CLASSIFYING OPEN PRINCIPAL FIBRATIONS

BY

DAVID A. EDWARDS AND HAROLD M. HASTINGS

ABSTRACT. Let G be a compact metric group. We shall construct classifying spaces for open principal G -fibrations over compact metric spaces.

1. Introduction. Let G be a compact metric group. J. Milnor [11] gave the first functorial construction of a universal G -bundle (classifying space) $EG \rightarrow BG$. R. Milgram [10] gave a later construction with better properties, some of which were developed by N. E. Steenrod [16]. If G is a Lie group, every principal fibration with completely regular total space and with fibre G is a fibre-bundle (A. Gleason [4]) (hence also open), so that BG classifies certain (open) principal G -fibrations. We shall extend this result to arbitrary compact metric groups G and compact metric spaces X . J. Cohen [1] extensively studied open principal fibrations, and proved, in particular, that every open principal fibration is an inverse limit of fibre-bundles (see §2). We shall use Cohen's result and some properties of Milgram's resolution (§4) to construct a universal open principal G -fibration

$$\hat{\xi}_G: G \rightarrow \hat{E}G \rightarrow \hat{B}G. \quad (1.1)$$

We describe $\hat{\xi}_G$ in §2. In §§5–6 we describe a natural isomorphism

$$\alpha: [X, \hat{B}G] \xrightarrow{\cong} k_G(X), \quad (1.2)$$

where $k_G(X)$ denotes the class of isomorphism classes of open principal G -fibrations over the compact metric space X .

(1.3) **REMARKS.** (a) J. P. May [9, see especially §§4–6] classified principal fibrations up to weak equivalence, a coarser equivalence relation. Compare weak homotopy equivalence versus homotopy equivalence for compact metric spaces.

(b) In §6 we shall see that an open principal G -fibration over a CW complex is a fibre-bundle, and hence that there is a natural *weak* homotopy equivalence $BG \rightarrow \hat{B}G$.

(c) This paper is an outgrowth of the authors' work [3] on strong pro-homotopy theory (Steenrod homotopy theory). See also Remark (6.6(b)).

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2. Open principal G -fibrations. We recall some of the theory of open principal G -fibrations due to J. Cohen [1]. Let G be a compact metric group. Suppose G acts principally (on the right) on a compact metric space E ; i.e., for any point y in E the orbit $\{yg | g \in G\}$ is homeomorphic to G . If the quotient map $p: E \rightarrow E/G$ is open and is a Hurewicz fibration (has the covering homotopy property), p is called an *open principal G -fibration*.

Open principal G -fibrations arise naturally as the limits of bundle maps. More precisely, let $\{G_n\}$ be a *Lie series* for G (L. Pontryagin [13, §46]), i.e., $\{G_n\}$ is a tower of Lie groups bonded by surjections π_n and $\lim\{G_n\} = G$. (We may use a tower for the Lie series of G because G is a compact metric group.) Assume, without loss of generality, that $G_0 = \{e\}$.

Cohen associates to an open principal G -fibration $G \rightarrow E \rightarrow X$ the tower

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \parallel \\
 G_{n+1} & \longrightarrow & E_{n+1} \equiv E \times_G G_{n+1} & \xrightarrow{p_{n+1}} & X \\
 \downarrow \pi_{n+1} & & \downarrow \pi'_{n+1} & & \parallel \\
 G_n & \longrightarrow & E_n \equiv E \times_G G_n & \xrightarrow{p_n} & X \\
 \downarrow & & \downarrow & & \parallel \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

of G_n -bundles over X . We shall frequently call such a tower a *principal G_n -fibration* over X and use the notation

$$\{\xi_n: G_n \rightarrow E_n \rightarrow X\}. \quad (2.2)$$

(2.3) REMARKS. The maps π_{n+1} and π'_{n+1} above are principal bundle maps with fibre $K_{n+1} \equiv \ker(\pi_{n+1}: G_{n+1} \rightarrow G_n)$.

