STRONG DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

BY

C. J. NEUGEBAUER

ABSTRACT. Let F be a differentiation basis in R^n , i.e., a family of measurable sets S contracting to 0 such that $\|M_F f\|_p < A_p \|f\|_p$, where M_F is the Hardy-Littlewood maximal operator. For $f \in \Lambda_\alpha^{pq}$, we let $E_F(f)$ be the complement of the Lebesgue set of f relative to F , and we show that E_F has L_α^{pq} -capacity 0, where L_α^{pq} is a capacity associated with Λ_α^{pq} in much the same way as the Bessel capacity B_α^p is associated with L_α^p .

1. With the Bessel potential space $L_\alpha^p(R^n)$ there is associated the Bessel capacity B_α^p which is defined for $E \subset R^n$ by

$$B_\alpha^p(E) = \inf \{ \|g\|_p^p : g \in L^p, g \geq 0, G_\alpha * g \geq 1 \text{ on } E \},$$

where G_α is the Bessel kernel given by $\widehat{G_\alpha}(x) = (1 + 4\pi^2|x|^2)^{-\alpha/2}$ [5, p. 132]. The capacity B_α^p is an outer measure on R^n and its relation to H^r , Hausdorff measure of dimension r , is given by the following [4]. If $p > 1$, $\alpha p < n$, then $H^{n-\alpha p}(E) = 0$ implies $B_\alpha^p(E) = 0$, and $B_\alpha^p(E) = 0$ implies $H^{n-\alpha p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$.

Let F be a family of measurable sets $S \subset R^n$ with $0 < |S| < \infty$, and let $S \rightarrow 0$ stand for the generic notation for the limit as $j \rightarrow \infty$ of any sequence $\{S_j\} \subset F$ such that, for any $\varepsilon > 0$, $S_j \subset \{|x| < \varepsilon\}$, $j \geq j(\varepsilon)$. With such a family we associate the Hardy-Littlewood maximal operator

$$M_F f(x) = \sup \left\{ \frac{1}{|S|} \int_{S+x} |f(y)| dy : S \in F \right\}.$$

The behavior of $M_F f$ is decisive in the study of the differentiability of the integral. If, for example, $\|M_F f\|_p \leq A_p \|f\|_p$ (there are many interesting families F with this property [3]), then the set

$$E_F(f) = \left\{ x : \frac{1}{|S|} \int_{S+x} |f(y) - f(x)| dy \not\rightarrow 0 \text{ as } S \rightarrow 0 \right\}$$

has measure 0 for $f \in L^p$. If $f \in L_\alpha^p$, then as shown in [2] the exceptional set $E_F(f)$ has even B_α^p -capacity 0.

The purpose of this paper is to prove an analogous result for the Lipschitz spaces $\Lambda_\alpha^{pq}(R^n)$ (see [5], [6]), using instead of B_α^p a "Lipschitz" capacity L_α^{pq} .

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In view of the inclusion relations

$$L_\alpha^p \subset \Lambda_\alpha^{pr} \subset \Lambda_\alpha^{pq} \subset L_\beta^p, \quad p > 1,$$

$r = \max(p, 2)$, $q > r$, $\beta < \alpha$ [6, pp. 441, 452] our differentiability result lies then intermediate to those of [2].

We recall that, if F is the family of all oriented rectangles, a nonregular family, then $\|M_F f\|_p \leq A_p \|f\|_p$, $1 < p < \infty$ [5]. For regular families F , i.e., those for which there is $c > 0$ such that every $E \in F$ lies in a sphere S with $|S| < c|E|$, the size of $E_F(f)$ for $f \in \Lambda_\alpha^{pq}$ can be studied with the techniques developed in [8], especially the mean value property $\int_{|y|<r} G_\alpha(x-y) dy < cG_\alpha(x)r^n$. One sees easily that this tool is no longer available for nonregular families, and its substitute $\|M_F f\|_p \leq A_p \|f\|_p$, which is not a condition on the kernel G_α , requires one to study the interplay between Λ_α^{pq} and M_F .

2. Let F be a family as in §1, and let $tF = \{tS: S \in F\}$, $0 < t \leq 1$, where $tS = \{ts: s \in S\}$. Let

$$M_t f(x) = \sup \left\{ \frac{1}{|tS|} \int_{tS+x} |f(y)| dy: S \in F \right\}.$$

LEMMA 1. If $\|M_F f\|_p \leq A_p \|f\|_p$, $f \in L^p$, then $\|M_t f\|_p \leq A_p \|f\|_p$, $0 < t \leq 1$.

PROOF. $|tS|^{-1} \int_{tS+x} |f(u)| du = |s|^{-1} \int_{S+x/t} |f(tv)| dv \leq M_F(\delta_t f)(x/t)$, where $(\delta_t f)(x) = f(tx)$. Hence

$$\begin{aligned} \|M_t f\|_p &\leq \left\{ \int_{R^n} M_F(\delta_t f) \left(\frac{x}{t} \right)^p dx \right\}^{1/p} \\ &= \left\{ t^n \int_{R^n} M_F(\delta_t f)(u)^p du \right\}^{1/p} \leq t^{n/p} A_p \left\{ \int_{R^n} |(\delta_t f)(x)|^p dx \right\}^{1/p} \\ &= A_p \|f\|_p. \end{aligned}$$

We note that A_p is independent of t .

