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MULTI-DIMENSIONAL QUALITY CONTROL PROBLEMS AND QUASI VARIATIONAL INEQUALITIES(1)

BY

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ABSTRACT. A machine can manufacture any one of n m-dimensional Brownian motions with drift λ_j, $P_x^{\lambda_j}$, defined on the space of all paths $x(t) \in C([0, \infty); R^m)$. It is given that the product is a random evolution dictated by a Markov process $\theta(t)$ with n states, and that the product is $P_x^{\lambda_j}$ when $\theta(t) = j$, $1 < j < n$. One observes the σ-fields of $x(t)$, but not of $\theta(t)$. With each product $P_x^{\lambda_j}$ there is associated a cost c_j. One inspects θ at a sequence of times (each inspection entails a certain cost) and stops production when the state $\theta = n$ is reached. The problem is to find an optimal sequence of inspections. This problem is reduced to solving a certain elliptic quasi variational inequality. The latter problem is actually solved in a rather general case.

Introduction. Let $\theta(t)$ be a Markov process with n states and transition probabilities $p_{i,j}(t)$. With each state i we associate an m-dimensional Brownian motion with drift λ_i, i.e., a probability $P_x^{\lambda_i}$ defined on the space Ω of continuous functions $x(t)$ from $[0, \infty)$ into R^m.

Let $K_1, \ldots, K_{n-1}$, $c_1, \ldots, c_n$ be nonnegative constants and define a function f by $f(i) = c_i$ if $i = 1, 2, \ldots, n$. Let $\tau = (\tau_1, \tau_2, \ldots)$ be an increasing sequence of "inspection times" and consider the cost function

$$J_x^i(\tau) = E^{i,x}\left[K_i + \sum_{j=1}^{n-1} K_j\left[\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\right]\right]$$

$$+ E^{i,x}\left[\int_0^{\tau_1} f(\theta(s))\,ds + \sum_{j=1}^{n-1}\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\,ds\right]. \quad (1)$$

Here $E^{i,x}$ indicates the expectation associated with the process $(x(t), \theta(t))$ when $x(0) = x$, $\theta(0) = i$; more precisely, one can show that there is a Markov process $P^{\theta,x}$ on the σ-fields of the functions $(x(t), \theta(t))$ such that

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$P^{\theta,x(s)}$ coincides with $P^{\lambda_l}_{x(s)}$ as long as $\theta(s) = l$ and $E^{l,x}$ is the expectation corresponding to $P^{l,x}$. In (1) it is understood that if $\theta(\tau_m) = n$ then $\tau_{m+1} = \infty$, and $K_j I_{\theta(\tau_l)=j} = 0$, $\int_{\tau_l}^{\tau_{l+1}} f = 0$ if $\tau_l = \infty$.

The concept "inspection time" means roughly that each set $(\tau_{h+1} \leqslant s)$ (τ_{h+1} in τ) depends only on the information of the σ-field of $x(t)$, $0 \leqslant t \leqslant s$, and on the knowledge of $\theta(\tau_j)$ for $1 \leqslant j \leqslant h$.

It is easy to see that $J_x^i(\tau)$ represents the cost incurred by

(i) inspection of θ at times τ_l, and

(ii) time spent in state k (provided $c_k > 0$).

The cost in (i) is K_l per inspection, provided that at the previous inspection θ was at state l. The cost in (ii) is a sum of terms $c_k \Delta_k$, where Δ_k is the time spent in state k.

The problem of minimizing the cost is a problem of quality control. The special case where $m = 1$, $n = 2$ was studied by the authors in an earlier paper [**1**].

The results in [**1**] consist of two parts:

(α) reducing the problem of finding a sequence of optimal inspection times to a problem of solving a certain quasi variational inequality (q.v.i.);

(β) solving the q.v.i.

In the present paper we shall generalize (α) and (β) to any n, m.

