ON THE SEIFERT MANIFOLD OF A 2-KNOT

BY

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ABSTRACT. From geometric facts about embeddings $S^2 \rightarrow S^4$ we study the relationship between the smallest number of normal generators (weight) of a group and its preabelian presentations.

Introduction. Let $f: S^2 \rightarrow S^4$ be an embedding or 2-knot: f always [9, Lemma 1] extends to an embedding $\bar{f}: S^2 \times D^2 \rightarrow S^4$. Let X be the closure of $S^4 - \text{Im } f$, a compact manifold called the *complement* of f . $\partial X = S^2 \times S^1$ and $\pi_1(X)$ is a finitely presented group G . Such G are called knot groups.

If $z_0 \in S^2$ and $z'_0 \in S^1$ we choose $(z_0, z'_0) \in \partial X$ as our basepoint. Let x be the element of G represented by $z_0 \times S^1$. Then [5, Theorem 1]

(1) $H_1(G) = G/G' = \mathbb{Z}$ with generator x (a meridian for f).

(2) $H_2(G) = 0$.

(3) x and all its conjugates generate G .

(G' is the commutator subgroup $[G, G]$ of G .)

In general if $S \subseteq G$, we write $N_G S$ for $\cap N$, where $N \triangleleft G$ and $N \supseteq S$. Then

(3) is $N_G\{x\} = G$, and $G' = N_G\{[g, h]: g, h \in G\}$.

Condition (2) is hard to handle geometrically in S^4 . We choose instead a somewhat stronger statement:

(2') G has a finite presentation with $r + 1$ generators and r relators.

Statement (2) is strictly stronger than (2') and knots exist whose group does not satisfy (2') [5, pp. 106–107]. The purpose of this note is to show:

THEOREM. *Let G be a finitely presented group satisfying (1), (2') and (3); then*

(i) *There exists a knot $f: S^2 \rightarrow S^4$ with group G .*

(ii) *There exists a submanifold $V \subset S^4$ of the form $(S^1 \times S^2 \# \cdots \# S^1 \times S^2)_0$ (that is, $S^1 \times S^2 \# \cdots \# S^1 \times S^2$ minus an open disk) with $\partial V = f(S^2)$, $\pi_1(V)$ free of rank r .*

(iii) *$\pi_1(V) = \pi_1(S^4 - V)$ is a free group Φ of rank r ; there exist monomorphisms $\nu_0, \nu_1: \Phi \rightarrow \Phi$ such that G has a presentation*

$$\langle x, \Phi: \nu_0(\varphi) = x\nu_1(\varphi)x^{-1}, \varphi \in \Phi \rangle.$$

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As a corollary we show that no such group G can be of the form $A * B$, $A \neq 1$, $B \neq 1$ (cf. [3]). We also obtain useful information about the cohomology of G .

1. Seifert manifolds of knots. Suppose $f: S^2 \rightarrow S^4$ is a knot. A Seifert manifold for f is any compact, framed 3-manifold $V \subseteq S^4$ with boundary $f(S^2)$.

If the group associated to f satisfies (2') then it has a presentation of the form

$$\langle x, b_1, \dots, b_r; b_1 B_1, \dots, b_r B_r \rangle \quad (B_1, \dots, B_r \in F'), \quad (1)$$

where we write F for the free group $\langle x, b_1, \dots, b_r \rangle$, J for $\langle x \rangle$ and L for $\langle b_1, \dots, b_r \rangle$ so that $F = J * L$. Let $\lambda: F \rightarrow L$ be the map defined by $\lambda(x) = 1$, $\lambda(b_j) = b_j$, $j = 1, \dots, r$.

Clearly any group with presentation (1) satisfies conditions (1) and (2'). For it to satisfy condition (3) it is necessary and sufficient that

$$L = N_L \{ \lambda(b_1 B_1), \dots, \lambda(b_r B_r) \}. \quad (2)$$

Conversely for any group with a presentation (1) and satisfying condition (2) we can find an embedding $f: S^2 \rightarrow \Sigma^4$, where Σ is a homotopy sphere and $G = \pi_1(\Sigma^4 - f(S^2))$. In [2] we prove we may assume $\Sigma = S^4$ but the paper contains some gaps. We wish to present that proof here with more detail.