For $0 < \alpha < 1$, the Lipschitz space $\Lambda_\alpha^{pq}(R^n)$ consists of all functions $f \in L^p(R^n)$ for which the norm

$$(i) \quad \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) + f(x-t) - 2f(x)\|_p^q}{|t|^{n+\alpha q}} dt \right\}^{1/q} = \|f\|_{\Lambda_\alpha^{pq}}$$

is finite [5, p. 151]; here $1 \leq p, q \leq \infty$. If $0 < \alpha < 1$, the above norm is equivalent to

$$(ii) \quad \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}} dt \right\}^{1/q}$$

[5, p. 153]. If $\alpha > 1$, Λ_{α}^{pq} is the collection of $f \in L^p(R^n)$ for which the norm $\|f\|_p + \sum_{j=1}^n \|\partial f / \partial x_j\|_{\Lambda_{\alpha-1}^{pq}} < \infty$ [5, p. 153], where $\partial f / \partial x_j$ is taken in the sense of distribution.

LEMMA 2. *Let $0 < \alpha < 1$ and $\|M_F f\|_p \leq A_p \|f\|_p$, $f \in L^p$. Then $\|M_f\|_{\Lambda_{\alpha}^{pq}} \leq A_p \|f\|_{\Lambda_{\alpha}^{pq}}$.*

PROOF. By Lemma 1 we only need to show that $\|M_F f\|_{\Lambda_{\alpha}^{pq}} \leq A_p \|f\|_{\Lambda_{\alpha}^{pq}}$. If $f_t(x) = f(x + t)$, one easily verifies that $|M_F f(x + t) - M_F f(x)| \leq M_F(f_t - f)(x)$, and the result follows from (ii).

3. If G_{α} is the Bessel potential of order $\alpha > 0$, and if $J_{\alpha} f = G_{\alpha} * f$, then $J_{\alpha}: \Lambda_{\beta}^{pq} \rightarrow \Lambda_{\alpha+\beta}^{pq}$ is an isomorphism and the norms $\|f\|_{\Lambda_{\beta}^{pq}}$, $\|J_{\alpha} f\|_{\Lambda_{\alpha+\beta}^{pq}}$ are equivalent [5, Chapter 5]. This result will be used frequently in the sequel.

For $0 < \alpha < \infty$, $1 \leq p, q < \infty$, we define, for $E \subset R^n$,

$$L_{\alpha}^{pq}(E) = \inf \{ \|g\|_{\Lambda_{\alpha}^{pq}} : g \geq 0 \text{ and } g \geq 1 \text{ on } E \}.$$

For $0 < \gamma < \alpha$ we define, for $E \subset R^n$,

$$B_{\gamma}^{pq}(E) = \inf \{ \|g\|_{\Lambda_{\gamma}^{pq}} : g \geq 0 \text{ and } G_{\alpha-\gamma} * g \geq 1 \text{ on } E \}.$$

It is easily verified that L_{α}^{pq} , B_{γ}^{pq} are capacities, i.e., they are monotone, countably subadditive, and assign 0 to $E = \emptyset$ (see [4, p. 251]).

The usefulness of B_{γ}^{pq} is exhibited in the following lemma.

LEMMA 3. $L_{\alpha}^{pq}(E) = 0$ if and only if there exists $0 < \gamma < \alpha$ such that $B_{\gamma}^{pq}(E) = 0$.

PROOF. ($\rightarrow$) Let $0 < \gamma < \alpha$ and let $\varepsilon > 0$. Choose $g \in \Lambda_{\alpha}^{pq}$ so that $g > 0$, $g \geq 1$ on E , and $\|g\|_{\Lambda_{\alpha}^{pq}} \leq \varepsilon$. If $g = G_{\alpha-\gamma} * \psi$, $\psi \in \Lambda_{\gamma}^{pq}$, then $\|\psi\|_{\Lambda_{\gamma}^{pq}} < K\varepsilon$, and $B_{\gamma}^{pq}(E) = 0$.

($\leftarrow$) If $B_{\gamma}^{pq}(E) = 0$, then there is $g \geq 0$ in Λ_{γ}^{pq} with $G_{\alpha-\gamma} * g \geq 1$ on E and $\|g\|_{\Lambda_{\gamma}^{pq}} \leq \varepsilon$. As before $\|G_{\alpha-\gamma} * g\|_{\Lambda_{\alpha}^{pq}} < K\varepsilon$, and $L_{\alpha}^{pq}(E) = 0$.

LEMMA 4. *Let $1 < p < \infty$. The relation between the Bessel capacity $B_{\alpha p}$ and the Lipschitz capacity L_{α}^{pq} is given by*

(i) $B_{\alpha p}(E) = 0$ implies $L_{\alpha}^{pq}(E) = 0$ $q > \max(p, 2)$.

(ii) $L_{\alpha}^{pq}(E) = 0$ implies $B_{\gamma p}(E) = 0$, $0 < \gamma < \alpha$.

PROOF. (i) We use Lemma 3 and verify that $B_{\gamma}^{pq}(E) = 0$ for $0 < \gamma < \alpha$. Since $B_{\alpha p}(E) = 0$, there is $g \geq 0$ in L^p such that $\|g\|_p \leq \varepsilon$ and $G_{\alpha} * g(x) \geq 1$, $x \in E$. If $\psi = G_{\gamma} * g$, then $\|\psi\|_{\Lambda_{\gamma}^{pq}} \leq M \|g\|_p \leq M \cdot \varepsilon$ [6, p. 452].

