ON THE RATIONALITY OF DIVISORS AND MEROMORPHIC FUNCTIONS⁽¹⁾

BY

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ABSTRACT. Let E be a holomorphic vector bundle over a connected complex manifold X and D a divisor on E . Let $A(D)$ be the set of all $x \in X$ for which $(\text{supp } D) \cap E_x$ is a proper algebraic set in E_x . The purpose of this paper is to prove that the following conditions are equivalent: (i) $A(D)$ has positive measure in X ; (ii) D extends to a unique divisor on the projective completion $\bar{E}$ of E ; (iii) D is locally given by the divisors of rational meromorphic functions defined over open sets in X . Similar results for meromorphic functions are derived. The proof requires an extension theorem for analytic set: Assume E is a holomorphic vector bundle over a pure p -dimensional complex space X and S an analytic set in E of pure codimension 1. Then the closure $\bar{S}$ of S in $\bar{E}$ is analytic if and only if $S \cap E_x$ is a proper algebraic set for all x in a set of positive $2p$ -measure in every branch of X .

1. Statement of results. Let $\pi: E \rightarrow X$ be a holomorphic $\mathbb{C}^q$ -bundle over a complex space X of dimension p . A meromorphic function m on E is said to be rational (over X) iff on each local trivialization of E , $E(U) \xrightarrow{\sim} U \times \mathbb{C}^q$, m is given by a quotient of (holomorphic) pseudopolynomials defined over U . If S is an analytic set in E , let $A(S)$ denote the set of all $x \in X$ such that $S \cap E_x$ is a proper algebraic set (possibly empty) in $E_x = \pi^{-1}(x)$. If D is a divisor on E , define $A(D) = A(\text{supp } D)$; D is said to be rational over X iff there is an open covering $\{U\}$ of X such that $D|_{E(U)}$ is the divisor of a rational meromorphic function defined on $E(U)$. A subset M of X is said to be of positive s -measure iff it is not of zero Hausdorff s -measure (in terms of local patches of X). The results of this paper are the following.

THEOREM 2.1. *Assume X is nonsingular and irreducible. Let D be a divisor on E . Then the following conditions are equivalent:*

- (i) $A(D)$ has positive $2p$ -measure;

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(ii) D extends to a unique divisor $\bar{D}$ on the projective completion $\bar{E}$ of E such that $\text{supp } \bar{D} \not\supseteq \bar{E}_\infty$.

(iii) D is rational over X .

THEOREM 2.2. *Let X be as above and m a meromorphic function on E . Let (m) denote the divisor associated to m .*

(1) *If $A((m))$ is of positive $2p$ -measure, then there exists a rational meromorphic function f on E with $(f) = (m)$.*

(2) *Let $R(m)$ be the set of all $x \in X$ such that $m|_{E_x}$ is a rational meromorphic function. Assume $R(m)$ has positive $2p$ -measure. Then m is rational over X . Furthermore, if X is a Stein manifold with $H^2(X, \mathbb{Z}) = 0$, there exist rational holomorphic functions P, Q (which are coprime) on E such that $m = P/Q$.*

The proof of the above theorems requires the following two facts:

THEOREM 1.1. *Assume X is a normal, irreducible complex space of dimension p and g a holomorphic function on E . Assume $A((g))$ has positive $2p$ -measure. Then there exist holomorphic functions $g_k: E \rightarrow \mathbb{C}$ for $k = 0, 1, \dots, r$, where g_k is homogeneous of degree k , and an invertible holomorphic function $u: E \rightarrow \mathbb{C}$ such that $\sum_{k=0}^r g_k = gu$.*

