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SETS OF DIVERGENCE ON THE GROUP 2^ω

BY

DAVID C. HARRIS[1] AND WILLIAM R. WADE

ABSTRACT. We show that there exist uncountable sets of divergence for $C(2^\omega)$. We also show that a necessary and sufficient condition that a set E be a set of divergence for $L^p(2^\omega)$, $1 < p < \infty$, is that E be of Haar measure zero.

1. **Introduction.** Let $\psi_0, \psi_1, \ldots$ represent the Walsh functions ordered according to Paley and defined on the group 2^ω (see Fine [2]). For each $f \in L^1(2^\omega)$ let

$$S_N(f) = \sum_{k=0}^{N-1} \hat{f}(k)\psi_k,$$

where $\hat{f}(k)$ is the kth Walsh Fourier coefficient of f, $k = 0, 1, \ldots$.

If $B \subseteq L^1(2^\omega)$, then a set $E \subseteq 2^\omega$ is said to be a *set of divergence for B* if there is a function $f \in B$ such that $S_N(f)$ diverges on E as $N \to \infty$.

It is known that every singleton in 2^ω (hence, by our Lemma 3, every countable subset of 2^ω) is a set of divergence for continuous functions, $C(2^\omega)$ (see [6] or [12]), and that every subset of 2^ω is a set of divergence for $L^1(2^\omega)$ [7]. Other results concerning sets of divergence on the group 2^ω can be found in [3] and [9].

Our first theorem was announced in [10]. It shows (see [8]) that the sets of divergence for $L^p(2^\omega)$, $1 < p < \infty$, coincide with those subsets of 2^ω of Haar measure zero.

THEOREM 1. *If E is a subset of the group 2^ω which is of Haar measure zero then there exists a function f belonging to $L^p(2^\omega)$ for $1 < p < \infty$ such that $S_{2^n}(f)$ diverges on E as $n \to \infty$.*

Our second theorem produces a new class of sets of divergence for $C(2^\omega)$ which contains uncountable sets of divergence for $C(2^\omega)$.

THEOREM 2. *For each $i = 0, 1$ and each nonnegative integer k set*

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$$I(i, k) = \{x \in 2^\omega : \psi_k(x) = (-1)^i\}. \tag{0}$$

Suppose that k_j, n_j and i_j are nonnegative integers which satisfy $i_j = 0$ or 1 and $2^{n_j} \leq k_j < 2^{n_j+1}$ for $j = 1, 2 \ldots$. If $E \subseteq 2^\omega$ is a set of Haar measure zero and if

$$E = \bigcap_{j=1}^{\infty} I(i_j, k_j),$$

then E is a set of divergence for $C(2^\omega)$.

This theorem is a partial answer to the following question: Is every subset $E \subseteq 2^\omega$ of Haar measure zero necessarily a set of divergence for $C(2^\omega)$? The trigonometric analogue to this question was answered in the affirmative over a decade ago by Kahane and Katznelson (see [4]), but in the Walsh case it continues to defy resolution.

2. **The proof of Kahane and Katznelson.** Our proof of Theorem 2 is similar to that of Kahane and Katznelson. We shall supply details only for those portions of the argument which are significantly different in the Walsh setting.

The facts are as follows. If B is a homogeneous Banach space in $L^1(2^\omega)$ then the Cesaro means, $\sigma_n(g)$, of the Walsh Fourier series of a function $g \in B$ converge to g in B as $n \to \infty$. From this it follows that if $g \in B$ then there is a function $f \in B$ and a sequence $0 \leq \Omega_0 \leq \Omega_1 \leq \cdots \leq \Omega_n \to \infty$ as $n \to \infty$ such that

$$\hat{f}(k) = \Omega_k \hat{g}(k), \qquad k = 0, 1, \ldots.$$

This is the requisite step in proving the following lemma.