(ii) If $0 < \gamma < \alpha$, we have $B_{\alpha-\gamma}^{pq}(E) = 0$, and hence there is $g \geq 0$ with $\|g\|_{\Lambda_{\alpha-\gamma}^{pq}} \leq \varepsilon$ and $G_{\gamma} * g(x) \geq 1$, $x \in E$. Since $\|g\|_p \leq \varepsilon$, we get $B_{\gamma p}(E) = 0$.

COROLLARY. *If $L_{\alpha}^{pq}(E) = 0$ and $\alpha p > 1$, then $H^{n-1}(E) = 0$.*

PROOF. Let $0 < \gamma < \alpha$ with $\gamma p > 1$. Since $B_{\gamma p}(E) = 0$, we see from [4, Theorem 22] that $H^{n-1}(E) = 0$.

The next lemma shows that $f \in \Lambda_\alpha^{pq}$ can be defined absolutely modulo sets of L_α^{pq} -capacity 0.

LEMMA 5. Let $f \in \Lambda_\alpha^{pq}$, $0 < \gamma < \alpha$, and $f = G_{\alpha-\gamma} * \psi$, $\psi \in \Lambda_\gamma^{pq}$. Then $G_{\alpha-\gamma} * |\psi|(x) < \infty$ for L_α^{pq} -a.e. x .

PROOF. Let $E = \{x: G_{\alpha-\gamma} * |\psi|(x) = \infty\}$. Then

$$\begin{aligned} B_\gamma^{pq}(E) &< B_\gamma^{pq}\{x: G_{\alpha-\gamma} * |\psi|(x) > k\} \\ &= B_\gamma^{pq}\left\{x: G_{\alpha-\gamma} * \frac{|\psi|}{k}(x) > 1\right\} \leq \frac{1}{k} \|\psi\|_{\Lambda_\gamma^{pq}} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

The result now follows from Lemma 3.

LEMMA 6. Let $\psi_j \rightarrow f$ (Λ_α^{pq}). Then there exists a subsequence $\{j_i\}$ such that $\psi_{j_i} \rightarrow f$ for L_α^{pq} -a.e. x .

PROOF. The proof is standard and we give it for the sake of completeness. Let $0 < \gamma < \alpha$, $f = G_{\alpha-\gamma} * g$, $\psi_j = G_{\alpha-\gamma} * \phi_j$, $g, \phi_j \in \Lambda_\gamma^{pq}$. Then

$$\begin{aligned} B_\gamma^{pq}\{x: |\psi_j - f|(x) > \varepsilon\} &\leq B_\gamma^{pq}\{x: G_{\alpha-\gamma} * (|\phi_j - g|/\varepsilon)(x) > 1\} \\ &\leq \|\phi_j - g\|_{\Lambda_\gamma^{pq}}/\varepsilon \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

If we select now $\{j_i\}$ such that for $A_i = \{x: |\psi_{j_i} - f|(x) > 1/2^i\}$ we have $B_\gamma^{pq}(A_i) < 1/2^i$, then on $A = \bigcap_{k \geq 1} \bigcup_{i \geq k} A_i$, $\psi_{j_i} \rightarrow f$ and $B_\gamma^{pq}(A) = 0$.

LEMMA 7. If $\|M_F f\|_p \leq A_p \|f\|_p$ and $0 < \alpha < 1$, then $\|\int_0^1 M_t g(x) dt\|_{\Lambda_\alpha^{pq}} \leq A_p \|g\|_{\Lambda_\alpha^{pq}}$ where M_F and M_t are the maximal operators associated with F and tF .

PROOF. Let $\psi(x) = \int_0^1 M_t g(x) dt$. Then $\|\psi\|_p \leq A_p \|g\|_p$ (Lemma 1), and

$$\begin{aligned} \|\psi(x + \tau) - \psi(x)\|_p &\leq \int_0^1 \|M_t(g_\tau - g)\|_p dt \\ &\leq A_p \|g(x + \tau) - g(x)\|_p, \end{aligned}$$

where $g_\tau(x) = g(x + \tau)$.

From Lemma 5 we deduce that for $g \in \Lambda_\alpha^{pq}$, $M_t g(x) \in L^1(dt, [0, 1])$ for L_α^{pq} -a.e. x .

4. We are now ready to state and prove our differentiability results. We let F be a family of sets as in §1, and we let $tF = \{tS: S \in F\}$, $0 < t < 1$.

THEOREM 1. Assume that for $f \in L^p(R^n)$, $\|M_F f\|_p \leq A_p \|f\|_p$. For $f \in \Lambda_\alpha^{pq}(R^n)$, let

$$E_t(f) = \left\{ x: \limsup_{S \rightarrow 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy > 0 \right\}.$$

Then $L_\alpha^{pq}(E_t) = 0$.

PROOF. We have to verify that, for $\sigma > 0$, $L_\alpha^{pq}(E_\sigma) = 0$, where

$$E_\sigma = \left\{ x \in E_t: \limsup_{S \rightarrow 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy > \sigma \right\}.$$

Let $0 < \beta < \min(1, \alpha)$. By Lemma 3 we need to show that $B_\beta^{pq}(E) = 0$. Let $\eta > 0$ be given.