LEMMA (1). *Let G be a finitely presented group satisfying conditions (1), (2') and (3). Then we may assume G has a presentation (1) such that $\lambda(b_1 B_1), \dots, \lambda(b_r B_r)$ actually generate L .*

PROOF. I. Let $\iota_0: D^3 \rightarrow D^5$ be the standard embedding of the 3-ball in D^5 and let $W_0 = D^5 - L(D^3)$. The boundary U_0 of W_0 is the complement of the trivial knot $f_0 = \iota_0|_{S^2}$, with group J . Let $\psi_j: S^0 \times D^4 \rightarrow U_0$ be mutually disjoint embeddings ($j = 1, \dots, r$). We attach 1-handles $h_j^1 \approx D^1 \times D^4$ to W_0 via ψ_j . Let ι' be the embedding $D^3 \rightarrow D^5 + \Sigma h_j^1$ and W' the complement of $\iota'(D^3)$. The fundamental group of $U' = \partial W'$ is now $F = J * L$, where the b_j correspond to the h_j^1 . Let now $\bar{\psi}_j: S^1 \times D^3 \rightarrow U'$ be mutually disjoint embeddings representing $b_j B_j \in \pi_1(U')$, that is, $\bar{\psi}_j(S^1 \times 0)$ is a loop in U' representing the word $b_j B_j$. Attaching 2-handles $h_j^2 \approx D^2 \times D^3$ to W' via the $\bar{\psi}_j$ we obtain an embedding $\iota'': D^3 \rightarrow T$, where $T = D^5 + \Sigma h_j^1 + \Sigma h_j^2$ is a contractible space with boundary Σ , a homotopy 4-sphere. Embedding $f = \iota''|_{S^2}$ is a knot whose group is presented by (1).

II. Recall now f_0 extends to an embedding $d_0: \Delta \rightarrow S^4$, where Δ is the 3-disk. Let $d': \dot{\Delta} \rightarrow U'$ be the composition $\dot{\Delta} \rightarrow U_0 \cup \varphi_j(S^0 \times D^4) \subset U'$ which we may assume to be transversal to the $\bar{\psi}_j$. Let $\sigma: J * L \rightarrow J$ be the map defined by $\sigma x = x$, $\sigma b_j = 1$ its kernel is a free product $\Pi^* L_n$ ($n \in \mathbb{Z}$) where $L_n = \langle b_{1n}, \dots, b_{rn} \rangle$, $b_{jn} = x^n b_j x^{-n}$.

We observe now that $d'\hat{\Delta}$ has a framing $\bar{v}'(z, t) \in U'$ ($z \in \hat{\Delta}$, $-1 < t < 1$) such that $\bar{v}'(z, 0) = d'z$. Define $v'_\varepsilon: \hat{\Delta} \rightarrow U' - d'\hat{\Delta}$ by $z \mapsto \bar{v}'(z, \varepsilon)$, $\varepsilon = \pm 1$. If $z_0 \in \hat{\Delta}$ is the basepoint, we adopt $u_0 = v'_1 z_0$ as the basepoint for $U' - d'\hat{\Delta}$. Let γ be a fixed arc from u_0 to $v'_{-1}(z_0)$ in $U' - d'\hat{\Delta}$. We are going to alter $\hat{\Delta}$ but we may assume γ remains in the complement of it throughout.

Suppose $j = 1$ for the time being, and let $b_1 B_1 = b_1 w_0 x^{\varepsilon_1} \cdots x^{\varepsilon_s} w_s$ where, if $1 < a < s$, $w_a \in L$, $\sum \varepsilon_a = 0$ and so $b_1 B_1 = b_{10} \prod_a w_{a n_a}$ ($w_{a n_a} = x^{n_a} w_a x^{-n_a}$) for certain $n_a \in \mathbb{Z}$. Let $M = \max\{n_a: n_a > 0\}$, $\mu = \min\{n_a: n_a < 0\}$. Define H_M^0 (resp. H_μ^0) to be the subgroup of L generated by the w_a for which $n_a = M$ (resp. $n_a = \mu$). Both subgroups are free of finite rank generated by, say, $h_1^0, \dots, h_{t_0}^0$ and $g_1^0, \dots, g_{u_0}^0$. It is understood that each h_p^0 (g_q^0) is one of the w_a ($n_a = M$) or w_a ($n_a = \mu$) for $p = 1, \dots, t_0$; $q = 1, \dots, u_0$.