Conversely, Cohen shows that applying the inverse limit functor $\lim$ to the principal $\{G_n\}$ -fibration (2.2) yields an open principal G -fibration where $G = \lim\{G_n\}$.

We shall call principal $\{G_n\}$ -fibrations over a compact metric space X *isomorphic* if the diagrams (2.1) are level-wise isomorphic over id_X . Let $k_{\{G_n\}}(X)$ denote the class of isomorphism classes of principal $\{G_n\}$ -fibrations over X and $k_G(X)$ the class of isomorphism classes of open principal G -fibrations over X . Summarizing, we have the following

(2.4) PROPOSITION (J. COHEN [1]). *Let G be a compact metric group with Lie series $\{G_n\}$ and let X be a compact metric space. Then there are natural isomorphisms*

$$k_G(X) \xrightleftharpoons[\lim]{\{\cdot \times_G G_n\}} k_{\{G_n\}}(X).$$

Therefore, (1.2) will follow if we can show

$$[X, \hat{B}G] \cong k_{\{G_n\}}(X). \quad (2.5)$$

We shall need the following notation. If H is a topological group, let

$$\xi_H: H \rightarrow EH \rightarrow BH$$

be Milgram's resolution (classifying space) [10] applied to H . See §4.

Let G be a compact metric group with Lie series $\{G_n\}$. Let

$$\xi_{\{G_n\}} \equiv \{G_n \rightarrow EG_n \rightarrow BG_n\},$$

$$\hat{E}G \equiv \lim\{EG_n\}, \hat{B}G \equiv \lim\{BG_n\},$$

$$\hat{\xi}_G \equiv G \rightarrow \hat{E}G \rightarrow \hat{B}G, \text{ and}$$

$$\hat{\xi}_{\{G_n\}} \equiv \{G_n \rightarrow E'G_n \rightarrow \hat{B}G_n\}$$

(take pullbacks of EG_n over $\hat{B}G$). (Note that $\lim\{E'G\} = \hat{E}G$.)

We shall call $\hat{\xi}_G$ the *standard universal principal open G -fibration*, see (1.2), (6.4).

There is a natural map $BG \rightarrow \hat{B}G$. In §6 we shall see that this map is a weak homotopy equivalence.

(2.6) PROPOSITION. *For any compact metric space X , any levelwise map of principal $\{G_n\}$ -fibrations, or any map of open principal G -fibrations over id_X is an isomorphism.*

PROOF. The first result follows easily from the analogous result in bundle theory; see, e.g., [6, p. 42]. The second result then follows by the *proof* of Proposition (2.4). $\square$

3. Compatible local sections. We shall need suitable versions of local sections and associated partitions of unity for principal $\{G_n\}$ -fibrations $\{G_n \rightarrow E_n \rightarrow X\}$ over compact metric spaces. An easy inductive argument, using local sections and associated partitions of unity for the principal bundle maps $\pi'_{n+1}: E_{n+1} \rightarrow E_n$ associated with a principal $\{G_n\}$ -fibration (see (2.1)–(2.3)), yields the following.

(3.1) PROPOSITION. *Let $\{G_n \rightarrow E_n \rightarrow X\}$ be a principal $\{G_n\}$ -fibration over a compact metric space X . Then, for each $n \geq 0$, there is a finite open cover*

$$\mathcal{U}_n = \{U_{(i_0, i_1, \dots, i_n)} | i_k \in I_k\},$$

a family of local sections.

$$\mathcal{S}_n = \{s_{(i_0, i_1, \dots, i_n)}: U_{(i_0, i_1, \dots, i_n)} \rightarrow E_n\},$$

and a partition of unity

$$\mathcal{H}_n = \{h_{(i_0, i_1, \dots, i_n)}: X \rightarrow [0, 1]\}$$

subordinate to $\mathcal{U}_n$, which satisfy the following compatibility conditions.