Let $C_0^\infty(R^n)$ be the space of infinitely differentiable functions with compact support. By [6, p. 444] there is a sequence $\{\psi_j\} \subset C_0^\infty$ such that $\psi_j \rightarrow f(\Lambda_\alpha^{pq})$. Choose now $\psi_j^\beta, f^\beta \in \Lambda_\beta^{pq}$ with $G_{\alpha-\beta} * \psi_j^\beta = \psi_j$ and $G_{\alpha-\beta} * f^\beta = f$. Then $\psi_j^\beta \rightarrow f^\beta(\Lambda_\beta^{pq})$, and from this we get that

$$B_\beta^{pq} \{x: |\psi - f|(x) > \delta\} \leq \|\psi^\beta - f^\beta\|_{\Lambda_\beta^{pq}} / \delta = o(1) \quad \text{as } j \rightarrow \infty.$$

If $\Sigma = tS$, we have

$$\begin{aligned} \frac{1}{|\Sigma|} \int_{\Sigma+x} |f(y) - f(x)| dy &\leq \frac{1}{|\Sigma|} \int_{\Sigma+x} |f(y) - \psi_j(y)| dy \\ &\quad + \frac{1}{|\Sigma|} \int_{\Sigma+x} |\psi_j(y) - \psi_j(x)| dy \\ &\quad + |\psi_j(x) - f(x)| \\ &= A_{1j}(x) + A_{2j}(x) + A_{3j}(x). \end{aligned}$$

(i) Since $\{x: M_t(f - \psi_j)(x) > \sigma/3\} \subset \{x: G_{\alpha-\beta} * M_t(f^\beta - \psi_j^\beta)(x) > \sigma/3\}$, we see that

$$\begin{aligned} B_\beta^{pq} \{x: M_t(f - \psi_j)(x) > \sigma/3\} &\leq (3/\sigma) \|M_t(f^\beta - \psi_j^\beta)\|_{\Lambda_\beta^{pq}} \\ &\leq (3/\sigma) A_p \|f^\beta - \psi_j^\beta\|_{\Lambda_\beta^{pq}} < \eta/2, \quad \text{if } j > j_1(\eta). \end{aligned}$$

The next to the last inequality follows from Lemma 2 (applicable since $\beta < 1$). Since $A_{1j}(x) \leq M_t(f - \psi_j)(x)$, there is a set E_{1j} such that $B_\beta^{pq}(E_{1j}) < \eta/3$, $j > j_1(\eta)$, and $x \notin E_{1j}$ implies $A_{1j}(x) < \sigma/3$.

(ii) Since $B_\beta^{pq} \{x: A_{3j}(x) > \sigma/3\} \leq (3/\sigma) \|\psi_j^\beta - f^\beta\|_{\Lambda_\beta^{pq}}$, there is $j_3(\eta)$ such that for $j > j_3(\eta)$, $A_{3j}(x) < \sigma/3$ except for $x \in E_{3j}$ with $B_\beta^{pq}(E_{3j}) < \eta/2$.

(iii) Let now $j_0 > \max[j_1(\eta), j_3(\eta)]$. Then there is $\tau(\sigma)$ such that $S \subset B(0, \tau)$ implies $|\psi_{j_0}(y) - \psi_{j_0}(x)| < \sigma/3$, $y \in tS + x$. If then $x \notin E_{1j_0} \cup E_{3j_0}$ and $S \subset B(0, \tau)$, we get $|\Sigma|^{-1} \int_{\Sigma+x} |f(y) - f(x)| dy \leq \sigma$, $\Sigma = tS$, and hence $E_\sigma \subset E_{1j_0} \cup E_{3j_0}$. Then $B_\beta^{pq}(E_\sigma) < \eta$, and the proof is complete.

THEOREM 2. Under the same hypothesis as for Theorem 1, we have for L_α^{pq} -a.e. x ,

$$(i) \quad \lim_{S \rightarrow 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy = 0, \text{ a.e. } t, 0 < t \leq 1.$$

$$(ii) \quad \lim_{S \rightarrow 0} \int_0^1 \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy dt = 0.$$

PROOF. (i) Let

$$E = \left\{ (t, x): \limsup_{S \rightarrow 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy > 0 \right\},$$

and let $E_x = \{t: (t, x) \in E\}$. We must show that $|E_x| = 0$ for L_{α}^{pq} -a.e. x .

Let $A_{\sigma} = \{x: |E_x| > \sigma\}$. Let $0 < \beta < \min(1, \alpha)$ and choose $\{\psi_j\} \subset C_0^{\infty}$ so that $\psi_j \rightarrow f(\Lambda_{\alpha}^{pq})$. By Lemma 6 we may assume that $\psi_j(x) \rightarrow f(x)$ for L_{α}^{pq} -a.e. x . Let E_1 be the exceptional set, i.e., $L_{\alpha}^{pq}(E_1) = 0$ and $x \notin E_1$ implies $\psi_j(x) \rightarrow f(x)$.

We need another exceptional set E_2 . Since $M_t(f - \psi_j)(x) \leq G_{\alpha-\beta} * M_t(f^{\beta} - \psi_j^{\beta})(x)$ (notation as in the proof of Theorem 1), and $\psi_j^{\beta} \rightarrow f^{\beta}(\Lambda_{\beta}^{pq})$, we see that

$$\int_0^1 M_t(f - \psi_j)(x) dt \leq G_{\alpha-\beta} * \int_0^1 M_t(f^{\beta} - \psi_j^{\beta})(x) dt \equiv \Psi_j(x).$$

By Lemma 7, $\int_0^1 M_t(f^{\beta} - \psi_j^{\beta})(x) dt \in \Lambda_{\beta}^{pq}$, and hence $\Psi_j \in \Lambda_{\alpha}^{pq}$.