Lemma 1. *A necessary and sufficient condition that a subset E of 2^ω be a set of divergence for a homogeneous Banach space $B \subseteq L^1(2^\omega)$ is the existence of a function $f \in B$ such that*

$$\sup_k |S_k(f, x)| = +\infty \quad \textit{for } x \in E.$$

This leads to the central reduction which Kahane and Katznelson used:

Lemma 2. *A necessary and sufficient condition that a subset E of 2^ω be a set of divergence for a homogeneous Banach space $B \subseteq L^1(2^\omega)$ is the existence of Walsh polynomials $P_1, P_2, \ldots$ which satisfy*

$$\sum_{n=1}^{\infty} \|P_n\|_B < \infty \tag{1}$$

and

$$\sup_n \left\{ \sup_k |S_k(P_n, x)| \right\} = +\infty \tag{2}$$

for all $x \in E$.

Necessity follows exactly the trigonometric proof. Sufficiency requires a slight change. Indeed, for each pair of integers k, m define $k \dotplus m$ by the equation

$$\psi_{k \dotplus m} = \psi_k \cdot \psi_m. \tag{3}$$

Recall from Paley's definition of the Walsh functions that $k \dotplus 2^l = k + 2^l$ if $k < 2^l$.

Now, suppose that the Walsh polynomials

$$P_n(x) = \sum_{k=0}^{m_n} a_k^{(n)} \psi_k(x), \qquad n = 1, 2, \ldots, \quad .$$

satisfy (1) and (2). Let $l_1 = m_1$. If l_{k-1} has been chosen, set $l_k = \max(l_{k-1}, m_k) + 1$. Then $l_1 < l_2 < \cdots$ and

$$2^{l_{n+1}} + k_1 \neq 2^{l_n} + k_0 \tag{4}$$

for any choices of $0 \leqslant k_0 \leqslant m_n$, $0 \leqslant k_1 \leqslant m_{n+1}$.

Set

$$f(x) = \sum_{n=0}^{\infty} \psi_{2^{l_n}}(x) P_n(x)$$

and observe by (1) that this series is Cauchy in B. In particular, $f \in B$, and since the B norm is stronger than the L^1 norm, this series is the Walsh Fourier series of f. Consequently, by (4) and an easy calculation, if $0 \leqslant j < 2^{l_{n+1}} - 2^{l_n}$ then

$$S_{2^{l_n}+j}(f, x) - S_{2^{l_n}}(f, x) \equiv \sum_{k=0}^{j} a_k^{(n)} \psi_k(x) \psi_{2^{l_n}}(x).$$

This allows us to conclude that

$$\sup_k |S_k(f, x)| \geqslant \tfrac{1}{2} \sup_k |S_k(P_n, x)|$$

for $n = 1, 2, \ldots$. Hence by (2), the Walsh Fourier series of f diverges on E as required.

The unanswered question concerning sets of divergence for $C(2^{\omega})$ should now be accessible. Indeed, we need only find Walsh's polynomials P_n which satisfy (1) and (2) for $B = C(2^{\omega})$. Yet, as we shall see in the next section, this is difficult even if we restrict ourselves to sets of the type which appear in Theorem 2. The problem is that (1) asks for $\|P_n\|_\infty \to 0$ as $n \to \infty$ while some partial sum of P_n is bounded away from one. This requires some sort of internal cancellation, whereas the square wave nature of the Walsh functions makes such cancellation seem highly unlikely.

We close this section with a corollary to Lemma 2.

LEMMA 3. *Let $B \subseteq L^1(2^\omega)$ be a homogeneous Banach space and suppose that $E_1, E_2, \ldots$ are all sets of divergence for B. Then $E \equiv \bigcup_{n=1}^{\infty} E_n$ is a set of divergence for B.*