With u_0 as basepoint, define maps $\bar{h}_p, \bar{g}_q: S^1 \times D^3 \rightarrow U' - d'\hat{\Delta}$ representing h_p^0, g_q^0 respectively. Then we may define new embeddings $\bar{h}'_p, \bar{g}'_q: S^1 \times D^3 \rightarrow U'$ representing $x h_p^0 x^{-1}$ and $x^{-1} g_q^0 x$ which are transversal to d' and where $\bar{h}'_p(S^1 \times D^3) \cap d'\hat{\Delta} = S^0 \times D^3$ for each p and a similar relation for $\bar{g}'_q$. The points in $(\bar{h}'_p)^{-1}(\bar{h}'_p(S^1 \times 0) \cap d'\hat{\Delta})$ partition S^1 in two intervals I_p, I'_p and if we join the endpoints of $\bar{h}'_p(I_p)$ and $\bar{h}'_p(I'_p)$ in $d'\hat{\Delta}$ one of the resulting loops, say the one coming from I_p , represents h_p^0 in U' . Similarly, let J_q be the corresponding interval for $\bar{g}'_q$. Write

$$\Delta' = \Delta - \bigcup_{p,q} \{ \bar{h}'_p(I_p \times D^3) \cup \bar{g}'_q(J_q \times D^3) \},$$

$$\Delta_1 = \Delta' \cup \bigcup_{p,q} \{ \bar{h}'_p(I_p \times D^3) \cup \bar{g}'_q(J_q \times D^3) \}.$$

Then d' can be extended to a (framed) embedding $d'_1: \Delta_1 \rightarrow U'$ with framing $\bar{v}_1$ and we may assume $z_0 \in \Delta_1$, $\gamma \subset U' - d'_1 \Delta_1$. The intersection points in $P_0 = \Delta \cap \bar{\psi}_1(S^1 \times 0)$ are in 1-1 correspondence with $\{1, \dots, s\} \times S^0$, whereas $P_1 = \Delta_1 \cap \bar{\psi}_1(S^1 \times 0)$ is in 1-1 correspondence with $\{\alpha: \mu \neq n_\alpha \neq M\} \times S^0$. Algebraically, this means the loop $\bar{\psi}_1(S^1 \times 0)$ represents a word $b_{10} w'_0 x^{\varepsilon_1} \cdots x^{\varepsilon_t} w'_t$ ($t < s$), $w'_t \in \pi_1(U' - d'_1 \Delta_1) = L * H_\mu^0 * H_M^0$ and so we can write $[\bar{\psi}_1(S^1 \times 0)] = b_{10} \prod_\beta w'_{\beta n_\beta}$ where if $1 < \beta < t$, $\mu + 1 < n_\beta < M - 1$.

III. We now repeat the procedure in II on the words w'_β with $n_\beta = M - 1$ or $\mu + 1$ provided $M - 1 > 0$, $\mu + 1 < 0$. If, say, $\mu + 1 = 0$ we work with $\{w'_\beta | n_\beta = M - 1\}$ only. Let H_{M-1}^0 and $H_{\mu+1}^0$ be the free group generated by $\{w'_\beta | n_\beta = M - 1\}$ and $\{w'_\beta | n_\beta = \mu + 1\}$ respectively and $\{h_1^1, \dots, h_{t_1}^1\}$, $\{g_1^1, \dots, g_{u_1}^1\}$ subsets of the above generating sets that freely generate the groups. Define $h_p^{(1)}, g_q^{(1)}: S^1 \times D^3 \rightarrow U' - d'_1(\Delta_1)$ representing the h_p^1, g_q^1 respectively and alter them to embeddings $\bar{h}_p^{(1)}, \bar{g}_q^{(1)}: S^1 \times D^3 \rightarrow U'$ representing $x h_p^1 x^{-1}$ and $x^{-1} g_q^1 x$, transversal to d'_1 and intersecting $d'_1(\Delta_1)$ in a copy of $S^0 \times D^3$. Again $(\bar{h}_p^{(1)})^{-1}(\bar{h}_p^{(1)}(S^1 \times 0) \cap d'_1(\Delta_1))$ partitions S^1 in two intervals