Next

$$\left\{ x: \int_0^1 M_t(f - \psi_j)(x) dt > \varepsilon \right\} \subset \left\{ x: G_{\alpha-\beta} * \frac{1}{\varepsilon} \int_0^1 M_t(f^{\beta} - \psi_j^{\beta})(x) dt > 1 \right\},$$

and hence

$$\begin{aligned} B_{\beta}^{pq} \left\{ x: \int_0^1 M_t(f - \psi_j)(x) dt > \varepsilon \right\} &\leq \frac{1}{\varepsilon} \left\| \int_0^1 M_t(f^{\beta} - \psi_j^{\beta})(x) dt \right\|_{\Lambda_{\beta}^{pq}} \\ &\leq \frac{A_p}{\varepsilon} \|f^{\beta} - \psi_j^{\beta}\|_{\Lambda_{\beta}^{pq}} = o(1) \text{ as } j \rightarrow \infty. \end{aligned}$$

We can now choose a subsequence $\{j_i\}$ so that $\int_0^1 M_t(f - \psi_{j_i})(x) dt \rightarrow 0$ for L_{α}^{pq} -a.e. x (see e.g. the proof of Lemma 6). We may assume that $\{j_i\} = \{j\}$. Let

$$E_2 = \left\{ x: \int_0^1 M_t(f - \psi_j)(x) dt \not\rightarrow 0 \right\}.$$

Then $L_{\alpha}^{pq}(E_2) = 0$.

We return now to the set A_{σ} introduced at the beginning of the proof, and we claim that, for every $\sigma > 0$, $A_{\sigma} \subset E_1 \cup E_2$.

If we deny this, then there is $x_0 \in A_{\sigma}$ and $x_0 \notin E_1 \cup E_2$. Then $|E_{x_0}| > \sigma$. If

$$E_{x_0\lambda} = \left\{ t: \limsup_{S \rightarrow 0} \frac{1}{|tS|} \int_{tS+x_0} |f(y) - f(x_0)| dy > \lambda \right\},$$

then there is $\lambda > 0$ with $|E_{x_0\lambda}| > \sigma$. We choose now j_0 so large that

$$(i) \int_0^1 M_t(f - \psi_{j_0})(x_0) dt < \sigma\lambda/3,$$

$$(ii) |\psi_{j_0}(x_0) - f(x_0)| < \lambda/3,$$

and we choose $\tau_0 > 0$ such that $S \subset B(0, \tau_0)$ implies

$$(iii) |\psi_{j_0}(y) - \psi_{j_0}(x_0)| < \lambda/3, y \in tS + x_0, 0 < t \leq 1.$$

Since for $t \in E_{x_0\lambda}$,

$$\lambda \leq M_t(f - \psi_{j_0})(x_0) + \frac{2\lambda}{3}, \quad S \subset B(0, \tau_0),$$

we see that $E_{x_0\lambda} \subset \{t: M_t(f - \psi_{j_0})(x_0) > \lambda/3\}$. From this $|E_{x_0\lambda}| < (3/\lambda) \int_0^1 M_t(f - \psi_{j_0})(x_0) dt < \sigma$, a contradiction.

To prove (ii), we first observe that $\int_0^1 M_t f^\beta(x) dt \in \Lambda_\beta^{pq}$ (Lemma 7), and hence $G_{\alpha-\beta} * \int_0^1 M_t f^\beta(x) dt < \infty$ for L_α^{pq} -a.e. x (Lemma 5). Since $M_t f(x) \leq G_{\alpha-\beta} * M_t f^\beta(x)$, we see that $M_t f(x) \in L^1(dt, [0, 1])$ for L_α^{pq} -a.e. x . Finally,

$$|f(x)| + M_t f(x) > \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| dy \rightarrow 0 \quad \text{as } S \rightarrow 0,$$

and we only need to apply the Lebesgue dominated convergence theorem to obtain (ii).

COROLLARY. *Under the same hypothesis,*

$$\lim_{S \rightarrow 0} \int_0^1 \cdots \int_0^1 \frac{1}{|t_1 t_2 \cdots t_k S|} \int_{t_1 t_2 \cdots t_k S + x} |f(y) - f(x)| dy dt_1 \cdots dt_k = 0$$

for L_α^{pq} -a.e. x .

5. In this section we will study higher order differentiability. Let F be a differentiation basis as in §1, and let $\delta(S)$ denote the diameter of S . We say that a function $f \in L^p(R^n)$ is in $t_k^p(x_0)$ with respect to F if there exists a polynomial $\Pi_{x_0}(y)$ of degree $\leq k$ such that

$$\left\{ \frac{1}{|S|} \int_{S+x_0} |f(y) - \Pi_{x_0}(y)|^p dy \right\}^{1/p} = o(\delta(S)^k), \quad \text{as } S \rightarrow 0.$$

If F is the family of all balls centered at the origin, this notion is due to Calderón and Zygmund [1], and for a general F was introduced in [2].