$$\begin{aligned}
(\mathcal{U}_n) \quad U_{(i_0, i_1, \dots, i_n, i_{n+1})} &\subset U_{(i_0, i_1, \dots, i_n)}, \\
(\mathcal{S}_n) \quad \pi'_{n+1} \circ s_{(i_0, i_1, \dots, i_n, i_{n+1})} &= s_{(i_0, i_1, \dots, i_n)}|: U_{(i_0, i_1, \dots, i_n, i_{n+1})} \rightarrow E_n, \\
(\mathcal{K}_n) \quad h_{(i_0, i_1, \dots, i_n)} &= \sum_{i_{n+1}} h_{(i_0, i_1, \dots, i_n, i_{n+1})}. \quad \square
\end{aligned}$$

4. Milgram's construction. Following N. E. Steenrod [16], G. Segal [17], S. Mac Lane [7], and [5], we describe the main properties of Milgram's resolution needed later. S. Mac Lane [7] and the second-named author [5] described Milgram's resolution $G \rightarrow EG \rightarrow BG$ of a topological group as the geometric realization (J. Milnor [12]) of a simplicial resolution $G \rightarrow \mathcal{E}G \rightarrow \mathcal{B}G$.

(4.1) *Standard simplicial resolutions; see, e.g., [7].* Given a topological group G , define a simplicial G -space (simplicial object over the category of right G -spaces) $\{\mathcal{E}G_n, d_i: \mathcal{E}G_n \rightarrow \mathcal{E}G_{n-1}, s_i: \mathcal{E}G_n \rightarrow \mathcal{E}G_{n+1} | n \geq 0, 0 \leq i \leq n\}$ by

$$\mathcal{E}G_n = G^{n+1} = \{(g_0, g_1, \dots, g_n)\},$$

$$d_i(g_0, g_1, \dots, g_n) = (g_0, g_1, \dots, g_i g_{i+1}, \dots, g_n),$$

$$s_i(g_0, g_1, \dots, g_n) = (g_0, g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

Regard G as a constant simplicial object, and let $\mathcal{B}G \equiv \mathcal{E}G \times_G *$ be the quotient simplicial space. The sequence $G \rightarrow \mathcal{E}G \rightarrow \mathcal{B}G$ forms the *standard simplicial resolution* of G .

(4.2) *Milgram's resolution.* Following G. Segal [17] and others, let R : simplicial spaces $\rightarrow CG$ (compactly generated spaces) be the extension of J. Milnor's geometric realization functor [12]. Then [16], [17], [7], [5], the sequence

$$(G \rightarrow EG \rightarrow BG) \equiv (G = RG \rightarrow R\mathcal{E}G \rightarrow R\mathcal{B}G)$$

is Milgram's resolution (classifying space) for G . The following results show the usefulness of Milgram's construction.

(4.3) **PROPOSITION (STEENROD [16]).** *Let G and H be topological groups. Then $E(G \times H) \cong EG \times EH$ as $(G \times H)$ -spaces, hence $B(G \times H) \cong BG \times BH$. $\square$*

(4.4) **PROPOSITION.** *Let $p: G \rightarrow H$ be a surjection of Lie groups with kernel K . Then the induced map $Bp: BG \rightarrow BH$ is a bundle map with fibre BK .*

PROOF (outlined). By L. Pontryagin [13, §44], G is locally isomorphic to $K \times H$ as sets under a local multiplication $K \times H \rightarrow G$. By the proof of (4.3) (in terms of simplicial G -spaces), BG is locally isomorphic to $BK \times BH$ near the basepoint. But the map Bp is homogeneous because the map $Ep: EG \rightarrow EH$ is a map of topological groups [16]. The conclusion follows. $\square$

We shall need the following formulas for classifying maps and homotopies between classifying maps. Compare, e.g., [6, pp. 54–57].