In §1 we state the quality control problem in precise terms. In §2 we recall some results of Shiryaev (Theorem 2.1) which assert that the process

$$p_j^\mu(t) = P^{\mu,x}[\theta(t) = j \mid \mathfrak{F}_t] \qquad (1 \leqslant j \leqslant n;\ \mathfrak{F}_t = \sigma(x(s), 0 \leqslant s \leqslant t))$$

is a solution of a system of stochastic differential equations whose coefficients can be expressed in terms of the λ_i and the infinitesimal generator $(q_{i,j})$ of the θ process. Here μ stands for the initial distribution of $\theta(0)$. We also give in §2 (Theorem 2.2) an explicit formula for the $p_j^\mu(t)$.

In the Appendix we give a derivation of the formula of Theorem 2.2. By the same method we also give a new proof of Theorem 2.1.

In §3 we prove several lemmas on transformations of expectations $E^{\mu,x}$ with integrands which involve stopping times. These lemmas are needed in the subsequent sections.

In §4 we write down the q.v.i.

$$M_{x,p} V(x,p) + \sum_{j=1}^{n} c_j p_j \geqslant 0, \qquad V(x,p) \leqslant K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j),$$

$$\left[M_{x,p} V(x,p) + \sum_{j=1}^{n} c_j p_j \right] \left[K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) - V(x,p) \right] = 0 \quad (2)$$

for $x \in R^m$, $p = (p_1, \ldots, p_n)$, $p_i > 0$, $\sum_{i=1}^n p_i = 1$; here e_j is the jth unit vector, $K(p)$ is a suitable function such that $K(p) = K_j$ if $p = e_j$, and $M_{x,p}$ is the infinitesimal generator of the (x, p) process. $M_{x,p}$ is a degenerate elliptic operator. We prove that if V is a solution of (2) then

$$V(x, p) = \inf_{\tau} J_x^p(\tau),$$

where $J_x^p(\tau) = J_x^i(\tau)$ when $p = e_i$. We also obtain an optimal sequence of inspections, defined as follows:

If at the lth inspection we find that $\theta = i$, then the $(l + 1)$th inspection is at the first time when the process $(x(t), p(t))$ hits the set

$$\left\{(x, p);\ V(x, p) = K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j)\right\};$$

here $p(0) = i$ and $x(0)$ is the position of x at the lth inspection. In view of Theorem 2.2, this optimal stopping rule can be expressed directly in terms of the process $x(t)$.

In §5 we consider other costs: (i) one which allows for the option of repairs, and (ii) a cost with a discount factor α, $\alpha > 0$; that is, in (1) we replace $f(s)$ by $f(s)e^{-\alpha s}$ and $I_{\theta(\tau_l)=j}K_j$ by $I_{\theta(\tau_l)=j}K_j e^{-\alpha\tau_l}$. Denote the optimal cost of the last cost function by V_α. The results of §4 extend to these two costs. In particular, the q.v.i. for V_α is the same as for V except that $M_{x,p}$ is replaced by $M_{x,p} - \alpha$. The results of §4 also extend to the case when the λ_j are functions of x. However, when the λ_j do not depend on x then we prove that also the optimal costs V, V_α do not depend on x.

In §6 we assume that the λ_j do not depend on x and that $\lambda_j - \lambda_1$ $(2 \leqslant j \leqslant n)$ generate the entire space R^m (this is equivalent to the statement that the elliptic operator M_p corresponding to the $p_j^\mu(t)$-process is nondegenerate in the set $p_i > 0$, $\Sigma p_i < 1$; it does degenerate, however, on the boundary of this set). We then prove that the q.v.i. for $V_\alpha = V_\alpha(p)$ has a unique regular solution which coincides with the optimal cost V_α; the optimal inspection rule stated above is then valid. We also show how to approximate V_α by solutions of partial differential equations.

In §7 we solve the q.v.i. (2) rather explicitly in the special case $m = 1$, $n = 2$ and general $q_{i,j}$.

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1. **The quality control problem.** Let $\theta(t)$ be a Markov process with n states $1, 2, \ldots, n$ and with transition probability matrix $P(t) = (p_{i,j}(t))$. We denote its generator by $Q = (q_{i,j})$, so that $P(t) = e^{tQ}$.