$I_p^{(1)}$ and $I_p^{(1)'}$ and we choose, as in II, $I_p^{(1)}$, that one that represents $h_p^{(1)}$. We may alter Δ_1 to Δ_2 and find a framed embedding $d_2: \hat{\Delta}_2 \rightarrow U'$ (with framing $\bar{\nu}_2$) so that $\gamma \subset U' - d_2(\Delta_2)$ and where $P_2 = d_2\Delta_2 \cap \psi_1(S^1 \times 0)$ is in 1-1 correspondence with $\{\alpha: \mu + 2 \leq n_\alpha \leq M - 2\}$. After $m = \max\{|\mu|, M\}$ we obtain a framed embedding $d_m: \Delta_m \rightarrow U'$ disjoint from $\psi_1(S^1 \times D^3)$ and Δ_m is obtained from the connected sum of S copies of $S^1 \times S^2$ by removing an open disk (or Δ_m is obtained from Δ by attaching S handles $D^1 \times S^2$). We may assume $z_0 \in \Delta_m$, $\gamma \subset U' - d_m\hat{\Delta}_m$.

IV. The fundamental group $\pi_1(\Delta_1)$ is free and its free generators are in 1-1 correspondence with h_p^0 and g_q^0 . Similarly $\pi_1(\Delta_2)$ is free in a set in 1-1 correspondence with h_p^η , g_q^θ , $0 \leq \eta, \theta \leq 1$. In general $\pi_1(\Delta_m)$ has free generators which we may label $h_p^{(\eta)}$, for $p = 1, \dots, t_\eta$, $1 \leq \eta \leq M$ and $g_q^{(\theta)}$, $q = 1, \dots, u_\theta$, $1 \leq \theta \leq |\mu|$.

On the other hand, $\pi_1(U' - d_m\hat{\Delta}_m)$ is also free generated by

$$\{b_j: 1 \leq j \leq r\} \cup \left\{ \left[\bar{h}_p^\eta(S^1 \times 0) \right]; p = 1, \dots, t_\eta, 1 \leq \eta \leq M \right\} \\ \cup \left\{ \left[\bar{g}_q^\theta(S^1 \times 0) \right]; q = 1, \dots, u_\theta, 1 \leq \theta \leq |\mu| \right\}$$

where, as usual, $[x]$ is the homotopy class of any loop x based on $u_0 = \gamma(0) \in U'$.

The framing $\bar{\nu}_m$ permits us to find [4, p. 573] a compact manifold Y which is a deformation retraction of $U' - d_m(\hat{\Delta}_m)$ and with boundary $V_0 \cup V_1$, where $V_t \approx \Delta_m$, and where map $f_t: \Delta_m \hookrightarrow V_t \hookrightarrow Y$ is homotopic to ν_{-1} if $t = 0$ and to ν_1 if $t = 1$. If we assume $\gamma \subset Y$ then $z_0 \in V_1$ and f_t defines homomorphisms $f_t: \pi_1(V) \rightarrow \pi_1(Y)$ by

$$f_{1*}(h_p^\eta) = w_{1p}, \quad f_{0*}(h_p^\eta) = [\gamma \bar{h}_p^\eta(S^1 \times 0) \gamma^{-1}], \\ f_{1*}(g_q^\theta) = [\bar{g}_q^\theta(S^1 \times 0)], \quad f_{0*}(g_q^\theta) = w'_{\theta q}, \quad (3)$$

where we adopt the following conventions: $w_{\eta p}$ (resp. $w'_{\theta q}$) is a word in the free group $\pi_1(U' - d_\eta(\hat{\Delta}_\eta))$ (resp. $\pi_1(U' - d_\theta(\hat{\Delta}_\theta))$) which is part of a set of free generators for $H_{M-\eta}$ (resp. $H_{\mu+\theta}$). Thus, for example w'_{1q} is one of the w_α ($n_\alpha = \mu$) of II and w_{2p} is one of the w_β ($n_\beta = M - 1$) of III. Since these $w_{\eta p}$, $w'_{\theta q}$ are free generators of $H_{M-\eta}$, $H_{\mu+\theta}$, the maps f_{t*} are monomorphisms. Furthermore if $x = h_p^\eta$ or g_q^θ , the $[f_{1*}(x)]^{-1}f_{0*}(x)$ generate a free factor Φ in $\pi_1(Y)$ since they contain letter x only once. Thus $\pi_1(Y) = L * \Phi$.

V. To work with all r words $b_j B_j$ simultaneously, we repeat the above procedure on Δ_m using an embedding $S^1 \times D^3 \rightarrow U' - d_m(\hat{\Delta}_m)$ representing $b_2 B_2$, etc.