If E is the trivial bundle $\mathbb{C}^p \times \mathbb{C}^q$ over $\mathbb{C}^p$, Theorem 1.1 has been obtained under different hypothesis by L. Ronkin [5] (where $A((g))$ is assumed to be a set of positive Γ -capacity in $\mathbb{C}^p$). In the present case, an elementary construction of a rational representation of (g) is given, making use of the extension of the zero set $g^{-1}(0)$ into the projective completion $\bar{E}$. The extension depends on a generalization of the Remmert-Stein theorem by Stoll [6]. For entire analytic sets, Stoll's theorem is also shown to be valid when the fiber dimension q is greater than 1;

THEOREM 1.2. *Assume X is a pure p -dimensional complex space and $E \rightarrow X$ a holomorphic $\mathbb{C}^q$ -bundle with $q \geq 1$. Let S be an analytic set in E of pure codimension 1. Then the closure $\bar{S}$ in $\bar{E}$ is analytic if and only if $A(S)$ has positive $2p$ -measure in every branch of X . Moreover, the analyticity of $\bar{S}$ implies $\text{codim } \bar{S}_\infty \geq 2$.*

It remains unknown if the (sufficiency part of) Theorem 1.2 holds in the case where S has higher codimension.

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2. The holomorphic case. We begin with a lemma without proof.

LEMMA 1. *Let X and Y be complex spaces, $f: X \times Y \rightarrow \mathbb{C}^n$ a holomorphic map, and S an analytic set in $X \times Y$. Assume Y is irreducible. Define*

$$c(f) = \{x \in X \mid f|_{\{x\}} \times Y = \text{constant}\}$$

and

$$d(S) = \{x \in X \mid \{x\} \times Y \subseteq S\}.$$

Then $c(f)$ and $d(S)$ are analytic in X .

If $\pi: W \rightarrow X$ is a holomorphic fiber space over X and $f: W \rightarrow \mathbb{C}$, define $f_x = f|_{\pi^{-1}(x)}$ for each $x \in X$ and $d(f) = \{x \in X \mid f_x \equiv 0\}$. Also denote the zero-multiplicity of a holomorphic function f at w by $\nu^0(f; w)$.

The proof of Theorem 1.1 will be given in two steps. In the first step we reduce the general case to the case of a $\mathbb{C}^1$ -bundle. For this purpose, (following an idea of Ronkin [5]) define $h: E \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$h(w, t) = g(tw) \quad ((w, t) \in E \times \mathbb{C}).$$

Let $D = (g)$. By Federer [2, Theorem 2.10.45], the set $\tilde{A}(D) = \pi^{-1}(A(D))$ is of positive measure in E . Clearly for each $w \in \tilde{A}(D)$, the function

$$h_w = h|_{\{w\} \times \mathbb{C}}$$

has a finite zero set. Assume the theorem holds for a $\mathbb{C}^1$ -bundle, then h admits a representation

$$h(w, t) = \Phi(w, t)K(w, t)$$

where $\Phi(w, t) = \sum_{k=0}^r g_k(w)t^k$ with coefficients g_k holomorphic on E ($g_r \not\equiv 0$), and K is invertible holomorphic on $E \times \mathbb{C}$. Furthermore, K can be chosen so that $K(w, 0) \equiv 1$ for all $w \in E$.⁽²⁾ Let $E_1 = E - g_r^{-1}(0)$. Define $\Psi: E_1 \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\Psi(w, t) = \prod_{z \in \mathbb{C}} (t - z)^{\nu^0(h_w; (w, z))}.$$

For each $(c, w) \in \mathbb{C}^* \times E_1$, $\Psi(cw, t) = \Psi(w, ct)/c^r$. Let $G = g_r K$. Then $\Psi G = h$ on $E_1 \times \mathbb{C}$, and $\Psi(w, t)G(w, 0) = \Phi(w, t)$. If $(c, w) \in \mathbb{C}^* \times E$ with $h(cw, t) \neq 0$, then

$$G(cw, t) = c^r G(w, ct).$$

Hence

$$\Phi(cw, t) = \Phi(w, ct) \quad ((c, w, t) \in \mathbb{C} \times E_1 \times \mathbb{C}).$$

It follows that each g_k is homogeneous of degree k along fibers of E , which proves the general case.