(4.5) *Alternate coordinates for Milgram's resolution.* If G is a topological

group, the map $G^{n+1} \rightarrow G^{n+1}$ given by

$$(g_0, g_1, \dots, g_n) \mapsto (g_0 g_1 \cdots g_n, g_1 g_2 \cdots g_n, \dots, g_n)$$

yields an isomorphism of G -spaces, where G acts on the rightmost factor of the "domain" G^{n+1} , and G acts diagonally (on the right) on the "range" G^{n+1} . This yields "delete-repeat" coordinates for EG and, hence, EG with these coordinates:

$$\mathcal{G}G = G^{n+1} = \{(g_0, g_1, \dots, g_n)\},$$

$$d_i(g_0, g_1, \dots, g_n) = (g_0, g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n), \text{ and}$$

$$s_i(g_0, g_1, \dots, g_n) = (g_0, g_1, \dots, g_{i-1}, g_{i-1}, g_i, g_i, g_{i+1}, \dots, g_n).$$

Thus [16] each point in EG may be represented uniquely as $((g_0, g_1, \dots, g_n), (t_0, t_1, \dots, t_n))$ where $g_i \neq g_{i+1}$ and $t_i \neq 0$ for all i .

(4.6) *Classifying maps.* As in [6, pp. 54–57], given a principal G -bundle $\xi: G \rightarrow E \rightarrow X$ over a compact metric space X , a finite open cover $\mathcal{U} = \{U_i\}$ of X , local sections $s_i: U_i \rightarrow E$, and a partition of unity $\{h_i\}$ subordinate to $\mathcal{U}$, there is a canonical classifying map $X \rightarrow BG$ for ξ given by an explicit formula. We omit the details.

(4.7) *Linear homotopies.* Compare [6, pp. 54–57]. The maps $f_{m,n}(G^{m+1} \times \Delta^m) * (G^{n+1} \times \Delta^n) \rightarrow EG$ defined by

$$\begin{aligned} & f_{m,n}(((g_0, g_1, \dots, g_m), (t_0, t_1, \dots, t_m)), s, \\ & \quad (g'_0, g'_1, \dots, g'_n), (t'_0, t'_1, \dots, t'_n)) \\ & = ((g_0, g_1, \dots, g_m, g'_0, g'_1, \dots, g'_n), \\ & \quad ((1-s)(t_0, t_1, \dots, t_m), s(t'_0, t'_1, \dots, t'_n))) \end{aligned}$$

define an equivariant map $\mu: EG * EG$ (diagonal action on the join) $\rightarrow EG$ whose restriction to each factor is the identity.

Now suppose that $f, f': X \rightrightarrows BG$ classify $G \rightarrow E \rightarrow X$. Let $\tilde{f}, \tilde{f}': E \rightrightarrows EG$ be the associated maps on total spaces. Define an equivariant homotopy $H: E \times I \rightarrow EG$ by the formula $H(y, s) = \mu(f(y), s, f'(y))$. Then the induced map $H: X \times I \rightarrow BG$ provides a canonical "linear homotopy" from f to f' .

5. The transformation α . Let $f: X \rightarrow \hat{B}G$ be a continuous map. We may associate to f the principal $\{G_n\}$ -fibration $f^* \xi_{\{G_n\}}$ over X where $\xi_{\{G_n\}}: \{G_n\} \rightarrow \{E'G_n\} \rightarrow \hat{B}G$ is the universal $\{G_n\}$ -fibration over BG ; see §2. This yields a function from the set of continuous maps $X \rightarrow \hat{B}G$ to $k_{\{G_n\}}(X)$. In this section we shall prove that homotopic maps $X \rightarrow \hat{B}G$ induce isomorphism principal $\{G_n\}$ -fibrations over X , and thus define a natural transformation of functors

$$\alpha: [\ , \hat{B}G] \rightarrow k_{\{G_n\}}(\), \quad \alpha_X: [c, \hat{B}G] \rightarrow k_{\{G_n\}}(X)$$

from compact metric spaces to pointed sets.

