: M \rightarrow H^n$ is generated by the family, and has generators H_c . In what follows, we shall sometimes suppress η , identifying M locally with its image in H^n .

Observe that by the Gauss equation, sectional curvatures of M satisfy $K_M \leq -1$, with equality holding if and only if the normalized variation field

$l_c^{-1}V_c$ is covariantly constant on H_c for all $c \in I$.

For each $c \in I$ the variation function \bar{l}_c has a unique minimum point, which by Lemma 2 it is reasonable to call the *striction point*. Note that the variation function on a given k -plane depends only up to a constant multiple on the choice of regular parametrization of the family (equivalently, of the base curve α), so the striction point is independent of that choice. If $\bar{l}_c(p) = 0$ for some $p \in H_{c_\infty}$, we say the family is *asymptotic* at p ; this terminology is justified by (3).

THEOREM 1. *Let $\eta: M \rightarrow H^n$ be the isometric immersion generated by a family $\{H_c\}_{c \in I}$ of hyperbolic k -planes without focal points. Then there is a uniquely determined curve $\sigma: I \rightarrow \bar{H}^n$ which satisfies: (i) $\sigma(c) \in \bar{H}_c$ for all $c \in I$; (ii) the generators are M -parallel along $\sigma|J$ when $\sigma|J$ lies in H^n ; (iii) the generators are asymptotic at $\sigma(c)$ when $\sigma(c) \in H_\infty^n$. The curve σ coincides with the curve of striction points.*

PROOF. Note that here we regard \bar{H}_c as lying in \bar{H}^n , in the natural way. Given $c_0 \in I$, it suffices to prove that the theorem holds for some open subinterval about c_0 . Therefore there is no loss of generality in assuming that M is coordinatized with respect to a base curve α orthogonal to the H_c . Thus we assume M carries coordinate vector fields X, X_1, \dots, X_k for which (suppressing η and η_*) the X_i are tangent to the H_c ; the restriction of X to each H_c is Jacobi in H^n ; along $\alpha, X = \alpha'$ and the X_i are covariantly constant in the normal bundle of α in M . Let γ be a geodesic in H_c satisfying $\gamma(0) = \alpha(c)$. By parallelism of $X_i, \nabla_{\alpha'(c)}(\sum c_i X_i)$ is orthogonal to H_c , where $\gamma'(0) = \sum c_i X_{i\alpha(c)}$ and ∇ is covariant differentiation in H^n . Therefore $\nabla_{\gamma'(0)}X$ is orthogonal to H_c . Since X is H^n -Jacobi along γ, X is orthogonal to H_c along γ . Therefore by the definition of variation field,

$$(4) \quad X|_{H_c} = V_c$$

and at any point of H_c ,

$$(5) \quad \langle \nabla_X X_i, X \rangle = \langle \nabla_X X, X \rangle = (X_i l_c) l_c.$$

Now let $\sigma(c)$ denote the striction point at c . If $\sigma|J$ lies in H^n then $\sigma|J$ is smooth (this follows from the implicit function theorem and the fact that $\sigma(c)$ is a nondegenerate critical point of l_c). Since $\{H_c\}$ has no focal points, $\sigma|J$ is regular. By (5), since $\sigma(c)$ is a critical point of l_c and H_c is totally geodesic,

$$\langle \nabla_{\sigma'} X_i, X \rangle = \langle \nabla_X X_i, X \rangle = 0.$$

Therefore the tangent k -planes to the H_c are M -parallel along $\sigma|J$, as required. Conversely, if $\sigma^*: J^* \rightarrow H^n$ is any curve in H^n satisfying $\sigma^*(c + \delta) \in H_{c+\delta}$ and parallelizing the H_c in M , then $\sigma^*(c)$ is critical for l_c , so $\sigma^*(c) = \sigma(c)$ for all $c \in J^*$.

By the Jacobi equation, if U_1, \dots, U_{n-k} are covariantly constant orthonormal fields defined on and orthogonal to H_c , then

$$(6) \quad V_c = \sum_{j=1}^{n-k} h_j U_j$$

where the h_j are hyperbolic functions on H_c , that is, where $(h_j \circ \gamma)'' = h_j \circ \gamma$ for every geodesic γ in H_c . If $\gamma(0) = q$, $V_{cq} = \sum a_j U_{jq}$ and $\nabla_{\gamma'(0)} V_c = \sum b_j U_{jq}$, then

$$(7) \quad h_j \circ \gamma(s) = a_j \cosh s + b_j \sinh s, \quad 1 \leq j \leq n - k.$$

If the striction point $\sigma(c)$ lies at H_∞^n and γ is a geodesic ray to $\sigma(c)$, the limit of $h_j \circ \gamma$ is finite for all j , by definition of $\sigma(c)$. But then $a_j = -b_j$ in (7), and the limit of h_j along γ vanishes for all j . Therefore $\bar{l}_c(\sigma(c)) = 0$. Conversely, if $\bar{l}_c(p) = 0$ for some $p \in H_{c\infty}$, obviously p is the striction point at c . This completes the proof of Theorem 1.

The proof of Theorem 1 has the following corollary, of which the second part is essentially due to Ferus [1]:

COROLLARY 1. *Let $\eta: M \rightarrow H^n$ be the isometric immersion generated by a family $\{H_c\}$ of hyperbolic k -planes without focal points. 1. The curvature vector in M of an orthogonal trajectory to the generators at any $q \in H_c$ is $(-l_c^{-1} \text{grad } l_c)_q$.*

2. If M has constant sectional curvature -1 , then the curvature in M of such a curve does not exceed 1.

PROOF. 1. Our notation will be that of the preceding proof. Since X is orthogonal to the H_c , the curvature vector in question is the orthogonal projection tangent to H_c of $\|X\|^{-2} \nabla_X X$ at q . But by (5) and (4),

$$\langle \|X\|^{-2} \nabla_X X, X_i \rangle = -l_c^{-1} X_i l_c = \langle -l_c^{-1} \text{grad } l_c, X_i \rangle.$$

2. If M has constant sectional curvature -1 , then in (6) we may take $h_1 = l_c$ and $h_j = 0$ for $j > 1$. Thus $l_c \circ \gamma(s) = a \cosh s + b \sinh s$ for any geodesic γ in H_c , by (7), where $|a^{-1}b| \leq 1$ since $l_c > 0$. If γ is chosen so that

$$\gamma(0) = q \quad \text{and} \quad \gamma'(0) = (-\|\text{grad } l_c\|^{-1} \text{grad } l_c)_q,$$

then $a = l_c(q)$, $b = -\|\text{grad } l_c\|(q)$, and $|a^{-1}b|$ is the required curvature at q by part 1.