3. **Fundamental lemmas.** Let M be a positive number and I be an interval in $[0, 1)$ with dyadic rational endpoints. A Walsh polynomial $P = \sum_{j=0}^{n} a_j \psi_j$ is said to have *dropped back from M on I* if $|P(x)| \leqslant 1$ for $x \in [0, 1)$ and if there is an integer $n_0 \leqslant N$ such that $|a_{n_0}| \leqslant 1$ and such that

$$\sum_{j=0}^{n_0} a_j \psi_j(x) \equiv M \quad \text{for } x \in I. \tag{5}$$

LEMMA 4. *If P is a Walsh polynomial which has dropped back from M on (α, β) then there is a Walsh polynomial which has dropped back from $M + \frac{1}{2}$ on $(\alpha/4, \beta/4)$.*

To prove Lemma 4 we may suppose that

$$P(x) = \sum_{j=0}^{2^n - 1} a_j \psi_j(x)$$

and that P satisfies (5) for $I = (\alpha, \beta)$. Consider the Walsh polynomial

$$Q(x) = \sum_{i=0}^{2^{n+1} - 1} b_i \psi_i(x)$$

whose coefficients are determined by the following process. Set $b_0 = a_0$, $b_{2n_0} = a_{n_0}$, $b_{2n_0+1} = 0$. In general, if $0 < i < 2^n$ and if $i \neq n_0$ then set

$$b_{2i} = b_{2i+1} \equiv \tfrac{1}{2} a_i. \tag{6}$$

Recall that if $x \in [0, \frac{1}{2})$ then

$$\psi_{2i}(x) = \psi_{2i+1}(x) \equiv \psi_i(2x).$$

Hence (6) implies that $Q(x) = P(2x)$ for $x \in [0, \frac{1}{2})$. In particular, if $x \in [0, \frac{1}{2})$ then

$$|Q(x)| \leqslant 1. \tag{7}$$

This inequality also holds for $x \in [\frac{1}{2}, 1)$. Indeed for such x, we know that

$$\psi_{2i}(x) + \psi_{2i+1}(x) \equiv 0.$$

Hence for such x, $|Q(x)| \equiv |b_{2n_0}| = |a_{n_0}|$.

By repeating this argument we can show that

$$\sum_{i=0}^{2n_0} b_i \psi_i(x) = \sum_{j=0}^{n_0} a_j \psi_j(2x)$$

for $x \in [0, \frac{1}{2})$. Since $(\alpha/2, \beta/2) \subseteq [0, \frac{1}{2})$, we combine this identity with (5) to conclude

$$\sum_{i=0}^{2n_0} b_i\psi_i(x) \equiv M \quad \text{for } x \in (\alpha/2, \beta/2). \tag{8}$$

We are now ready to define the coefficients of P^*. Set $c_0 = b_0$, $c_{4m_0+2} = \frac{1}{2}\psi_{4m_0+2}([\alpha + \beta]/8)$, and $c_{4m_0+3} = -c_{4m_0+2}$. For all other indices $k \in [0, 2^{n+1})$ set

$$c_{2k} = c_{2k+1} = \tfrac{1}{2} b_k.$$

Finally, let

$$P^*(x) = \sum_{k=0}^{2^{n+2}-1} c_k\psi_k(x).$$

As before, (7) leads to

$$|P^*(x)| \leqslant 1 \quad \text{for } x \in \left[0, \tfrac{1}{2}\right). \tag{9}$$

This time, however, if $x \in [\frac{1}{2}, 1)$ then

$$|P^*(x)| = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1.$$

Hence (9) holds for $x \in [0, 1)$.

Again, as before, if $x \in [0, \frac{1}{2})$ then

$$\sum_{k=0}^{4n_0+1} c_k\psi_k(x) \equiv \sum_{i=0}^{2n_0} b_i\psi_i(2x).$$

Consequently, by (8), if $x \in (\alpha/4, \beta/4)$ then

$$\sum_{k=0}^{4n_0+2} c_k\psi_k(x) \equiv M + \tfrac{1}{2}\, \psi_{4n_0+2}(x)\psi_{4n_0+2}([\alpha + \beta]/8).$$

Since ψ_{4n_0+2} is constant on $(\alpha/4, \beta/4)$ we have proved that P^* has dropped back from $M + \frac{1}{2}$ on $(\alpha/4, \beta/4)$.