Let $\lambda_1, \ldots, \lambda_n$ be given distinct m-vectors and define a function $g(\theta)$ by $g(i) = \lambda_i$. We shall be working in this paper with the process $(x(t), \theta(t))$ which, formally, is given by

$$dx(t) = g(\theta(t))\, dt + dw(t) \tag{1.1}$$

where $w(t)$ is an m-dimensional Brownian motion. This process is called *random evolution* and it can be constructed in various ways. We describe one simple way:

Let Q be the measure associated with the standard Brownian motion $w(t)$, that is,

$$Q(w(0) = 0) = 1,$$

$$Q[w(t+s) \in A | \mathcal{G}_t] = \int_A \frac{1}{(2\pi s)^{n/2}} \exp\left[- \frac{|y - w(t)|^2}{2s} \right] dy$$

for any Borel set A in $\mathfrak{B}(R^m)$. Let R^θ be the measures associated with the Markov process $\theta(t)$. We take the processes $\theta(t)$ and $w(t)$ to be independent and, in fact, to exist in two different probability spaces Ω' and Ω, respectively.

Let $\hat{P}^\theta = Q \times R^\theta$. Consider the process

$$\tilde{x}(t) = x + w(t) + \int_0^t g(\theta(s))\, ds \tag{1.2}$$

with $\theta(0) = \theta$, and write $\tilde{x}(t) = \tilde{x}_{x,\theta}(t)$. Define

$$\begin{aligned} \tilde{P}^{\theta,x}&(\tilde{x}_{x,\theta}(t_1) \in A_1, \ldots, \tilde{x}_{x,\theta}(t_m) \in A_m, \theta(t_1) \in B_1, \ldots, \theta(t_m) \in B_m) \\ &= \hat{P}^\theta \Big(x + w(t_1) + \int_0^{t_1} g(\theta(s))\, ds \in A_1, \ldots, x + w(t_m) \\ &\qquad + \int_0^{t_m} g(\theta(s))\, ds \in A_m, \theta(t_1) \in B_1, \ldots, \theta(t_m) \in B_m \Big), \end{aligned} \tag{1.3}$$

and extend $\tilde{P}^{\theta,x}$ as measures on the σ-field $\mathcal{Q}$ of $(\theta(u), w(u))$, $0 \leqslant u < \infty$.

LEMMA 1.1. *$\tilde{P}^{\theta,x}$ is a Markov process with respect to the σ-fields $\mathcal{Q}_t = \sigma((x(u), \theta(u)), 0 \leqslant u \leqslant t)$.*

PROOF. Set

$$G(x, \theta) = \tilde{P}^{\theta,x}(\tilde{x}(t-s) \in A, \theta(t-s) \in B). \tag{1.4}$$

We shall prove that

$$\tilde{P}^{\theta,x}[\tilde{x}(t) \in A, \theta(t) \in B | \mathcal{Q}_s] = G(\tilde{x}(s), \theta(s)). \tag{1.5}$$

We can write

$$\tilde{x}(t) = \tilde{x}(s) + (w(t) - w(s)) + \int_s^t g(\theta(u))\, du. \tag{1.6}$$

In order to evaluate the left-hand side of (1.5) we first evaluate

$$I \equiv \tilde{P}^{\theta,x}[g_1(\tilde{x}(s)) g_2(w(t) - w(s)) g_3(\theta(u)) g_4(\theta(t)) | \mathcal{Q}_s]$$

where $s \leqslant u \leqslant t$ and the g_i are bounded Borel measurable functions. Using

(1.3) and the Markov property of both $w(t)$ and $\theta(t)$, we find that

$$I = \hat{P}^{\theta}\big[g_1(\tilde{x}(s))g_2(w(t) - w(s))g_3(\theta(u))g_4(\theta(t))|\mathcal{Q}_s\big]$$

$$= g_1(\tilde{x}(s))\hat{P}^{\theta(s)}\big[g_2(w(t-s))g_3(\theta(u-s))g_4(\theta(t-s))\big].$$

Setting

$$\hat{G}(x, \theta) = \tilde{P}^{\theta,x}\big[g_1(x)g_2(w(t-s))g_3(\theta(u-s))g_4(\theta(t-s))\big],$$

we then have

$$I = \hat{G}(\tilde{x}(s), \theta(s)).$$

The same kind of result holds also for linear combinations of functions of the form $g_1g_2g_3g_4$. By approximation, it therefore also holds for the left-hand side of (1.5). Since

$$\tilde{x}(t-s) = x + (w(t-s) - w(0)) + \int_0^{t-s} g(\theta(u))\,du,$$

by comparing with (1.6), we see that the function G which is to appear on the right-hand side of (1.5) is given by (1.4). This shows that $\tilde{P}^{\theta,x}$ is a Markov process with respect to $\mathcal{Q}_t$.