REMARK 1. By Corollary 1.1, an orthogonal trajectory is an M -geodesic if and only if it coincides with the striction curve.

PROPOSITION 1. *A family $\{H_c\}$ of hyperbolic k -planes is H^n -parallel along an arbitrary regular base curve σ in H^n if and only if for each c , the variation field at c is covariantly constant on metric spheres in H_c about $\sigma(c)$.*

PROOF. If V_c is covariantly constant on metric spheres about some $q \in H_c$, then for h_j as in (6), $(h_j \circ \gamma)'(0) = 0$ for any geodesic γ in H_c with $\gamma(0) = q$; hence $b_j = 0$ for all j , in (7). It follows that V_c is covariantly constant on metric spheres in H_c about q if and only if

$$(8) \quad V_{cp} = a \cosh D(p, q)U_p$$

where U is a covariantly constant unit normal to H_c .

Now suppose $V_{\sigma(c)} \neq 0$. Locally the family generates an imbedded submanifold which, as in the proof of Theorem 1, carries coordinate vector fields X, X_1, \dots, X_k . Here the restriction of X to each member of the family coincides with the variation field, the X_i are tangent to the family, and (5) is satisfied. If V_c is covariantly constant on spheres about $\sigma(c)$, then $\sigma(c)$ is critical for the length function l_c . Furthermore, on H_c , $\nabla_X X$ is proportional to X by (8). Therefore $\nabla_X X_i = 0$ at $\sigma(c)$ by (5), so that $\nabla_{\sigma'(c)} X_i$ is tangent to H_c , as required. Conversely, if $\nabla_{\sigma'(c)} X_i$ is tangent to H_c , then so is $\nabla_X X_i$ at $\sigma(c)$, since $\sigma'(c)$ is transverse to H_c by assumption. Since $\langle \nabla_X X, X_j \rangle = -\langle \nabla_X X_j, X \rangle = 0$, it follows that $\nabla_X X = 0$ at $\sigma(c)$. Therefore by (7), V_c is covariantly constant on spheres about $\sigma(c)$.

Suppose instead that $V_{\sigma(c)} = 0$. Define a map $\eta: I \times \mathbb{R}^k \rightarrow H^n$ by (1), for any choice of frame over σ . It suffices to show that V_c is covariantly constant on spheres about $\sigma(c)$ if and only if the k vectors

$$Y_i = \nabla_{\sigma'(c)}(\eta_* X_i|_{x^1 = \dots = x^k = 0}) = \nabla_{\eta_* X_i}(\eta_* X|_{x=c}) \quad \text{at } \sigma(c)$$

are tangent to H_c . If V_c is constant on spheres about $\sigma(c)$, then V_c vanishes identically, by (8). Since V_c is the component of $\eta_* X|_{x=c}$ orthogonal to H_c , it follows that the Y_i are tangent to H_c . Conversely, if the Y_i are tangent to H_c , then for any geodesic γ with $\gamma(0) = \sigma(c)$, we have $b_j = 0$ in (7). Since $a_j = 0$ by assumption, V_c vanishes identically. Thus Proposition 1 is proved.

We shall call a family $\{H_c\}_{c \in I}$ of hyperbolic k -planes parallel in \overline{H}^n along a curve $\sigma: I \rightarrow \overline{H}^n$ if for each c , $\sigma(c) \in \overline{H}_c$, and the variation field at c is covariantly constant on metric spheres about $\sigma(c)$ if $\sigma(c) \in H_c$, or on horospheres about $\sigma(c)$ if $\sigma(c) \in H_{c\infty}$. (In the latter case, the variation function will be shown to vanish at $\sigma(c)$.) By Proposition 1, this definition is simply an extension of the usual one.

PROPOSITION 2. *Let $\eta: M \rightarrow H^n$ be the isometric immersion generated by a family $\{H_c\}_{c \in I}$ without focal points, and let $\sigma: I \rightarrow \overline{H}^n$ be the striction curve of the family. Then M has constant sectional curvature -1 if and only if the family is parallel in \overline{H}^n along σ .*

PROOF. Recall that $K_M = -1$ if and only if for each c , $l_c^{-1}V_c$ is covariantly constant. We claim that this occurs if and only if V_c is covariantly constant on spheres about $\sigma(c)$. Indeed, if $\sigma(c) \in H_c$ and V_c is constant on metric

spheres about $\sigma(c)$, then $l_c^{-1}V_c$ is constant by (8). If $\sigma(c) \in H_\infty^n$, then $l_c^{-1}V_c$ is constant on each geodesic ray to $\sigma(c)$ by the remarks following (7); thus constancy on horospheres about $\sigma(c)$ implies constancy on all of H_c .

For the converse, consider any H^n -Jacobi field on H_c of the form lU , where $l > 0$, U is a given covariantly constant unit normal to H_c , and l has a given value at some fixed point of H_c . Such a field is determined by choice of minimum point $q \in \overline{H}_c$ for l , since l is hyperbolic. Since l is a constant multiple of $\cosh D(\cdot, q)$ when q is finite, then l is constant on metric spheres about q when $q \in H_c$, and by a limit argument, on horospheres about q when $q \in H_{c\infty}$.

THEOREM 2. *A regular family $\{H_c\}_{c \in I}$ of hyperbolic k -planes is without focal points and generates an immersed submanifold of constant curvature -1 if and only if the family is parallel in \overline{H}^n along some curve $\sigma: I \rightarrow \overline{H}^n$. If such a parallelizing curve σ exists, it is unique.*

PROOF. It suffices, by Proposition 2, to show that if the family is regular and parallel in \overline{H}^n along some σ , then there are no focal points and σ is the striction curve of the family. But if V_c is covariantly constant on metric spheres about some $q \in H_c$, then V_c is given by (8). Thus V_c never vanishes because it does not vanish identically, and q is the striction point at c . If V_c is constant on horospheres about $q \in H_{c\infty}$, then again V_c never vanishes, because otherwise the zero set of V_c would be a hyperbolic r -plane for $0 < r < k$, and no such r -plane contains a horosphere of H_c . Furthermore, since the striction point is the *unique* minimum point of the length function \bar{l}_c , this point is certainly q .