The following result is an immediate consequence of Lemma 4.

LEMMA 5. *If m is any positive integer and if $I = [0, 2^{-m})$ then there is a Walsh polynomial P which has dropped back from $1 + \frac{1}{4}\log(1/m(I))$ on I.*

We need to extend Lemma 5 to sets of the form $I(i, k)$ (see (0)). To do this we introduce additional notation and a technical lemma.

A Walsh function ψ_k is said to *belong to the nth layer* for some nonnegative integer n if $2^n \leqslant k < 2^{n+1}$. A subset W of the Walsh functions is said to be *layered* if no two functions in W belong to the same layer.

Let $W = \{\psi_{w_j}: j = 0, 1, \ldots, n\}$ be layered with $0 = w_0 < w_1 < \cdots <$

w_n. For each integer $j = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_r}$ where $0 \leq j_1 < j_2 < \cdots < j_r \leq n$, we shall define p_j by the following equation:

$$\psi_{p_j} = \psi_{w_{j_1}} \psi_{w_{j_2}} \cdots \psi_{w_{j_r}} \tag{10}$$

and denote $\{\psi_{p_j} : j = 0, 1, \ldots, 2^n - 1\}$ by P_W.

It is clear that P_W is closed under multiplication, and since W is layered, that P_W contains 2^n distinct elements. We shall now show that if W is chosen carefully then the ordering on P_W induced by (10) is the usual ordering.

LEMMA 6. *If $W' = \{\psi_{w_j'}: j = 0, 1, \ldots, n\}$ is layered then there exists a layered set $W = \{\psi_{w_j}: j = 0, 1, \ldots, n\}$ such that $P_W = P_{W'}$ and such that p_j increases with j, where the p_j are defined by* (10).

To begin to prove Lemma 6 we observe that $P_{W^*} = P_{W'}$ if we form W^* from W' by replacing a $\psi_{w_j'}$ by any ψ_l provided that $\psi_l \in P_{W'}$ and that ψ_l and $\psi_{w_j'}$ belong to the same layer. Indeed, having made such a choice, P_{W^*} still has 2^n distinct elements and $P_{W^*} \subseteq P_{W'}$.

Consequently, set $W = \{\psi_{w_j}: j = 0, 1, \ldots, n\}$ where ψ_{w_j} is the Walsh function of smallest index in $P_{W'}$ which belongs to the same layer as $\psi_{w_j'}$. We need to show that the corresponding p_j increase in j. We shall do this by induction on n. If $n = 1$ then there is nothing to prove.

For the inductive step, set $V = \{\psi_{w_j}: j = 0, 1, \ldots, n - 1\}$ and suppose that P_V is ordered correctly. Then to show that the p_j increase in j we need only consider the case when ψ_{p_j} belongs to the same layer as ψ_{w_n}.

Toward this, let $j < k$ and set

$$\psi_{p_j} = \psi_{w_n} \cdot \psi_{p_l}, \qquad \psi_{p_k} = \psi_{w_n} \cdot \psi_{p_m} \tag{11}$$

where $\psi_{p_l}, \psi_{p_m} \in P_V$. We claim that neither ψ_{p_l} nor ψ_{p_m} have any (Rademacher) factors in common with ψ_{w_n}. Indeed, let ϕ_i be the Rademacher factor of $\psi_{p_l} \cdot \psi_{p_m}$ of highest index. If ϕ_i is a factor of ψ_{w_n} then $\psi_{w_n} \cdot \psi_{p_l} \cdot \psi_{p_m} \in V$ belongs to the same layer as ψ_{w_n}, but has index smaller than w_n. This contradicts the choice of w_n. If ϕ_i is a factor of ψ_{p_l} then since $\phi_i^2 = 1$, it cannot be a factor of ψ_{p_m}. Hence $p_l > p_m$. By our inductive hypothesis, then, we conclude that $l > m$. This contradicts the assumption that $j < k$. It follows, then, that ϕ_i must be a factor of ψ_{p_m} and cannot be a factor of ψ_{p_l}. By equation (11), then, $p_j < p_k$, as required.