THEOREM 3. *Let $\|M_F f\|_p \leq A_p \|f\|_p$. If $f \in \Lambda_\alpha^{pq}(R^n)$, and k is a nonnegative integer $< \alpha$, then $f \in t_k^1(x)$ with respect to F for L_α^{pq} -a.e. x .*

Of course, the special case $k = 0$ is Theorem 1. The proof of Theorem 3 proceeds along the lines of the corresponding theorem for L_α^p [2]. We shall see that with the help of Theorem 2 it is possible to omit the hypothesis $tF \subset F$, $0 < t \leq 1$, made in [2]. We need

LEMMA 8. *Let $f \in \Lambda_\alpha^{pq}(R^n)$, $1 < \alpha < \infty$. Then for L_α^{pq} -a.e. x , f is absolutely*

continuous on H^{n-1} -a.e. ray from x , and on such a ray fix $f(x+z) - f(x) = \int_0^1 \nabla f(x+tz) \cdot z \, dt$.

PROOF. The proof is precisely the same as the proof of the corresponding lemma for L_α^p [2] as soon as we establish the existence of a sequence $\{f_j\} \subset C_0^\infty$ such that for L_α^{pq} -a.e. x ,

$$G_1 * |\nabla f_j - \nabla f|(x) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Let $0 < \gamma < \min(1, \alpha - 1)$, and choose $g \in \Lambda_\gamma^{pq}$ such that $f = G_{\alpha-\gamma} * g$. Let $\{f_i\} \subset C_0^\infty$ such that $f_i \rightarrow f(\Lambda_\alpha^{pq})$, and let $g_i \in \Lambda_\gamma^{pq}$ with $f_i = G_{\alpha-\gamma} * g_i$. Then $g_i \rightarrow g(\Lambda_\gamma^{pq})$. By Lemma 6, $f_k(x) \rightarrow f(x)$ for L_α^{pq} -a.e. x , for some subsequence $\{f_k\}$. Since $\alpha > 1$, $\nabla f_k \rightarrow \nabla f(\Lambda_\alpha^{pq-1})$, and hence $\nabla f_k - \nabla f = G_{\alpha-\gamma-1} * h_k$, $h_k \rightarrow 0(\Lambda_\gamma^{pq})$. Since $\gamma < 1$, we easily see that $|h_k| \rightarrow 0(\Lambda_\gamma^{pq})$, and hence for some subsequence $\{h_i\}$, $G_{\alpha-\gamma} * |h_i|(x) \rightarrow 0$ for L_α^{pq} -a.e. x . Since $G_1 * G_{\alpha-\gamma-1} * |h_i|(x) = G_{\alpha-\gamma} * |h_i|(x)$, the sequence $\{f_j\}$ has the desired property.

PROOF OF THEOREM 3. Since $\alpha \leq 1$ is Theorem 1, we may assume that $1 < \alpha$. Let $f \in \Lambda_\alpha^{pq}$, and let $\Pi_x(y) = \sum_{0 < |\beta| < k} (D^\beta f(x)/\beta!)(y-x)^\beta$; we observe that $D^\beta f \in \Lambda_{\alpha-|\beta|}^{pq}$.

Set $R_x(y) = f(y) - \Pi_x(y)$, and let $|\gamma| = k$. Then $D^\gamma R_x(y) = D^\gamma f(y) - D^\gamma f(x)$. By Theorem 1, for $L_{\alpha-k}^{pq}$ -a.e. x ,

$$\frac{1}{|S|} \int_{S+x} |D^\gamma R_x(y)| \, dy \rightarrow 0 \quad \text{as } S \rightarrow 0,$$

and by Theorem 2,

$$\int_0^1 \frac{1}{|tS|} \int_{tS+x} |D^\gamma R_x(y)| \, dy \, dt \rightarrow 0.$$

Let now $|\nu| = k-1$. For $L_{\alpha-k}^{pq}$ -a.e. x , $D^\nu f$ is AC on H^{n-1} -a.e. ray from x , and on such a ray $D^\nu R_x(y)$ is also AC from which

$$|D^\nu R(x+z) - D^\nu R_x(x)| \leq \int_0^1 |\nabla(D^\nu R_x)(x+tz) \cdot z| \, dt.$$

We note that $D^\nu R_x(x) = 0$. As in [2] for any S (not necessarily in F)

$$(i) \quad \frac{1}{\delta(S)} \frac{1}{|S|} \int_{S+x} |D^\nu R_x(z)| \, dz \leq \int_0^1 \frac{1}{|tS|} \int_{tS+x} |\nabla(D^\nu R_x)(z)| \, dz \, dt.$$

If $S \in F$ and $S \rightarrow 0$, the integral goes to 0 for $L_{\alpha-k}^{pq}$ -a.e. x .

For the next step of the proof we need

$$(ii) \quad \int_0^1 \frac{1}{\delta(\tau S)} \frac{1}{|\tau S|} \int_{\tau S+x} |D^\nu R_x(z)| \, dz \, d\tau \rightarrow 0$$

as $S \rightarrow 0$, $S \in F$, for $L_{\alpha-k}^{pq}$ -a.e. x . From (i), replacing S by τS , we obtain

$$\begin{aligned} & \int_0^1 \frac{1}{\delta(\tau S)} \frac{1}{|\tau S|} \int_{\tau S+x} |D^\eta R_x(z)| dz d\tau \\ & \leq \int_0^1 \int_0^1 \frac{1}{|t\tau S|} \int_{t\tau S+x} |\nabla(D^\eta R_x)(z)| dz dt d\tau \end{aligned}$$

and the corollary to Theorem 2 proves (ii).