VI. Summarizing: we have found a 3-manifold V obtained from Δ by attaching handles, that is V is of the form $S^1 \times S^2 \# \dots \# S^1 \times S^2 - \hat{D}^3$ and a framed embedding $d: \hat{V} \rightarrow U'$. The embeddings $\bar{\psi}: S^1 \times D^3 \rightarrow U'$ are

disjoint from $d(\dot{V})$. If $\Phi = \pi_1(V)$, $\pi_1(U' - d(V)) = \pi_1(Y)$ has the form $L * \Phi$ and we have maps $f_1, f_0: \Phi \rightarrow L * \Phi$ where the $y_j = [f_1(x_j)]^{-1}f_0(x_j)$ generate the free factor Φ if the x_j do. Recall $U' = \partial W'$ (see I) and so we can find an embedding $\iota': S^2 \rightarrow \partial(D^3 + \Sigma h_j^1)$ such that U' is the complement of $\iota'(S^2)$; $d: \dot{V} \rightarrow U'$ can be extended to an embedding $V \rightarrow U' \cup \iota(S^2)$. The $\bar{\psi}_j$ represent in $U' - d_m(\dot{V})$ elements of the form $b_j w_{j0} v_{j1} \cdots v_{js} w_{js}$ ($w_{jl} \in L$, $v_{jl} \in \Phi$) where the w_{jl} correspond to those factors w_{an} with $n_a = 0$. Fix j , by increasing the rank of Φ by 2 we may assume the $\bar{\psi}_j$ represent words $b_j v_j$ where v_j is in $\Phi_1 = \Phi * \langle c, d: \rangle$ where $L * \Phi_1 = \pi_1(U' - d'(V'))$, V' is obtained from V by taking connected sum with $S^1 \times S^2 \# S^1 \times S^2$. In fact, write $xb_j B_j x^{-1} = x b_j x^{-1} \cdot x B_j x^{-1}$. To distinguish those four factors we write $x_1 b_j x_2 x_3 B_j x_4$. Deform $\bar{\psi}_j$ to $\bar{\psi}'_j$ so that it represents $xb_j B_j x^{-1}$. Thus $\bar{\psi}'_j(S^1 \times D^3)$ intersects $d_m(V)$ and $\bar{\psi}'_j^{-1}(\bar{\psi}'_j(S^1 \times D^3) \cap d_m \dot{V}) = \bigcup_{i=1}^4 p_i \times D^3$ where $p_i \in S^1$ corresponds to factor x_i ($i = 1, 2, 3, 4$). The p_i partition S^1 in four intervals, I_1, I_2, I_3, I_4 , with $I_1 = \{p_4, p_1\}$, $I_2 = \{p_1, p_2\}$, etc. Finally, let $\frac{1}{2}D^2 = \{z \in R^4: |z| < \frac{1}{2}\}$. Define

$$V'_0 = V - \left\{ \bar{\psi}'_j(I_1 \times \dot{D}^3) \cup \bar{\psi}'_j(I_3 \times \frac{1}{2}\dot{D}^3) \right\},$$

$$V' = V'_0 \cup \left\{ \bar{\psi}'_j(I_1 \times \dot{D}^3) \cup \bar{\psi}'_j((S^1 - I^3) \times \frac{1}{2}\dot{D}^3) \right\}.$$

This V' satisfies the desired condition. By doing this for all j , we obtain a manifold $V'' \subset U'$ of the form $\#(S^1 \times S^2) - \dot{D}^3$ such that if Y_1 is a deformation retraction of $U' - V''$ which is a compact manifold with $\partial Y_1 = V''_0 \cup V''_1$, then

- (a) $\pi_1(V'') = \Phi$ is free; let $\{x_1, \dots, x_n\}$ be a set of generators.
- (b) $\pi_1(Y_1) = L * \Phi$ and $V''_i \subset Y_1$ define two monomorphisms $f_i: \pi_1(V'') \rightarrow \pi_1(Y_1)$ such that $y_i = f_1(x_i)f_0(x_i^{-1})$ are free generators of free factor Φ in $\pi_1(Y_1)$.
- (c) The $\bar{\psi}_j$ represent words $b_j v_j$, $v_j \in \Phi$ in $\pi_1(Y_1)$.