REMARK 2. A submanifold of the type considered in Theorem 2 is always isometric to an open subset of H^{k+1} (see Proposition 3).

REMARK 3. We shall see in §4 that if \overline{H}^n is given the manifold-with-boundary structure induced by the projective model of H^n , then the parallelizing curve σ of Theorem 2 is smooth in \overline{H}^n . Furthermore, when σ lies at infinity there are constraints on the derivatives of σ . For example, if $k = 1$, σ' vanishes at infinity (Corollary 2); thus if σ lies at infinity on an interval, it is constant there, and the immersion takes the form of an *asymptotic cone*.

2. Foliations of H^n by hyperplanes. Let \mathcal{F} denote a foliation of H^n by hyperbolic $(n - 1)$ -planes. We shall require the following important theorem of Ferus concerning such foliations:

THEOREM 3 [1]. *1. For any $p \in H^n$ and any regular curve α in H^n with curvature $\kappa < 1$, the function $D(\alpha(t), p)$ has at most one critical point, namely a strict relative minimum point, and is unbounded if α has infinite length. In consequence:*

2. The family of orthogonal hyperplanes to a regular curve with $\kappa < 1$ is without self-intersections.

3. For a regular curve with $\kappa \leq 1$ and unbounded length in both directions, the family of orthogonal hyperplanes foliates H^n .

REMARK 4. A family $\{H_c\}$ of hyperplanes in H^n is without focal points if and only if it is without self-intersections. Corollary 1.2 and Theorem 3.2 imply that if there are no focal points then there are no self-intersections. On the other hand, since the variation field for an arbitrary hyperplane family, possibly with focal points, is of the form hU where U is covariantly constant and h is hyperbolic, the focal set in H_c is nonempty only if it is an $(n - 2)$ -plane on either side of which h takes opposite signs. Observing that h is given by the expression in (3), where D is interpreted as *directed* distance toward U , one may show that each focal point is the limit of intersection points of H_c and $H_{c+\delta}$ as $\delta \rightarrow 0$. Thus our remark is verified.

Given a foliation \mathcal{F} of H^n by hyperplanes, one always may choose a regular curve $\alpha: I \rightarrow H^n$ of infinite length which is everywhere orthogonal to \mathcal{F} . By Corollary 1.2 and Theorem 3.3, the family of hyperplanes orthogonal to α is all of \mathcal{F} . Therefore \mathcal{F} corresponds to a regular one-parameter family $\{H_c\}_{c \in I}$ (without focal points) of hyperplanes in H^n , and by Theorem 2, possesses a uniquely determined parallelizing curve $\sigma: I \rightarrow \overline{H^n}$.

The following theorem gives a necessary and sufficient condition that a given curve σ in H^n and initial transverse hyperplane determine by parallel translation a foliation of H^n .

THEOREM 4. Let $\sigma: \mathbf{R} \rightarrow H^n$ be a unit-speed curve, H_0 a hyperbolic $(n - 1)$ -plane through $\sigma(0)$, and H_c the parallel translate of H_0 along σ to $\sigma(c)$.

1. $\{H_c\}$ is without focal points if and only if H_c is transverse to $\sigma'(c)$ for all $c \in \mathbf{R}$.

2. Assuming that $\{H_c\}$ is without focal points, let $\theta(c) \in (0, \pi/2]$ be the angle between H_c and $\sigma'(c)$. Fix $q \in H^n$. Then $\{H_c\}$ foliates H^n if and only if

$$\int_0^\infty f(x) dx = \infty \quad \text{and} \quad \int_{-\infty}^0 f(x) dx = \infty,$$

where $f(x) = \sin \theta(x) \cosh D(q, \sigma(x))$. In particular:

3. $\{H_c\}$ foliates H^n if θ is bounded away from zero.

PROOF. 1. By (8) and the definition of variation field, $V_{\sigma'} = a \cdot \cosh D(p, \sigma(c)) U_p$, where U is a covariantly constant unit normal to H_c and $a = \langle \sigma'(c), U \rangle = \sin \theta(c)$. Thus V_c is nowhere vanishing if $\theta(c) \neq 0$ and vanishes identically if $\theta(c) = 0$.

2. Let $\alpha: I \rightarrow H^n$ be a maximal curve satisfying $\alpha(c) \in H_c$ and $\alpha'(c) = V_{c\alpha(c)}$ for all $c \in I$. Since $\{H_c\}_{c \in I}$ is without focal points and α is an

orthogonal trajectory, the curvature of α does not exceed 1. The length of α is unbounded in both directions if and, by Theorem 3.3, only if $I = \mathbf{R}$ and $\{H_c\}$ foliates H^n .

Suppose $\{H_c\}$ does not foliate H^n . We may assume α has finite length in the positive direction. Finiteness and maximality of α imply that the domain of α includes $[d, \infty)$ for some d . Then

$$\begin{aligned} \|\alpha'(x)\| &= \sin \theta(x) \cosh D(\alpha(x), \sigma(x)) \\ &\geq \sin \theta(x) \cosh [D(q, \sigma(x)) - D(q, \alpha(x))] \\ &\geq Lf(x) \end{aligned}$$

for some $L > 0$ independent of $x \in [d, \infty)$, since $D(q, \alpha(x))$ is bounded on $[d, \infty)$. Therefore $\int_d^\infty f(x) dx < \infty$.

Now, suppose $\{H_c\}$ foliates H^n . Choose α to pass through q . By Theorem 3.1, $D(\alpha, \sigma(x)) = D(\alpha(x), \sigma(x)) \leq D(q, \sigma(x))$ for each x . Therefore $\|\alpha'(x)\| \leq f(x)$, and since α has unbounded length in both directions, the required divergence of integrals follows.