Let Ω be the space of all continuous functions $t \to x(t)$ from $[0, \infty)$ into R^m and let $\hat{\mathcal{F}}_t$, $\mathcal{F}$ be the σ-fields generated by $x(u)$ for $0 \leqslant u \leqslant t$ and $0 \leqslant u < \infty$, respectively. We denote functions $x(t)$ also by ω and write $x(t) = x(t, \omega) = \omega(t)$.

Let Ω' be the space of all right continuous functions $\theta(t)$ from $t \in [0, \infty)$ into the set $\{1, 2, \ldots, n\}$ which have left limits $\theta(t-0)$ for each $t > 0$. Denote by $\mathcal{H}_t$ and $\mathcal{H}$ the σ-fields generated by $\theta(u)$ for $0 \leqslant u \leqslant t$ and $0 \leqslant u < \infty$, respectively.

In what follows we shall choose the random evolution model constructed in Lemma 1.1 with the specific Ω and Ω' which we have just defined. Since the process $\tilde{x}(t)$ is continuous, the measures $\tilde{P}^{\theta,x}$ can be redefined as measures $P^{\theta,x}$ in $\Omega' \times \Omega$ by

$$P^{\theta,x}\big[x(t_1, \omega) \in A_1, \theta(t_1) \in B_1, x(t_2, \omega) \in A_2, \theta(t_2) \in B_2, \ldots\big]$$

$$= \hat{P}^{\theta,x}\big[\tilde{x}(t_1, \omega) \in A_1, \theta(t_1) \in B_1, \tilde{x}(t_2, \omega) \in A_2, \theta(t_2) \in B_2, \ldots\big].$$

Notation.

$$\hat{\mathcal{F}}_t^s = \sigma(x(u), s \leqslant u \leqslant t), \qquad \hat{\mathcal{M}}_t^s = \sigma((\theta(u), x(u)), s \leqslant u \leqslant t),$$

$$\mathcal{F}_t^s = \hat{\mathcal{F}}_{t+0}^s, \quad \mathcal{M}_t^s = \hat{\mathcal{M}}_{t+0}^s, \quad \mathcal{F}_t = \mathcal{F}_t^0, \quad \mathcal{M}_t = \mathcal{M}_t^0.$$

Notice that $\mathcal{F}_t$, $\mathcal{M}_t$ are right continuous σ-fields.

From Lemma 1.1 we deduce (using the Feller property) that the $P^{\theta,x}$ form a Markov process with respect to $\mathcal{M}_t$.

Let ϕ_s be the shift operator, mapping Ω into Ω:

$$\phi_s\colon x(t) \to (t + s).$$

DEFINITION. A sequence $\tau = (\tau_1, \tau_2, \ldots)$ is called a *sequence of inspection times* if $\theta(\tau_m) = n$ implies $\tau_l = \infty$ for all $l > m$, and

$$\tau_1 = \sigma_1, \quad \tau_{m+1} = \tau_m + \sum_{l=1}^{n-1} I_{\theta(\tau_m)=l}\sigma_{m+1,l}(\phi_{\tau_m}) \qquad (m \geqslant 1), \tag{1.7}$$

where σ_1 and the $\sigma_{m,l}$ are (finite valued) stopping times with respect to $\mathcal{F}_t$. Each τ_m is called an *inspection time*.