(5.1) PROPOSITION. Let X be a compact metric space and let $\xi = \{\xi_n: G_n \rightarrow$

$E_x \rightarrow X \times I$ be a principal $\{G_n\}$ -fibration over $X \times I$. Let $\xi^0 = \{\xi_n^0\}$ and $\xi^1 = \{\xi_n^1\}$ be the restrictions of ξ to $X \times 0$ and $X \times 1$, respectively. Then there is an isomorphism $\xi^0 \rightarrow \xi^1$.

PROOF (outlined). Use Proposition (3.1) and the usual proof that the restrictions of an ordinary bundle over $X \times I$ to $X \times 0$ and $X \times 1$ are isomorphic (e.g., Husemoller [6, pp. 54–57]) to inductively define compatible bundle maps $f_n: \xi_n \rightarrow \xi_n^1 \subset \xi_n$. The required isomorphism is given by $\{f_n\}: \xi_n^0 \rightarrow \xi_n^1$. $\square$

(5.2) COROLLARY. Let $f, g: X \rightarrow \hat{B}G$ be homotopic maps. Then f and g induce isomorphic principal $\{G_n\}$ -fibrations over X .

PROOF. Let $H: X \times I \rightarrow \hat{B}G$ be a homotopy from f to g . Then $f^*\xi = (H^*\xi)^0 \cong (H^*\xi)^1$ (by (5.1)) $= g^*\xi$. $\square$

This yields a well-defined natural transformation $\alpha: [, \hat{B}G] \rightarrow k_{\{G_n\}}()$.

6. Proof of (1.2). We shall prove our main result, that the natural transformation $\alpha: [X, \hat{B}G] \rightarrow k_{\{G_n\}}(X)$ is an isomorphism for compact metric X , by constructing classifying maps for principal $\{G_n\}$ -fibrations.

(6.1) Construction of classifying maps. Let $\xi = \{\xi_n: G_n \rightarrow E_n \rightarrow X\}$ be a principal $\{G_n\}$ -fibration over a compact metric space X .

Use Proposition (3.1) to obtain compatible finite open covers $\mathcal{U}_n$, families of local sections $\mathcal{S}_n$, and associated partitions of unity $\mathcal{H}_n$. Use (4.6) to obtain classifying maps $f_n: X \rightarrow BG_n$ for each bundle $\xi_n: G_n \rightarrow E_n \rightarrow X$ from this data. A lengthy but straightforward computation shows that the diagrams

$$\begin{array}{ccccc}
 E_{n+1} & \xrightarrow{\cong} & f_{n+1}^* EG_{n+1} & \longrightarrow & EG_{n+1} \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 X & \xrightarrow{\cong} & X & \xrightarrow{\quad} & BG_{n+1} \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & E_n & \xrightarrow{\cong} & f_n^* EG_n & \longrightarrow & EG_n \\
 & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & X & \xrightarrow{\quad} & X & \xrightarrow{\quad} & BG_n
 \end{array}$$

commute. Our construction thus yields the following.

(6.2) PROPOSITION. Let $\xi = \{\xi_n: G_n \rightarrow E_n \rightarrow X\}$ be a principal $\{G_n\}$ -fibration over a compact metric space X . Then there is a classifying map $f: X \rightarrow \hat{B}G$ with $\xi \cong f^*\xi_{\{G_n\}}$.

PROOF. Let $f = \lim_n \{f_n\}: X \rightarrow \hat{B}G \equiv \lim_n \{BG_n\}$. The conclusion follows. $\square$

(6.3) PROPOSITION. Isomorphic principal $\{G_n\}$ -fibrations over a compact metric space have homotopic classifying maps.