LEMMA 2. Assume X and Y are normal, irreducible complex spaces and f, g are holomorphic functions on $X \times Y$. Let $\pi: X \times Y \rightarrow X$ be the projection and define $f_x(g_x)$ with respect to π . Assume $d(f) = \emptyset$. Suppose there exists a dense

⁽²⁾Suggested by Professor R. Narasimhan.

subset X_0 of X_{reg} such that for all $x \in X_0$ and $w \in \{x\} \times Y_{\text{reg}}$,

$$\nu^0(f_x; w) \leq \nu^0(g_x; w).$$

Then g/f is holomorphic on $X \times Y$.

PROOF. It suffices to prove the lemma for the case where both X and Y are nonsingular. Let $Z = f^{-1}(0) \neq \emptyset$. The projection $\pi_0 = \pi: Z \rightarrow X$ is open. Let B be a connectivity component of Z_{reg} and $b \in B$. Suppose $\nu^0(f; b) > \nu^0(g; b)$. Then there is an open neighborhood U of b and a holomorphic function h on U such that $f|_U = gh$ and $h|_{B \cap U} \equiv 0$. This implies $\pi(B \cap U) \cap X_0 = \emptyset$, a contradiction. Therefore $\nu^0(f; w) \leq \nu^0(g; w)$ on Z_{reg} , from which the lemma follows.

LEMMA 3. Assume X is a connected Stein manifold with $H^2(X, \mathbb{Z}) = 0$. Let $\pi: W \rightarrow X$ be a locally trivial holomorphic fiber space with smooth connected fibers. Let g be a holomorphic function $\not\equiv 0$ on W . Then there exist holomorphic functions $g': W \rightarrow \mathbb{C}$ and $\varphi: X \rightarrow \mathbb{C}$ such that $g = (\pi^*\varphi)g'$ and $\text{codim } d(g') \geq 2$.

PROOF. Define $Z = g^{-1}(0)$. Let N be the union of all branches of $d(g)$ of codimension 1 in X . If B is a branch of N , then $\pi^{-1}(B)$ is analytic and irreducible in Z , hence by [1, Lemma 1.27], $\pi^{-1}(B)$ is a branch of Z . For each $x \in N_{\text{reg}}$ and $w \in Z_{\text{reg}}$ with $\pi(w) = x$, define $\delta(x) = \nu^0(g; w)$. If $x \in X - N$, define $\delta(x) = 0$. Since $\nu^0(g; \cdot)$ is constant on each component of Z_{reg} , the function δ is well defined on $X - N_{\text{sing}}$ and locally constant on N_{reg} . Hence there is a unique divisor D on X whose multiplicity function coincides with δ on $X - N_{\text{sing}}$. Because $H^1(X, \mathcal{O}_X^*) = 0$, there exists a holomorphic function $\varphi: X \rightarrow \mathbb{C}$ with $(\varphi) = D$. Let $V = Z_{\text{sing}} \cup \pi^{-1}(N_{\text{sing}})$. Then V is analytic in W of codimension ≥ 2 . For each $w \in W - V$,

$$\nu^0(g; w) \geq \nu^0(\pi^*\varphi; w).$$

Hence $g/\pi^*\varphi$ extends to a holomorphic function $g': W \rightarrow \mathbb{C}$. Obviously $g = (\pi^*\varphi)g'$. Since g and $\pi^*\varphi$ have the same vanishing order at every point of $Z_{\text{reg}} \cap \pi^{-1}(N_{\text{reg}})$, $\text{codim } d(g') \geq 2$. Q.E.D.