If now η is a multi-index, $|\eta| = k - 2$, we get as before

$$|D^\eta R_x(x+z)| \leq \int_0^1 |\nabla(D^\eta R_x)(x+tz) \cdot z| dt$$

and hence for any S ,

$$\frac{1}{\delta(S)} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)| dz \leq \int_0^1 \frac{1}{|tS|} \int_{tS+x} |\nabla(D^\eta R_x)(z)| dz dt.$$

If we divide this by $\delta(S) \geq \delta(tS)$, we finally obtain

$$\begin{aligned} & \frac{1}{\delta(S)^2} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)| dz \\ & \leq \int_0^1 \frac{1}{\delta(tS)} \frac{1}{|tS|} \int_{tS+x} |\nabla(D^\eta R_x)(z)| dz. \end{aligned}$$

We apply now (ii).

For the step treating a multi-index of length $k - 3$, we need to know that for $L_{\alpha-k}^{pq}$ -a.e. x

$$\int_0^1 \frac{1}{\delta(\sigma S)^2} \frac{1}{|\sigma S|} \int_{\sigma S+x} |D^\eta R_x(z)| dz d\sigma \rightarrow 0$$

as $S \rightarrow 0$, $S \in F$. This integral is majorized by

$$\begin{aligned} & \int_0^1 \int_0^1 \frac{1}{\delta(t\sigma S)} \frac{1}{|t\sigma S|} \int_{t\sigma S+x} |\nabla(D^\eta R_x)(z)| dz dt d\sigma \\ & \leq \sum \int_0^1 \int_0^1 \int_0^1 \frac{1}{|\tau t\sigma S|} \int_{\tau t\sigma S+x} |\nabla(D^\rho R_x)(z)| dz dt d\sigma d\tau, \end{aligned}$$

where Σ extends over all ρ with $|\rho| = k - 1$. By the corollary to Theorem 2 this is $o(1)$ as $S \rightarrow 0$, $S \in F$, for $L_{\alpha-k}^{pq}$ -a.e. x . The proof of Theorem 3 is now complete.

6. We let F be a family as in §1 and we shall assume now that

$$(i) \quad \|M_F f\|_p \leq A_p \|f\|_p, \quad 1 < p < \infty.$$

The family of all oriented rectangles is an example satisfying (i). If (i) holds, then for a.e. x ,

$$\lim_{S \rightarrow 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|' dy = 0,$$

$f \in L^p$ and $1 < r < p$. This is no longer true if $r = p$ as the example due to Saks [7] shows.

If $f \in \Lambda_\alpha^{pq}$, then $f \in L_\beta^p$, $\alpha > \beta$, and hence by Sobolov's theorem $f \in L^r$, where $1/r = 1/p - \beta/n > 1/p - \alpha/n$. Here, of course, we assume that $\alpha p < n$. As in Theorem 1, we wish to study the size of the exceptional set (for which (ii) does not hold) under the hypothesis that $f \in \Lambda_\alpha^{pq}$. For this purpose we need a lemma.

LEMMA 9. Let $f \in \Lambda_\alpha^{pq}$, $0 < \alpha < 1$, and let $1 < r < p$. Then

$$\| |f|^r \|_{\Lambda_\alpha^{pq/r}} \leq A_r \|f\|_p^{r-1} \|f\|_{\Lambda_\alpha^{pq}}.$$

PROOF. Recall that

$$\|f\|_{\Lambda_\alpha^{pq}} = \|f\|_p + \left\{ \int_{R^n} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{n+\alpha q}} dt \right\}^{1/q}.$$

Now $|f(x+t)|^r - |f(x)|^r = (|f(x+t)| - |f(x)|)r \cdot \chi^{r-1}$, where χ is between $|f(x+t)|$, $|f(x)|$. Hence, if $\psi(x, t) = \max(|f(x+t)|, |f(x)|)$, we get

$$| |f(x+t)|^r - |f(x)|^r | \leq r |f(x+t) - f(x)| \psi(x, t)^{r-1}.$$

For $\beta < p$ we have

$$\| |f(x+t)|^r - |f(x)|^r \|_\beta \leq r \left\{ \int |f(x+t) - f(x)|^p \psi(x, t)^{\beta(r-1)} dx \right\}^{1/\beta}.$$

If we let $s = p/\beta$, $1/t = 1 - 1/s = (p - \beta)/p$, and apply Hölder's inequality, we get

$$\begin{aligned} & \| |f(x+t)|^r - |f(x)|^r \|_\beta \\ & \leq r \left[\left\{ \int |f(x+t) - f(x)|^p dx \right\}^{\beta/p} \left\{ \int \psi(x, t)^{\beta(r-1)p/(p-\beta)} \right\}^{(p-\beta)/p} \right]^{1/\beta}. \end{aligned}$$

We let now $\beta = p/r$. Then $\beta(r-1)p/(p-\beta) = p$, and $p \cdot (p-\beta)/p\beta = r-1$, and hence

$$\| |f(x+t)|^r - |f(x)|^r \|_{p/r} \leq r \|f(x+t) - f(x)\|_p \| \psi(x, t) \|_p^{r-1}.$$

Since $\psi(x, t) \leq |f(x)| + |f(x+t)|$, we see that $\| \psi(x, t) \|_p \leq 2 \|f\|_p$. Since, clearly, $\| |f|^r \|_{p/r} = \|f\|_p \|f\|_p^{r-1}$, the proof of the lemma is complete.