LEMMA 7. *Suppose that $W = \{\psi_{w_j}: j = 0, 1, \ldots, n\}$ is layered and that*

$$E_n = \bigcap_{j=1}^{n} I(i_j, w_j)$$

where $i_j = 0$ or 1. Then there is a Walsh polynomial from P_W which has dropped back from $1 + \frac{1}{4} \log_2 (1/m(E_n))$ on E_n.

PROOF. By multiplying coefficients of the polynomial by -1 where needed, we may suppose that all $i_j = 0$. In view of Lemmas 5 and 6 it suffices to partition [0, 1] into sets K_l, $l = 0, 1, \ldots, 2^n - 1$, such that $K_0 = E_n$ and such that

$$\psi_{p_j} = 1 \text{ on } K_l \text{ if and only if } \psi_j = 1 \text{ on } \left[\frac{l}{2^n}, \frac{l+1}{2^n} \right) \tag{12}$$

for $j = 0, 1, \ldots, 2^n - 1$.

Toward this for each $l = 0, 1, \ldots, 2^n - 1$ write out its dyadic expansion

$$l = i_1 2^{n-1} + i_2 2^{n-2} + \cdots + i_{n-1} 2 + i_n$$

and set

$$K_l = \bigcap_{j=1}^{n} I(i_j, w_j)$$

(see (0)). Clearly, $K_0 = E_n$ and $K_m \cap K_l = \varnothing$ if $m \neq l$.

Also, $\psi_{w_j} = 1$ on K_l if and only if $i_l = 0$. Hence

$$\psi_{w_j} = 1 \text{ on } K_l \text{ if and only if } \phi_j = 1 \text{ on } \left[\frac{l}{2^n}, \frac{l+1}{2^n} \right)$$

where ϕ_j is a Radamacher function. Finally, then, since P_W is built from ψ_{w_j} by (10) in precisely the same manner that Walsh functions are built from Radamacher functions, we conclude that (12) holds for $j = 0, 1, \ldots, 2^n - 1$.

4. **Proofs of the theorems.** To prove Theorem 2 fix $n > 0$ and set $w_j = k_j$ for $j = 0, 1, \ldots, 2^n$. Apply Lemma 7 to choose a Walsh polynomial Q_n which has dropped back from $1 + \frac{1}{4} \log_2(1/m(E_{2^n}))$ on E_{2^n}.

Now set $P_n = Q_n/n^2$, $n = 1, 2, \ldots,$ and observe that since $\|P_n\|_\infty \leqslant n^{-2}$, we have $\sum_{n=1}^{\infty} \|P_n\|_\infty < \infty$. Also,

$$\sup_k |S_k(P_n, x)| \geqslant 2^{n-2}/n^2$$

for $n = 1, 2, \ldots$ and for $x \in E_{2^n}$. Since $E \subset E_{2^n}$ for $n = 1, 2, \ldots$ we have

$$\sup_n \left\{ \sup_k |S_k(P_n, x)| \right\} = +\infty$$

for $x \in E$. Consequently, E is a set of divergence for $C(2^\omega)$ by Lemma 2.

To prove Theorem 1 we translate the Haar series result of Prohorenko [5] to Walsh series in the usual manner (see [11]), which allows us to conclude that if $E \subseteq 2^\omega$ is a set of Haar measure zero then $E \sim D$ is a set of divergence for $L^P(2^\omega)$, $1 < p < \infty$, where D are those points in 2^ω which terminate in 1's or in 0's. But D is countable and, therefore, it too is a set of divergence for $L^P(2^\omega)$. Finally, then, by Lemma 3, E itself is a set of divergence for $L^P(2^\omega)$, $1 < p < \infty$.

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916 (Current address of D. C. Harris)

Current address (W. R. Wade): Fulbright Professor, Faculty of Mathematics and Mechanics, Moscow State University, Moscow, B-224, U.S.S.R.