EXAMPLE 1. Let $\sigma: \mathbf{R} \rightarrow H^n$ be any unit-speed curve with total curvature

$$\int \|\nabla_{\sigma'} \sigma'\| < \pi,$$

with each of

$$\int_0^\infty \|\nabla_{\sigma'} \sigma'\| \quad \text{and} \quad \int_{-\infty}^0 \|\nabla_{\sigma'} \sigma'\|$$

less than $\pi/2$; and let H_0 be orthogonal to σ . Then the family of parallel translates foliates H^n .

Indeed, suppose I is a maximal interval about 0 on which the family is transverse to σ , and let U be the unit normal to the family along $\sigma|I$ in the direction of $\sigma'(c)$. Then

$$|\langle \sigma', U \rangle| = |\langle \nabla_{\sigma'} \sigma', U \rangle| \leq \|\nabla_{\sigma'} \sigma'\| \cos \theta,$$

whence $|\theta'| \leq \|\nabla_{\sigma'} \sigma'\|$, where θ is the angle between σ' and H_c . Therefore our hypothesis implies that θ is bounded away from 0 on I . But then $I = \mathbf{R}$ and the claim follows by Theorem 4.3.

3. Isometric immersions of H^{k+1} into H^n . First we take note of the global structure of an immersed submanifold of H^n of the type considered in Theorem 2:

PROPOSITION 3. Let $\eta: M \rightarrow H^n$ be the isometric immersion generated by a family of hyperbolic k -planes without focal points. If $K_M = -1$ then M is isometric to an open submanifold of H^{k+1} .

PROOF. Since M and H^{k+1} are analytic and locally isometric, and since M

is simply connected, there exists an isometric immersion $\iota: M \rightarrow H^{k+1}$ ([3, pp. 255 and 263]). By assumption, M is foliated by a family $\{\tilde{H}_c\}$ of complete, totally geodesic submanifolds (where $\eta(\tilde{H}_c) = H_c$). But then $\iota(\tilde{H}_c)$ is a k -plane in H^{k+1} for each c . Since ι is an immersion, the family $\{\iota(\tilde{H}_c)\}$ is without focal points. By Remark 3, $\{\iota(H_c)\}$ is without self-intersections, and it follows that ι is one-one.

REMARK 5. Consider an isometric immersion $\eta: M \rightarrow H^n$ such as is described in Proposition 3, and identify M with an open submanifold of H^{k+1} . Then M is foliated by a family $\{\tilde{H}_c\}$ of hyperbolic hyperplanes, where $\eta(\tilde{H}_c) = H_c$. Denote by $\tilde{\sigma}: I \rightarrow \bar{H}^{k+1}$ and $\sigma: I \rightarrow \bar{H}^n$ the parallelizing curves of the families $\{\tilde{H}_c\}$ and $\{H_c\}$ respectively (the existence of $\tilde{\sigma}$ and σ is guaranteed by Theorem 2), and by $\eta_c: (\tilde{H}_c) \rightarrow \bar{H}_c$ the natural identification extending $\eta|_{\tilde{H}_c}$. If \tilde{V}_c and V_c are the variation fields at c for the respective families, then $\eta_*\tilde{V}_c$ is orthogonal to H_c since η is isometric. Therefore $\eta_*\tilde{V}_c = V_c$, and $\eta_c \circ \tilde{\sigma}(c) = \sigma(c)$ for all $c \in I$.

The following construction of a class of isometric immersions of H^{k+1} into H^n depends upon Theorem 4:

THEOREM 5. Let $\sigma: \mathbf{R} \rightarrow H^n$ be a unit-speed curve, H_0 a hyperbolic k -plane through $\sigma(0)$, and H_c the parallel translate of H_0 along σ to $\sigma(c)$. Let $\theta(c)$ be the angle between $\sigma'(c)$ and H_c , for each $c \in \mathbf{R}$. If θ is bounded away from zero, then the family $\{H_c\}$ generates an isometric immersion $\eta: H^{k+1} \rightarrow H^n$. The immersion η is umbilic-free if and only if the curvature vector in H^n of σ at c is transverse to the span of $\sigma'(c)$ and H_c for each $c \in \mathbf{R}$.

PROOF. The family $\{H_c\}$ is regular since it is transverse to σ . Therefore by Theorem 2, $\{H_c\}$ generates an isometric immersion $\eta: M \rightarrow H^n$ where M has curvature -1 . By Proposition 3, M may be identified with an open submanifold of H^{k+1} . Since η is an isometric immersion, we have a unit-speed curve $\tilde{\sigma}: \mathbf{R} \rightarrow M$, and a family $\{\tilde{H}_c\}$ of hyperbolic k -planes which foliates M and is parallel along $\tilde{\sigma}$, and whose angle with $\tilde{\sigma}$ is bounded away from zero (where $\eta \circ \tilde{\sigma} = \sigma$ and $\eta(\tilde{H}_c) = H_c$). But then by Theorem 4.3, $\{\tilde{H}_c\}$ foliates H^{k+1} , and so $M = H^{k+1}$.

Let K be the open submanifold of H^{k+1} on which the nullity of the second fundamental form of η takes its minimum value ν . Since all tangent vectors to the \tilde{H}_c are nullity vectors, $k < \nu < k + 1$. Recall that the nullity foliation of K is well known to have complete leaves: for the present situation, namely immersion of H^{k+1} into H^n , completeness was proved in [5]. The transversality hypothesis on the curvature vectors of σ implies that nullity on K is k and $\tilde{\sigma}$ lies in K . Therefore K contains the complete k -plane \tilde{H}_c for every $c \in \mathbf{R}$, and so $K = H^{k+1}$ and η has no umbilics. Conversely, if there are no

umbilics, then $\nu = k$ along $\bar{\sigma}$, and the transversality condition follows. This completes the proof of Theorem 5.