Attaching the two handles h_j^2 to U' along the $\bar{\psi}_j$ adds the relations $b_j = \bar{v}_j^{-1}$. Let $Y'' = \partial T - (Y_1 \cap T)$; then $\pi_1(Y'') = \Phi$.

VII. We have found an embedding $d'': V'' \rightarrow \partial T = \Sigma$ such that both V'' and Y'' ($\sim \Sigma - d''(V'')$) have free fundamental group Φ with generators $\{x_i\}$ and $\{y_i = f_1(x_i^{-1})f_0(x_i)\}$ respectively. We can present our group G by

$$\langle x, y_1, \dots, y_n: f_1(x_i^{-1})x f_0(x_i)x^{-1}, i = 1, \dots, n \rangle. \quad (4)$$

In fact $\Sigma - d''(S^2)$ has fundamental group G . On the other hand $\Sigma - d''(S^2)$ is obtained from Y'' by identifying $\dot{V}''_0$ to $\dot{V}''_1$ and so by the van Kampen theorem, G has presentation (4). Clearly presentation (4) satisfies condition (2).

The result in the introduction follows from this proof.

THEOREM (2). *Let G be a group satisfying conditions (1), (2') and (3); then there exists a knot $f: S^2 \rightarrow S^4$ with associated group G .*

PROOF. Repeat part I of the proof of Lemma (1) using a presentation that satisfies condition (2). Using the same notation we may assume that in T the attaching maps $\bar{\psi}_j$ isotope to maps so that $\bar{\psi}_j(S^1 \times 0)$ intersects the transversal disk of h_k^1 in one point if $j = k$ or it is empty if $j \neq k$. This is the geometric meaning of condition (2). In particular ∂T is the standard sphere.

2. Commutator subgroup. In view of our above results we may write a structure theorem for $G' = [G, G]$.

Let Φ_m be a group isomorphic to Φ ($m \in \mathbb{Z}$) and $a_m: \Phi \rightarrow \Phi_m$ an explicit isomorphism. Consider $P = \prod_m^* \Phi_m$, an infinite free product. Write

$$\dots *_{\Phi} \Phi_{-1} *_{\Phi} \Phi_0 *_{\Phi} \Phi_1 *_{\Phi} \dots \quad (5)$$

for the quotient $P/N_P\{a_m f_1(x_i)[a_{m+1} f_0(x_i)]^{-1}: m \in \mathbb{Z}, i = 1, \dots, n\}$, that is the infinite free product of copies of Φ with amalgamations by f_0 and f_1 ([6], as in [4] and [7]).

PROPOSITION (3). *If G satisfies conditions (1), (2') and (3), G is isomorphic to the group presented by (5), where Φ, f_0 and f_1 are as in Lemma (1).*

For the proof of this proposition construct $\tilde{X}$, the universal abelian (infinite cyclic) covering space of X by taking copies Y_m'' of Y'' (cf. Lemma 1) with boundary $V_0''(m) \cup V_1''(m)$. Then $\tilde{X}$ is obtained by identifying $V_1''(m)$ to $V_0''(m+1)$ for all m . An application of the van Kampen theorem and a direct limit yield the desired result since $\pi_1(Y_m'') = \Phi_m$.

COROLLARY (4). *G' has cohomological dimension ≤ 3 . If G' is finitely presented then G' is free.*

THEOREM (5). *G' has cohomological dimension ≤ 2 .*

PROOF. Let M be a (left) G' -module. Then (5) defines structures of Φ_m -module for M . If G is presented by (4) $x\Phi_m x^{-1} = w\Phi_{m+1}w^{-1}$ for some $w \in G'$ and so $H^*(\Phi_m; M)$ is canonically isomorphic to $H^*(\Phi_{m+1}; M)$ and so $H^*(\Phi; M)$ is independent of m ([1, p. 197], [3, Lemma 2]). Write J for the free group with generator x . As in [4], $H^*(G'; M)$ fits in a Mayer-Vietoris sequence of J -modules, where differentials d_0 and d_1 are both $f_1^* \otimes 1 - f_0^* \otimes x$:

$$\begin{aligned} 0 \rightarrow H^0(G'; M) &\rightarrow H^0(\Phi; M) \otimes Z \xrightarrow{d_0} H^0(\Phi; M) \otimes ZJ \\ &\rightarrow H^1(G'; M) \rightarrow H^1(\Phi; M) \otimes Z \xrightarrow{d_1} H^1(\Phi; M) \otimes ZJ \\ &\rightarrow H^2(G'; M) \rightarrow 0. \end{aligned}$$