Let $\pi: E \rightarrow X$ be a holomorphic vector bundle over a complex space X . The associated projective bundle $\mathbf{P}(E)$ is a holomorphic fiber bundle whose fiber over $x \in X$ is the projective space of lines in E_x . Define $C_X = X \times \mathbb{C}$, $\tilde{E} = C_X \oplus E$ and $\bar{E} = \mathbf{P}(\tilde{E})$. Then there exist biholomorphic imbeddings of E , resp. $\mathbf{P}(E)$, into $\bar{E}$, such that $\bar{E} = E \cup \mathbf{P}(E)$ and $E \cap \mathbf{P}(E) = \emptyset$. The fiber bundle $\bar{\pi}: \bar{E} \rightarrow X$ is called the projective completion of E . For each subset V of $\bar{E}$, define $V_\infty = V \cap \mathbf{P}(E)$.

PROOF OF THEOREM 1.1 FOR THE CASE $q = 1$. Let $S = g^{-1}(0)$. If $M \subseteq E$ and $a \in X - d(S)$, define

$$n_g(M_a; 0) = \sum \{ \nu^0(g_a; w) | w \in M \cap E_a \}.$$

If M is a relatively compact open set with $\partial M \cap g_a^{-1}(0) = \emptyset$, then $n_g(M_x; 0)$ is constant for all x in a neighborhood of a [7, Theorem 2.7].

If $A(S)$ has interior points, the Remmert-Stein extension theorem implies the closure $\bar{S}$ of S in $\bar{E}$ is analytic of pure codimension 1.⁽³⁾ In case $A(S)$ is known to have positive measure in X , a generalization of the extension theorem by Stoll [6, Satz 2] yields the analyticity of $\bar{S}$. Consequently $X - d(S) = A(S)$. Define $\Delta(g) = \bar{\pi}(\bar{S}_\infty)$. Since $\text{codim}(\bar{S}_\infty) \geq 2$ (see Theorem 1.2), the analytic set $\Delta(g)$ is nowhere dense in X . It follows that $n_g(E_x; 0) = \text{constant} = r \geq 0$ for all $x \in X - \Delta(g)$.

Let $U \subseteq A(S)$ be an open set over which E is trivial and identify $E(U)$ with $U \times \mathbb{C}$. Let $V = U - \Delta(g)$. Assume $r > 0$. The function

$$G_V(x, t) = \prod_{z \in \mathbb{C}} (t - z)^{\nu^0(g_x; (x, z))} \quad ((x, t) \in V \times \mathbb{C})$$

is holomorphic on $V \times \mathbb{C}$. For every $x \in V$, let $U_{(x)} = V$ and $G_{(x)} = G_V$. Assume $a \in U \cap \Delta(g)$. There exist open neighborhoods Q of a in U and $Y = \mathbb{C}(\rho)$ ($\rho > 0$) of the origin in $\mathbb{C}$ such that

$$h(x, t) = \prod_{z \in Y} (t - z)^{\nu^0(g_x; (x, z))}$$

is holomorphic on $Q \times \mathbb{C}$.⁽⁴⁾ Then $\tilde{g} = g/h$ is holomorphic on $Q \times \mathbb{C}$ and nonvanishing on $Q \times Y$. Moreover, the function

$$\begin{aligned} f(x, t) &= \prod_{z \in \mathbb{C}} (t - z)^{\nu^0(\tilde{g}_x; (x, z))} \\ &= f_0(x) + \cdots + f_{l-1}(x)t^{l-1} + f_l(x)t^l \end{aligned}$$

is holomorphic on $(Q \cap V) \times \mathbb{C}$ (where $f_l = 1$). By Riemann's extension theorem, each quotient f_j/f_0 extends to a holomorphic function ζ_j on Q . Furthermore, $Q \cap \Delta(g) = \{x \in Q | \zeta_l(x) = 0\}$. Define $U_{(a)} = Q$ and

$$G_{(a)}(x, t) = h(x, t)(\zeta_0(x) + \zeta_1(x)t + \cdots + \zeta_l(x)t^l)$$

on $U_{(a)} \times \mathbb{C}$. Then $G_{(a)}$ is rational holomorphic over $U_{(a)}$. By Lemma 2, $g/G_{(x)}$ is invertible holomorphic on $U_{(x)} \times \mathbb{C}$ for every $x \in U$. Let $G_{(x)} = 1$ if $r = 0$.