Finally we consider umbilic-free isometric immersions $\eta: H^{n-1} \rightarrow H^n$. For such an immersion, H^{n-1} is foliated by hyperbolic $(n - 2)$ -planes tangent to the nullity spaces of the second fundamental form. As in the previous section, the foliating $(n - 2)$ -planes may be regarded as a regular one-parameter family $\{\tilde{H}_c\}_{c \in I}$ in H^{n-1} . Since η is totally geodesic on each \tilde{H}_c , η is generated by a family $\{H_c\}_{c \in I}$, without focal points, of $(n - 2)$ -planes in H^n . Then Theorem 2 immediately yields:

THEOREM 6. *Let $\eta: H^{n-1} \rightarrow H^n$ be an umbilic-free isometric immersion and $\{H_c\}_{c \in I}$ be the family of generators of η . Then there is a uniquely determined curve $\sigma: I \rightarrow \bar{H}^n$ along which the generators are parallel in \bar{H}^n .*

4. Parallelization at H^n_∞ . In order to study the behavior at infinity of the parallelizing curve of a family, we shall give \bar{H}^n the differentiable manifold-with-boundary structure of \bar{D}^n induced by the projective model. By working with the projective model, we shift our attention to families of Euclidean k -planes in E^n and their intersection properties. Although the methods of this section are Euclidean, note that all the results except Theorem 7 are results about \bar{H}^n carrying the given differentiable structure.

g^* or \cdot will denote the standard Riemannian metric on E^n , and D^* will denote Euclidean distance. Any family $\{H_c^*\}$ of Euclidean k -planes in E^n has a variation field V_c^* at c , which is defined just as before but in terms of g^* , and which is a g^* -Jacobi field on H_c^* . Thus

$$(9) \quad V_c^* = \sum_{j=1}^{n-k} h_j^* U_j^*$$

where the U_j^* are constant orthonormal fields defined on and orthogonal to H_c^* , and the h_j^* are affine functions on H_c^* . The focal set F_c of $\{H_c^*\}$ at c is the intersection of the zero sets of the h_j^* , and if nonempty is an r -plane (possibly "at infinity") for some $r \geq 2k - n$. (Since the solution set of any homogeneous linear equation on a projective space is a projective hyperplane, we are justified in adopting the convention that any affine function on a Euclidean space vanishes on a hyperplane, which lies "at infinity" if the function is constant.) Since two affine functions are constant multiples of each other if and only if they have the same zeros, F_c is a $(k - 1)$ -plane if and only if we may assume $h_j^* = 0$ for all $j > 1$ in (9).

For the rest of this section, H^n will be identified with the Euclidean open unit disc D^n carrying the projective hyperbolic metric g . Unparametrized H^n -geodesics are thus identified with line segments in D^n , and totally geodesic submanifolds of H^n with totally geodesic submanifolds of D^n .

Asymptotic geodesics in H^n correspond to segments approaching the same point of $\partial D^n = S^{n-1}$. Thus H_∞^n is identified with S^{n-1} , and \bar{H}^n with \bar{D}^n .

To each k -plane of a given family $\{H_c\}$ in H^n may be associated the complete Euclidean k -plane H_c^* satisfying $H_c = H_c^* \cap D^n$. The hyperbolic family $\{H_c\}$ is without focal points if and only if for each c , the focal set of $\{H_c^*\}$ does not enter D^n . (Note that focal points depend only on differentiable structure, not on metric structure.) Such a family $\{H_c\}$ generates an immersed submanifold of H^n . The g -induced metric on this submanifold has curvature -1 if and only if for each c , the hyperbolic variation field V_c is tangent to a fixed hyperbolic $(k + 1)$ -plane through H_c ; hence if and only if the Euclidean variation field of $\{H_c^*\}$ is tangent to a fixed Euclidean $(k + 1)$ -plane through H_c^* . Equivalent conditions on $\{H_c^*\}$ are that the metric induced by g^* on the submanifold generated by $\{H_c^* \cap D^n\}$ is flat; or (by the second paragraph of this section) that the focal sets F_c of $\{H_c^*\}$ are all $(k - 1)$ -planes.

By Theorem 2, a hyperbolic family $\{H_c\}$ generating an immersion of curvature -1 has a unique parallelizing curve in \bar{H}^n . The following theorem relates the parallelizing curve of $\{H_c\}$ in \bar{H}^n to the focal family of $\{H_c^*\}$ in E^n via the classical notion of duality of points and hyperplanes. For any $p \in \bar{D}^n$, let p^* denote the inversion of p through S^{n-1} ; then the hyperplane through p^* and orthogonal to the line Opp^* is the dual to p through S^{n-1} .

THEOREM 7. *Let $\{H_c\}_{c \in I}$ be a regular family of hyperbolic k -planes which is parallel in \bar{H}^n along a curve $\sigma: I \rightarrow \bar{H}^n$, and let $\{H_c^*\}$ be the associated Euclidean family. Then the focal set F_c of $\{H_c^*\}$ is the $(k - 1)$ -plane dual to $\sigma(c)$ through $H_c^* \cap S^{n-1}$.*

PROOF. By Theorem 2 and remarks above, the hyperbolic and Euclidean families $\{H_c\}$ and $\{H_c^*\}$ have variation fields lU and l^*U^* respectively at c , where U is a covariantly constant unit normal to H_c , with respect to g ; U^* is a constant unit normal to H_c^* , with respect to g^* ; l is positive, and hyperbolic on g -geodesics; and l^* is affine on g^* -geodesics. Moreover, U and U^* are tangent to the same $(k + 1)$ -plane P through H_c^* . By definition of variation field, the following equation holds on $H_c = H_c^* \cap D^n$:

$$(10) \quad l^* = lU \cdot U^*.$$

Let q be the center and r_1 the radius of $H_c^* \cap D^n$, and r_2 the radius of $P \cap D^n$. By (10) and hyperbolicity of l , if p is a fixed point of H_c then

$$(11) \quad l^*(p) = r_2^{-1}(ar_1^2 + br_1D^*(q, p))$$

where $a = l(q)$, $b = Y(p)l$ and $Y(p)$ is the g -unit vector at q tangent to the g -ray γ from q toward p . Since l^* is affine, formula (11) in fact holds at any $p \in H_c^*$. Therefore we may choose p so that $p \in F_c$ and $D^*(q, p)$ realizes

distance from q to F_c . Then $D^*(q, p) = -ab^{-1}r_1$ since $l^*(p) = 0$; and b (< 0) is minimized since $D^*(q, p)$ is minimized. By definition of b , it follows that the ray γ from q toward p is tangent to $-(\text{grad}_g l)_q$.