Let C_1, C_2 be two capacities on R^n , and let $[C_1, C_2](E) = \inf \{ C_1(E') + C_2(E'') \}$, where the inf is extended over all E', E'' with $E = E' \cup E''$. It is easy to check that this is a capacity, and that $[C_1, C_2](E) \leq \min(C_1(E), C_2(E))$. If $\{C_\alpha\}$ is a collection of capacities on R^n , then $C(E) = \sup_\alpha C_\alpha(E)$ is also a capacity on R^n .

Let $f \in \Lambda_\alpha^{pq}$, $\alpha p < n$. As we have seen, $f \in L^r$ for $1/r > 1/p - \alpha/n$. If $p < r < s$, $1/s = 1/p - \alpha/n$, let for $r < t < s$, $\alpha_t = \alpha - n/p + n/t$. We

define now the capacity $C_{r\alpha}$ by

$$C_{r\alpha}(E) = \begin{cases} [L_\alpha^{pq}, L_\alpha^{p/rq}](E), & 1 \leq r < p, \\ \sup_{t>r} [L_\alpha^{tq}, L_\alpha^{t/rq}](E), & p \leq r < s. \end{cases}$$

THEOREM 4. Assume that $\|M_F f\|_p \leq A_p \|f\|_p$, $1 < p < \infty$. Let $f \in \Lambda_\alpha^{pq}$, $0 < \alpha < 1$, $\alpha p < n$, and let $1/r > 1/p - \alpha/n$. Then for $C_{r\alpha}$ -a.e. x ,

$$\frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r dy \rightarrow 0 \quad \text{as } S \rightarrow 0.$$

PROOF. We assume first that $1 \leq r < p$. Let

$$E = \left\{ x: \limsup_{S \rightarrow 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r dy > 0 \right\},$$

and let $0 < \beta < \alpha$. By Lemma 3 it suffices to show that there exists a decomposition $E = E' \cup E''$ such that $B_\beta^{pq}(E') = 0$ and $B_\beta^{p/rq}(E'') = 0$.

Let $\{\psi_j\} \subset C_0^\infty$ such that $\psi_j \rightarrow f(\Lambda_\alpha^{pq})$. Then for each j

$$\begin{aligned} \left\{ \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r dy \right\}^{1/r} &\leq \{M_F(|f - \psi_j|^r)(x)\}^{1/r} \\ &\quad + \left\{ \frac{1}{|S|} \int_{S+x} |\psi_j(y) - \psi_j(x)|^r dy \right\}^{1/r} + |\psi_j(x) - f(x)| \\ &= A_{1j}(x) + A_{2j}(x) + A_{3j}(x). \end{aligned}$$

We claim that there is $j_1 < j_2 < \dots$ such that $M_F(|f - \psi_{j_i}|^r)(x) \rightarrow 0$ as $i \rightarrow \infty$ for $B_\beta^{p/rq}$ -a.e. x .

By Lemma 9 $|f - \psi_i|^r \in \Lambda_\alpha^{p/rq}$, and hence $|f - \psi_j|^r = G_{\alpha-\beta} * g_j$, $g_j \in \Lambda_\beta^{p/rq}$. Since

$$\{x: A_{1j}(x) \geq \varepsilon^r\} \subset \{x: G_{\alpha-\beta} * M_F g_j(x) \geq \varepsilon^r\},$$

we obtain

$$B_\beta^{p/rq} \{x: A_{1j}(x) \geq \varepsilon^r\} \leq \frac{1}{\varepsilon^r} \|M_F g_j\|_{\Lambda_\beta^{p/rq}} \leq \frac{A}{\varepsilon^r} \|g_j\|_{\Lambda_\beta^{p/rq}} \quad (\text{Lemma 2}).$$

By Lemma 9, $|f - \psi_j|^r \rightarrow 0$ ($\Lambda_\alpha^{p/rq}$), and hence $g_j \rightarrow 0$ ($\Lambda_\beta^{p/rq}$). The claim now follows (see Lemma 6).

Let $E'' = \{x: \limsup_{i \rightarrow \infty} M_F(|f - \psi_{j_i}|^r)(x) > 0\}$. Then $B_\beta^{p/rq}(E'') = 0$. We may assume that $\psi_{j_i}(x) \rightarrow f(x)$ for L_α^{pq} -a.e. x , and if E' is the exceptional set, we have $B_\beta^{pq}(E') = 0$. We show that $E \subset E' \cup E''$.

Let $x \notin E' \cup E''$, and let $\varepsilon > 0$. Choose j_0 such that $A_{1j_0}(x) < \varepsilon/3$, $A_{3j_0}(x) < \varepsilon/3$, and then choose τ_0 such that $S \subset B(0, \tau_0)$ implies $A_{2j_0}(x) < \varepsilon/3$. It follows then that $x \notin E$.

If $p \leq r < s$, $1/s = 1/p - \alpha/n$, let $r < t < s$, and let $\alpha_t = \alpha - n/p +$

n/t . By [6, p. 441], $f \in \Lambda_{\alpha}^{tq}$. Consequently by the first part of the proof, $[L_{\alpha}^{tq}, L_{\alpha}^{t/rq}](E) = 0$, and the proof of the theorem is complete.

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MATHEMATICAL SCIENCES DIVISION, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907