Let Ω be a domain in X_{reg} biholomorphic to an open ball in $\mathbb{C}^p$. There exist holomorphic functions $\varphi: \Omega \rightarrow \mathbb{C}$ and $g': \Omega \times \mathbb{C} \rightarrow \mathbb{C}$ with $\text{codim } d(g') \geq 2$ and $\varphi(x)g'(x, t) = g(x, t)$ for $(x, t) \in \Omega \times \mathbb{C}$. For each $x \in \Omega - d(g')$, there is a rational holomorphic function $G'_{(x)}$ for which $g'/G'_{(x)}$ is invertible holomorphic on $U_{(x)} \times \mathbb{C}$. Hence there exists a rational holomorphic function

⁽³⁾Since it can be shown that $\bar{\pi}(\bar{S}_\infty) \cap U = \emptyset$ for some nonvoid open set $U \subseteq X$.

⁽⁴⁾Cf. [8, Theorem 1.2.20] or [7, Theorem 3.7].

$G'_\Omega: \Omega \times \mathbb{C} \rightarrow \mathbb{C}$ for which g'/G'_Ω is invertible holomorphic. Define $G_\Omega = (\pi^* \varphi)G'_\Omega$ on $E(\Omega)$. Observe that if $x \in \Omega - \Delta(g')$, $n_{g'}(E_x; 0) = r$.

Choose an open covering $\{\Omega_j\}$ of X_{reg} such that $g|E(\Omega_j) = G_j \cdot H_j$ where G_j is rational holomorphic over Ω_j and H_j invertible holomorphic on $E(\Omega_j)$ with $H_j = 1$ on the zero section of $E(\Omega_j)$. Then there exist holomorphic functions g_k on E_{reg} for $k = 0, 1, \dots, r$ such that

$$\left(\sum_{k=0}^r g_k \right) = (g)|_{E_{\text{reg}}},$$

and each g_k is homogeneous of degree k along fibers of E . Since X is normal, the above equation extends to E . Q.E.D.

PROOF OF THEOREM 1.2. It is easy to show that the analyticity of $\bar{S}$ implies $X - d(S) = A(S)$ and $\text{codim } d(S) \geq 1$. Conversely, assume $A(S)$ has positive $2p$ -measure in every branch of X . If $q = 1$, the analyticity of $\bar{S}$ follows from Satz 2 of Stoll [6]. Now assume $q > 1$ and X is nonsingular. Let X_ν be a connectivity component of X . There exists an open set $\Omega \subseteq X_\nu$ biholomorphic to an open ball in $\mathbb{C}^p$ such that $A(S) \cap \Omega$ has positive $2p$ -measure. Let D be the divisor on $E(\Omega)$ associated to the analytic set $S(\Omega) = S \cap E(\Omega)$. By Theorem 1.1, there exists a holomorphic function $g = \sum_{k=0}^r g_k$ on $E(\Omega)$, where each g_k is homogeneous of degree k (with $g_r \neq 0$), such that $(g) = D$. The equation

$$\sum_{k=0}^r z^{r-k} g_k(w) = 0$$

defines an analytic set V in $\bar{E}(\Omega)$ of pure codimension 1. Moreover, $V \cap E(\Omega) = S(\Omega)$ and hence $\bar{S} \cap \bar{E}(\Omega) = V$. The Remmert-Stein extension theorem implies $\bar{S}$ is analytic in $\bar{E}$ of pure codimension 1.