Now, $\sigma(c)$ is the striction point of $\{H_c\}$, that is, the minimum point of \bar{l} . Since l is hyperbolic, $\sigma(c)$ clearly lies on this same ray γ ; and since $(l \circ \gamma)(s) = a \cosh s + b \sinh s$, then

$$D(q, \sigma(c)) = \tanh^{-1}(-a^{-1}b) \quad \text{and} \quad D^*(q, \sigma(c)) = -a^{-1}br_1.$$

Therefore p is the inversion of $\sigma(c)$ through $H_c^* \cap S^{n-1}$. This completes the proof of Theorem 7.

Note that it is a consequence of Theorem 7 that σ is smooth in \bar{H}^n . The following corollary gives certain properties which σ must possess at H_∞^n . Theorem 8 will show that these are in a sense exhaustive. (In the rest of this section, we use the following convention, based on the canonical identification with \mathbb{R}^n of each tangent space to \mathbb{R}^n . If β denotes a Euclidean curve, $\beta'(c)$, $\beta''(c)$, and $\beta^{(3)}(c)$ denote elements of the tangent space at $\beta(c)$, unless these symbols occur in an algebraic expression, in which case they denote elements of \mathbb{R}^n .)

COROLLARY 2. *Let $\{H_c\}_{c \in I}$ be a family of hyperbolic k -planes which is parallel in \bar{H}^n along a smooth curve $\sigma: I \rightarrow \bar{H}^n$, and suppose $\sigma(c) \in H_\infty^n$ for some c . Then $\sigma'(c)$ is tangent to $H_{c\infty}$, and $\sigma''(c)$ is tangent to \bar{H}_c .*

PROOF. Note that we allow the possibility that $\{H_c\}$ is not regular at c , in which case the variation fields of both $\{H_c\}$ and $\{H_c^*\}$ vanish. By Theorem 7 (or trivially, in the nonregular case), $\sigma(c)$ lies in the focal set F_c of $\{H_c^*\}$, and so the variation field for $\{H_c^*\}$ vanishes at $\sigma(c)$. It follows from the definition of variation field and the fact that $\sigma(d) \in H_d^*$ for all $d \in I$ that $\sigma'(c)$ is tangent to H_c^* . Since σ is not transverse to S^{n-1} , $\sigma'(c)$ is tangent to $H_{c\infty} = H_c^* \cap S^{n-1}$.

Denote by $\sigma^*(d)$ the inversion of $\sigma(d)$ through $H_d^* \cap S^{n-1}$. By Theorem 7, $\sigma^*(d) \in F_d$ for all $d \in I$, hence $\sigma^{*'}(d)$ is tangent to H_d^* . In particular, since $\sigma^*(c) = \sigma(c)$ and σ^* is not transverse to S^{n-1} , $\sigma^{*'}(c)$ is tangent to $H_c^* \cap S^{n-1}$, and thus to F_c (since either $F_c = H_c^*$ or F_c is the hyperplane of H_c^* tangent to S^{n-1} at $\sigma(c)$). Setting $\zeta = \sigma^* + \sigma^{*'}$, we have $\zeta(d) \in H_d^*$ for all d , and $\zeta(c) \in F_c$. Therefore $\zeta'(c)$ is tangent to H_c^* . But $\sigma^{*''}(c) = \zeta'(c) - \sigma^{*'}(c)$, so $\sigma^{*''}(c)$ is also tangent to H_c^* . Moreover, it may be verified by direct computation that because σ and σ^* are inverse curves and $\sigma^{*''}(c)$ is tangent to H_c^* , then $\sigma''(c)$ is tangent to H_c^* . Corollary 2 follows.

We are interested in regular families, since they always generate submanifolds. The following condition will be useful:

COROLLARY 3. *A parallel family $\{H_c\}$ along a regular curve $\sigma: I \rightarrow \bar{H}^n$ is*

regular if and only if $\sigma'(c)$ is transverse to H_c whenever $\sigma(c) \in H_c$ and $\sigma^{(3)}(c)$ is transverse to \bar{H}_c whenever $\sigma(c) \in H_{c\infty}$.

PROOF. (Note that by Theorem 2, for a parallel family regularity is equivalent to having no focal points.) When σ lies in H^n , the proof is the same as for Theorem 4.1. If $\sigma(c) \in H_\infty^n = S^{n-1}$, then $\sigma \cdot \sigma'' < 0$ at c because σ is regular; therefore $\sigma(c) + k\sigma''(c)$ lies in D^n for some $k > 0$, and hence in $H_c = H_c^* \cap D^n$ by Corollary 3.2. Now consider the curve $\zeta = \sigma + k\sigma''$. If $\sigma^{(3)}(c)$ is transverse to H_c^* , then so is $\zeta'(c)$, and it follows that $\{H_c\}$ is regular at c . If $\sigma^{(3)}(c)$ is tangent to H_c^* , then $\{H_c\}$ focuses at $\zeta(c)$, and so $\{H_c\}$ is not regular at c . Thus Corollary 3 is proved.

A parallelizing curve σ in \bar{H}^n of a regular family of k -planes, while always regular where σ lies in H^n , need not be regular at H_∞^n . The regularity hypothesis on σ is necessary for the two uniqueness results which follow. Note that for a regular curve in D^n , the osculating 2-planes with respect to g and g^* agree, when they exist. Furthermore, for a regular curve in \bar{D}^n and lying on S^{n-1} at c , the osculating 2-plane at c with respect to g^* exists and enters D^n . Therefore one may speak without ambiguity of the *osculating hyperbolic 2-plane* at c of any regular curve σ in \bar{H}^n such that $\sigma(c) \in H_\infty^n$. For any parallel family along σ , the k -plane at c must contain this 2-plane, by Corollary 2.

THEOREM 8. *Let $\sigma: I \rightarrow H_\infty^n$ be a regular curve, where $0 \in I$, and H_0 be any hyperbolic k -plane containing the osculating hyperbolic 2-plane of σ at $\sigma(0)$. Then H_0 extends uniquely to a parallel family $\{H_c\}_{c \in I}$ along σ .*

PROOF. We reduce to a linear problem, that of parallel translation in the Euclidean normal bundle of σ (regarded as a curve lying on S^{n-1}). If $\{H_c\}$ is parallel along σ , then the corresponding Euclidean family $\{H_c^*\}$ has the following properties (by Corollary 2 and Theorem 7):

- (i) H_c^* contains the osculating 2-plane of σ at $\sigma(c)$, and
- (ii) the focal set F_c is either H_c^* or the hyperplane of H_c^* tangent to S^{n-1} at $\sigma(c)$.