PROOF. Let $\xi = \{\xi_n: G_n \rightarrow E_n \rightarrow X\}$ and $\xi' = \{\xi'_n: G_n \rightarrow E'_n \rightarrow X\}$ be isomorphic principal $\{G_n\}$ -fibrations over X "classified" by f and mapping X to $\hat{B}G$, and $\xi \cong f^* \hat{\xi}_G$, and $\xi' \cong f'^* \hat{\xi}_G$. Because $\xi \cong \xi'$, $f' (= f' \circ \text{id}_X)$ also classifies ξ . We obtain a commutative diagram

$$\begin{array}{ccccc}
 \vdots & & & & \vdots \\
 \downarrow & & \xrightarrow{\tilde{f}_{n+1}} & & \downarrow \\
 E_{n+1} & \xrightarrow{\quad} & E'G_{n+1} & \xrightarrow{\quad} & EG_{n+1} \\
 \downarrow & & \downarrow \tilde{f}_n & & \downarrow \\
 E_n & \xrightarrow{\quad} & E'G_n & \xrightarrow{\quad} & EG_n \\
 \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \xrightarrow{\quad f \quad} & \hat{B}G & &
 \end{array}$$

in which $\{\tilde{f}_n: E_n \rightarrow EG_n\}$ is a compatible family of equivariant maps and a similar diagram involving $f': X \rightarrow \hat{B}G$ and $\{\tilde{f}'_n: E'_n \rightarrow EG_n\}$. Now use Proposition (4.8) to obtain compatible, equivariant "linear" homotopies $\{\tilde{H}_n: E_n \times I \rightarrow EG_n\}$ with $H_n|_0 = \tilde{f}_n$ and $H_n|_1 = \tilde{f}'_n$. Passing to quotient spaces yields compatible homotopies $\{H_n: X \times I \rightarrow \hat{B}G_n\}$. Finally, the required homotopy is given by $H = \lim\{H_n\}: X \times I \rightarrow \hat{B}G$ ($H|_0 = f$ and $H|_1 = f'$). $\square$

(6.4) *Proof of (1.2).* Propositions (6.2)–(6.3) yield a well-defined function $\beta: k_G(X) \rightarrow [X, \hat{B}G]$, where β associates to a bundle ξ its classifying map $\beta(\xi) \in [X, \hat{B}G]$. By construction, $\xi \cong \alpha(\beta(\xi)) \equiv \beta(\xi)^* \hat{\xi}_G$. Also, if $[f] \in [X, \hat{B}G]$, $\beta(\alpha(f))$ and f both classify $f^* \hat{\xi}_G$, so $[\beta(\alpha(f))] = [f]$ by (6.7). Therefore $\alpha: [X, \hat{B}G] \rightarrow k_G(X)$ and β are inverse isomorphisms. $\square$

(6.5) PROPOSITION. (a) *Every open principal G -fibration over a CW complex is a principal G -bundle.* (b) *The natural map $BG \rightarrow \hat{B}G$ is a weak homotopy equivalence.*

PROOF. By (1.2) or a direct argument similar to (5.1), k_G is a homotopy functor, so each open principal G -fibration over a contractible space is a principal G -bundle. (a) now follows. (b) now follows from studying open principal G -fibrations over spheres. $\square$

(6.6) REMARKS. (a) The above classification results can be extended to open principal G -fibrations over paracompact spaces.

(b) The classifying space $\hat{B}G = \lim\{BG_n\}$ can be interpreted as the

homotopy inverse limit [6, §4], $\text{holim}\{BG_n\}$, of the tower of fibrations $\{BG_n\}$; see (4.6). In this sense $\hat{B}G$ is a kind of completion of BG ; consider, for example, profinite groups G .

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DEPARTMENT OF MATHEMATICS, SUNY, BINGHAMTON, NEW YORK 13901

DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NEW YORK 11550 (Current address of H. M. Hastings)

Current address (D. A. Edwards): Department of Mathematics, University of Georgia, Athens, Georgia 30602