I. For $m \in H^0(\Phi; M) = M^\Phi$, $d_0(m \otimes 1) = m \otimes (1 - x)$ and so in $\text{coker } d_0$ we equate the actions of $f_0(x_i)$ and $f_1(x_i)$ (x_i generate Φ). Since $y_i = f_1(x_i^{-1})f_0(x_i)$ also generate Φ , $\text{coker } d_0 = M^\Phi$, and our sequence reduces to

$$0 \rightarrow M^\Phi \rightarrow H^1(G'; M) \rightarrow M_\Phi \otimes ZJ \xrightarrow{d_1} M_\Phi \otimes ZJ \rightarrow H^2(G'; M) \rightarrow 0 \quad (6)$$

since $H^1(\Phi; M) = M_\Phi = M \otimes_\Phi Z$ (cf. [1, p. 197]). Tensoring (6) over J with Z we get

$$M_\Phi \xrightarrow{\bar{d}_1} M_\Phi \rightarrow H^2(G'; M)_J \rightarrow 0, \quad (7)$$

an exact sequence of abelian groups.

II. To calculate $\bar{d}_1$, let $R = Z\Phi$ and $I = \ker(R \rightarrow Z)$ [1, X.4]. If $\xi_i = x_i - 1$ and $\eta_i = y_i - 1$, I is a free R -module in the ξ_i or the η_i ($i = 1, \dots, n$). For a left Φ -module M define $\iota^*: M \rightarrow \text{Hom}_\Phi(I, M)$ by $\iota^*(m)(w - 1) = wm - m$. Then (loc. cit.) $H^1(\Phi; M) = \text{coker } \iota^*$.

Let $g \in \text{Hom}_\Phi(I; M)$. Then

$$\begin{aligned} (f_1^* - f_0^*)g(\xi_i) &= g(f_1(x_i) - f_0(x_i)) \\ &= f_1(x_i)g(1 - f_1(x_i^{-1})f_0(x_i)) = -f_1(x_i)g(\eta_i) \end{aligned}$$

and since $f_1(x_i)$ is a unit in R , $f_1^* - f_0^* = \bar{d}_1$ is an isomorphism $\text{Hom}_\Phi(I; M) \rightarrow \text{Hom}_\Phi(I; M)$ which induces an isomorphism $M_\Phi \rightarrow M_\Phi$ and so, by the exactness of (7)

$$H_2(G'; M)_J = 0. \quad (8)$$

III. By the Lyndon spectral sequence [1, XVI. 7],

$$H^3(G; M) = H^1(J; H^2(G'; M)) = H^2(G'; M)_J = 0$$

by (8). Q.E.D.

COROLLARY (6). For a group G satisfying conditions (1), (2'), (3) and for all G' -modules M , $H_2(G'; M)_J = 0$.

PROPOSITION (7). A group satisfying conditions (1), (2') and (3) is never a nontrivial free product. (Compare with [3].)

PROOF. If it is, we may assume $G = H * J$. Then $G' = \prod_m^* H_m$ where $H_m \approx H$. For all H -modules M , $H^2(H; M) = H^2(G'; M)_J = 0$ by Corollary (6), so that c.d. $H < 1$, that is, H is free [8, 0.3]. By condition (1), $H/H' = 0$ so H itself must be trivial.

3. Generalizations. Let mS^2 be the disjoint union of m copies of S^2 : $mS^2 = S_1^2 + \dots + S_m^2$. An m -link is an embedding $f: mS^2 \rightarrow S^4$. If $G = \pi_1(S^4 - \text{Im } f)$ then [5, Theorem 3]

(4) $H_1(G; Z)$ is free of rank m with generators $x_1, \dots, x_m$,

$$(5) H_2(G; Z) = 0,$$

$$(6) N_G\{x_1, \dots, x_m\} = G,$$

where the x_i are meridians for $f|S_i^2$.

Again if we change (5) by

(5') G has a presentation with $r + m$ generators and r relations, we obtain

THEOREM (8). *For any group G satisfying conditions (4), (5') and (6) there exists an embedding $f: mS^2 \rightarrow S^4$ with associated group G ; the commutator subgroup G^1 has $c \cdot d \leq 2$ and G is never a nontrivial free product with $m + 1$ factors.*

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