Conversely, if $\{H_c^*\}$ is a Euclidean family satisfying (i) and (ii), then each H_c^* enters D^n by (i), and we claim that the corresponding hyperbolic family $\{H_c\}$ is parallel along σ . Where the families are not regular, constancy of the hyperbolic variation field on horospheres is obvious. Elsewhere, since each F_c is a $(k - 1)$ -plane not entering D^n , $\{H_c\}$ generates a submanifold of curvature -1 in H^n ; thus by Theorems 2 and 7, $\{H_c\}$ is parallel along σ . Therefore it suffices, for the theorem, to demonstrate existence and uniqueness of a family $\{H_c^*\}$ of Euclidean k -planes satisfying (i) and (ii), when H_0^* is given.

Suppose such a family $\{H_c^*\}$ exists. Let ∇ be the Euclidean vector bundle over σ whose $(k - 2)$ -dimensional fiber at each c is tangent to $H_c^* \cap S^{n-1}$

and orthogonal to $\sigma'(c)$. Then H_c^* is tangent to $[\mathcal{V}, \sigma', \sigma'']_c$ (where bracket denotes span). Let v be a smooth section of \mathcal{V} . Setting $\zeta = \sigma + v$, we have $\zeta(c) \in F_c$ for all c , hence $\zeta'(c)$ tangent to H_c^* . Therefore $v'(c) \in [\mathcal{V}, \sigma', \sigma'']_c$. But $v' \cdot \sigma = 0$ since $v \cdot \sigma = v \cdot \sigma' = 0$, and so $v'(c) \in [\mathcal{V}, \sigma']_c$. Therefore \mathcal{V} has parallel fiber in the Euclidean normal bundle of σ . Since \mathcal{V} is uniquely determined by H_0^* and σ , so is $\{H_c^*\}$.

On the other hand, for given H_0^* let \mathcal{V}_0 consist of the vectors at $\sigma(0)$ tangent to $H_0^* \cap S^{n-1}$ and orthogonal to σ , and \mathcal{V} be the parallel extension of \mathcal{V}_0 in the Euclidean normal bundle of σ . Then \mathcal{V} is always tangent to S^{n-1} . Letting H_c^* be the k -plane through $\sigma(c)$ tangent to $[\mathcal{V}, \sigma', \sigma'']_c$, we have a family satisfying (i). For (ii), it suffices to verify that $[\mathcal{V}, \sigma']_c$ is always tangent to the focal set F_c of $\{H_c^*\}$. But for any curve of the form $\zeta = \sigma + v + f\sigma'$ with v a section of \mathcal{V} , then $\zeta'(c) \in [\mathcal{V}, \sigma', \sigma'']_c$ and so $\zeta'(c)$ is tangent to H_c^* . Therefore $\zeta(c) \in F_c$, as required. Thus we have constructed a family $\{H_c^*\}$ satisfying (i) and (ii).

COROLLARY 4. *Let $\sigma: I \rightarrow \bar{H}^n$ be a regular curve, where $0 \in I$, and H_0 be a hyperbolic k -plane at $\sigma(0)$. If there exists a parallel family along σ extending H_0 , then that family is unique.*

PROOF. Note that if two hyperbolic k -planes intersect, one may speak of the hyperbolic angle between them; this angle is measured in the $(n - r)$ -plane which is g -orthogonal to the intersection r -plane P at any $p \in P$, and is independent of the choice of p . We shall show that any two parallel families $\{H_{1c}\}$ and $\{H_{2c}\}$ of k -planes along a given regular curve $\sigma: I \rightarrow \bar{H}^n$ have constant intersection dimension, and form a constant angle.

Note that $H_{1c} \cap H_{2c} \neq \emptyset$. Indeed, if $\sigma(c) \in H^n$ then $\sigma(c)$ lies in this intersection, and if $\sigma(c) \in H_\infty^n$ then this intersection contains a 2-plane at least. Let I_0 be a maximal subinterval of I on which the intersection dimension takes its minimum for I . Then I_0 is open in I . The angle between the two families is smooth on I_0 , and certainly is constant on any subinterval on which σ lies entirely in H^n . Furthermore, it follows from the proof of Theorem 8 that the angle is constant on any subinterval J of I_0 on which σ lies entirely in H_∞^n : In the notation of that proof, suppose H_{ic} corresponds to $[\mathcal{V}_i, \sigma', \sigma'']_c$, for each $c \in J$. It may be verified that the hyperbolic angle between H_{1c} and H_{2c} is equal to the Euclidean angle between $[\mathcal{V}_1, \sigma']_c$ and $[\mathcal{V}_2, \sigma']_c$, since $[\mathcal{V}_i, \sigma']_c$ is tangent to $H_{ic}^* \cap S^{n-1}$ at $\sigma(c)$. In turn, the Euclidean angle between the $[\mathcal{V}_i, \sigma']$ equals that between the \mathcal{V}_i since the \mathcal{V}_i are orthogonal to σ' . This angle is constant on J because the \mathcal{V}_i are parallel in the Euclidean normal bundle of $\sigma|_J$.

Therefore the hyperbolic angle between the two families has vanishing

derivative on a dense subset of I_0 , hence is constant on I_0 . It follows that I_0 is closed in I , and $I_0 = I$.

REMARK 6. Given a regular curve $\sigma: I \rightarrow \overline{H}^n$, and an initial k -plane H_0 , there may be no parallel family on σ extending H_0 . For example, if $\sigma(0) \in H^n$, the parallel translate of H_0 along σ as σ approaches H_∞^n may have more than one limit k -plane, or none at all.

5. **Examples.** By way of illustration, we examine families of hyperbolic k -planes which have parallelizing curves in \overline{H}^n of special type. Our conventions are the same as in §4.

EXAMPLE 2. A family $\{H_c\}_{c \in I}$ without focal points is parallel along an orthogonal trajectory $\sigma: I \rightarrow H^n$ if and only if the corresponding Euclidean k -planes H_c^* all pass through a fixed $(k - 1)$ -plane not meeting \overline{D}^n . This may be seen as follows.