If X is singular, let $M = \bar{E} - \mathbf{P}(E(X_{\text{sing}}))$ and $N = \bar{S} \cap M$. The analyticity of $\bar{S} \cap \bar{E}(X_{\text{reg}})$ in $\bar{E}(X_{\text{reg}})$ implies that of N in M , and N has pure codimension 1. Since $\dim N > \dim \mathbf{P}(E(X_{\text{sing}}))$, it follows that $\bar{S} = \bar{N}$ is analytic in $\bar{E}$. Now suppose $\bar{S}_\infty$ contains a branch B of $\bar{S}$. Then the projection map $\tau: B \rightarrow X$ is proper, holomorphic, of pure rank p . Hence $X' = \tau(B)$ is a branch of X , and for every $x \in X'$, $(\bar{E}_\infty)_x = B \cap \tau^{-1}(x) \subseteq \bar{S}_x$. On the other hand, it can be shown that X' contains an open set Ω for which $\bar{E}(\Omega)_\infty \not\subseteq \bar{S}(\Omega)$. Hence a contradiction. Thus $\text{codim } \bar{S}_\infty \geq 2$.

3. Application to divisors and meromorphic functions.

PROOF OF THEOREM 2.1. It is easy to show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Assume D satisfies (i) and let $S = \text{supp } D$. By Theorem 1.2, $\bar{S}$ is analytic in $\bar{E}$ of pure codimension 1. Let U be an open set in X biholomorphic to an open ball in $\mathbb{C}^p$. Then $E(U)$ is a connected Stein manifold with $H^2(E(U), \mathbb{Z}) = 0$. There exist relatively prime holomorphic functions f_U and g_U on $E(U)$ such that

$$(f_U/g_U) = D|E(U).$$

For every $x \in A(D) \cap U$, $(f_U)_x^{-1}(0)$ and $(g_U)_x^{-1}(0)$ are algebraic sets in E_x of pure codimension 1, provided nonempty. Then $d(S)$ is nowhere dense analytic in X and $U - d(S) = A(D) \cap U$. Theorem 1.1 implies there are relatively prime rational holomorphic functions P_U and Q_U on $E(U)$ such that

$$(P_U/Q_U) = D|E(U).$$

Thus the theorem is proved.

Observe also that the polynomial bidegree of a relatively prime rational representation (P_U, Q_U) of D on $E(U)$ is independent of the open set U .

PROOF OF THEOREM 2.2. (1) By Theorem 2.1, there exists an open covering $\{U\}$ of X such that

$$m|E(U) = g_U h_U$$

where g_U is a rational meromorphic function over U and h_U is invertible holomorphic on $E(U)$ with $h_U \equiv 1$ on the zero section of $E(U)$. Consequently there is a rational meromorphic function f on E for which $(f) = m$.

(2) Suppose $m \not\equiv 0$. There exists an open set Ω in X for which (i) $m|E(\Omega)$ is given by a quotient f/g of relatively prime holomorphic functions f, g ; (ii) $\Omega \cap R(m)$ has positive $2p$ -measure. Let $N = \{x \in \Omega | f_x \cdot g_x \equiv 0\}$. Lemma 1 implies $(\Omega - N) \cap R(m)$ has positive $2p$ -measure. Hence there exists a rational meromorphic function f' on E with $(f') = (m)$. Define $u = m/f'$. The analytic set $c(u) = R(u)$. Since $R(u)$ has positive $2p$ -measure in X , $c(u) = X$. Therefore there exists a holomorphic function φ on X for which $m = (\pi^* \varphi) f'$. Hence m is rational over X .

Assume X is Stein with $H^2(X, \mathbb{Z}) = 0$. There exist relatively prime holomorphic functions g_j ($j = 1, 2$) on E such that $m = g_1/g_2$. Each $A((g_j))$ has positive $2p$ -measure. Hence there is a rational holomorphic function f_j on E (given by Theorem 1.1) with $(f_j) = (g_j)$. Then $m/(f_1/f_2)$ is invertible holomorphic on E . As above, there exists an invertible holomorphic function Ψ on X for which

$$m = (\pi^* \Psi) \frac{f_1}{f_2}. \quad \text{Q. E. D.}$$

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