Note that if σ is both parallelizing and orthogonal, then σ lies in a hyperbolic $(n - k)$ -plane orthogonal to the H_c . To see this, observe that the family $\{H_c^\perp\}$ of hyperbolic $(n - k)$ -planes, where $H_c^\perp \cap H_c = \sigma(c)$ and H_c^\perp is orthogonal to H_c , is parallel along and tangent to σ . Thus the variation field at c of $\{H_c^\perp\}$ vanishes at $\sigma(c)$, and, by Theorem 2, vanishes identically. Therefore the H_c^\perp coincide, in an $(n - k)$ -plane containing σ . Now, it can be shown that a hyperbolic k -plane is orthogonal to a given hyperbolic $(n - k)$ -plane H if and only if the corresponding Euclidean k -plane lies in the pencil through a certain Euclidean $(k - 1)$ -plane, not meeting \overline{D}^n , determined by H . Thus, since the H_c are all orthogonal to a fixed $(n - k)$ -plane, it follows that the H_c^* all pass through a fixed $(k - 1)$ -plane.

The converse follows from Theorem 7 and the fact that, if $\{H^*\}$ now denotes the collection of Euclidean k -planes passing through a fixed $(k - 1)$ -plane F and entering D^n , the dual points of F in all the H^* form a hyperbolic $(n - k)$ -plane which is g -orthogonal to each $H = H^* \cap D^n$.

EXAMPLE 3. A family $\{H_c\}$ has constant parallelizing curve in H_∞^n if and only if the H_c^* all pass through a fixed $(k - 1)$ -plane tangent to S^{n-1} . Here it suffices to verify that if the parallelizing curve σ is constant at H_∞^n , then the focal family $\{F_c\}$ of $\{H_c^*\}$ is constant. If $\{F_c\}$ is not constant, then at some c the variation field of $\{F_c\}$ does not vanish identically on F_c . This field is defined on the hyperplane F_c of H_c^* and is tangent to H_c^* . Since all $F_{c+\delta}$ lie in the tangent plane to S^{n-1} at the image point of σ , it follows that H_c^* is tangent to S^{n-1} , which is impossible since H_c^* enters D^n .

EXAMPLE 4. The osculating Euclidean 2-planes of a regular curve σ lying on S^{n-1} and having nowhere vanishing torsion (second order curvature function) correspond to a regular family of hyperbolic 2-planes which is parallel along σ . Indeed, parallelism along σ follows from the proof of Theorem 8. Regularity is a consequence of Corollary 3 and the torsion

assumption. Conversely, all regular hyperbolic 2-plane families which are parallel along a regular curve in H_∞^n are obtained in this way, by Corollaries 2 and 3.

Now suppose $n = 3$, and consider all foliations of H^3 by 2-plane families whose parallelizing curves are regular curves in H_∞^3 . These foliating families correspond precisely to the families of osculating 2-planes of regular Euclidean curves lying on S^2 , for which torsion is nonvanishing and curvature (denoted here by κ^*) becomes infinite at both endpoints. For example, a loxodrome is such a curve. It is only necessary to verify that for a unit-speed curve σ on S^2 , the osculating planes approach tangency to S^2 at an endpoint if and only if curvature becomes infinite. This follows by differentiating $\sigma \cdot \sigma' = 0$, to obtain $|\sigma \cdot \sigma'' / \|\sigma''\| = 1/\kappa^*$.

EXAMPLE 5. Let $\{H_c^*\}$ be any regular family of $(n - 1)$ -planes in E^n such that each member intersects D^n . The equation of H_c^* is $y \cdot N(c) - r(c) = 0$, where N and r denote (smoothly varying) unit normal and distance from the origin respectively. Then the focal $(n - 2)$ -plane F_c in H_c^* satisfies $y \cdot N'(c) - r'(c) = 0$. (Indeed, $y \in F_c$ if and only if there is a curve β satisfying $\beta(c) = y$, $\beta(c + \delta) \in H_{c+\delta}^*$, and $\beta'(c) \cdot N(c) = 0$; but the last two conditions imply $\beta(c) \cdot N'(c) - r'(c) = 0$.) If we assume that $\{F_c\}$ never enters D^n , then $\{H_c^*\}$ determines a regular hyperbolic family which is parallel along some curve σ in \bar{H}^n . By Theorem 7, the equations for $\{F_c\}$ yield a formula for σ :

$$(12) \quad \sigma = rN + (1 - r^2)(r')^{-1}N'.$$

Here r' never vanishes because $\{F_c\}$ never enters D^n .

This formula may be used to construct examples, such as the following one for $n = 2$. The curve $\sigma^*(c) = (1 + c^2, c^3)$ lies outside S^1 and meets S^1 at $c = 0$, in a cusp tangent to the x -axis. The tangent lines H_c^* to σ^* form a regular Euclidean family (even though σ^* is not regular). The focal sets of $\{H_c^*\}$ are given by $F_c = \sigma^*(c)$, and hence lie outside D^2 . Thus for a certain symmetric interval I about 0, the tangent lines to $\sigma^*|I$ determine a family $\{H_c\}$ of hyperbolic lines foliating H^2 . The parallelizing curve σ may be computed from (12) to be

$$\sigma(c) = (3c^4 + 2c^2 + 2)^{-1}(c^6 + 3c^4 + 2, -c^5 - 4c^3).$$

Note that $\sigma'(0) = 0$, as required by Corollary 3, and that σ meets S^1 at $c = 0$, in a cusp tangent to the x -axis.

REMARK 7. Finally, we mention one further possible direction for investigation: With regard to §4, one might ask precisely which curves $\sigma: I \rightarrow \bar{H}^n$ (taking values in both H^n and H_∞^n) allow the existence of parallel families.

REFERENCES

1. D. Ferus, *On isometric immersions between hyperbolic spaces*, Math. Ann. **205** (1973), 193–200.
2. P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.
3. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, Interscience, New York, 1963.
4. K. Nomizu, *Isometric immersions of the hyperbolic plane into the hyperbolic space*, Math. Ann. **205** (1973), 181–192.
5. B. O'Neill and E. Stiel, *Isometric immersions of constant curvature manifolds*, Michigan Math. J. **10** (1963), 335–339.
6. E. Portnoy, *Developable surfaces in hyperbolic space*, Pacific J. Math. **57** (1975), 281–288.

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