LEMMA 1.2. *Each τ_{m+1} is a stopping time with respect to the σ-field*

$$\mathcal{F}_{m,t} = \sigma(\mathcal{F}_t, \theta(\tau_1), \ldots, \theta(\tau_m)).$$

PROOF. The proof is by induction. Assume that τ_m is a stopping time with respect to $\mathcal{F}_{m-1,t}$. We shall need the rule:

$$A \in \mathcal{F}_t \text{ implies } \phi_{\tau_m}^{-1}(A) \in \mathcal{F}_{t+\tau_m},$$

which follows by checking it first for cylinder sets (making use of the fact that $x(s + \tau)$ is $\mathcal{F}_{t+\tau}$ measurable). Taking $A = \{\sigma_{m+1,l} \leqslant t\}$, we conclude that

$$\phi_{\tau_m}^{-1}(\sigma_{m+1,l} \leqslant t) \in \mathcal{F}_{t+\tau_m},$$

i.e.,

$$\{\sigma_{m+1,l}(\phi_{\tau_m}) \leqslant t\} \in \mathcal{F}_{t+\tau_m}.$$

Thus $\sigma_{m+1,l}(\phi_{\tau_m})$ is $\mathcal{F}_{t+\tau_m}$ stopping time. Now, according to [7, p. 74], if $\mathcal{B}_t$ is a right continuous increasing family of σ-subfields of $\mathcal{M}_t$ and $\hat{\tau}$ is $\mathcal{B}_t$ stopping time, then $\hat{\sigma} + \hat{\tau}$ is $\mathcal{B}_t$ stopping time if and only if $\hat{\sigma}(\phi_{\hat{\tau}})$ is $\mathcal{B}_{t+\hat{\tau}}$ stopping time. Applying this result with $\mathcal{B}_t = \mathcal{F}_{m-1,t}$ we find that

$$\{\tau_m + \sigma_{m+1,l}(\phi_{\tau_m}) \leqslant t\} \in \mathcal{F}_{m-1,t}$$

for all $t \geqslant 0$. Writing

$$\{\tau_{m+1} \leqslant t\} = \bigcup_{l=1}^{n-1} \{\theta(\tau_m) = l\} \cap \{\tau_m + \sigma_{m+1,l}(\phi_{\tau_m}) \leqslant t\},$$

it follows that $\{\tau_{m+1} \leqslant t\} \in \mathcal{F}_{m,t}$.

COROLLARY 1.3. *Each τ_{m+1} is $\mathcal{M}_t$ stopping time.*

In the problem to be introduced below it seems natural to take τ_{m+1} as any stopping time with respect to $\mathcal{F}_{m,t}$. However, in order to be able to apply the strong Markov property for suitable functionals we have restricted the τ_{m+1} to be as in (1.7).

Let $K_1, \ldots, K_{n-1}$ be positive constants and define a function $K(\theta)$ by $K(i) = K_i$. Let $c_1, \ldots, c_n$ be nonnegative constants and define a function $f(\theta)$ by $f(i) = c_i$. We now introduce the cost function

$$J_x^i(\tau) = E^{i,x}\left[K_i + \sum_{j=1}^{n-1} K_j\left[\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\right]\right]$$
$$+ E^{i,x}\left[\int_0^{\tau_1} f(\theta(s))\,ds + \sum_{j=1}^{n-1}\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\,ds\right], \tag{1.8}$$

or, more briefly,

$$J_x^i(\tau) = E^{i,x}\left[\sum_{l=0}^{\infty} I_{\theta(\tau_l)\neq n}\left[K(\theta(\tau_l)) + \int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\,ds\right]\right], \tag{1.9}$$

where $\tau_0 = 0$, $\theta(0) = i$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_l, \ldots)$ is a sequence of inspections. It is understood here that if $\theta(\tau_m) = n$ for some m, then $K(\theta(\tau_l)) + \int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\,ds = 0$ if $l > m$.

In order to motivate the interest in (1.8), we take $i = 1$ and rewrite (1.8) in extended form:

$$\begin{aligned}
J_x^1(\tau) = E^{1,x}\Bigg[& K_1 + \int_0^{\tau_1} f(\theta(s))\,ds \\
& + I_{\theta(\tau_1)=1}\Bigg[K_1 + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds \\
& \qquad + I_{\theta(\tau_2)=1}\left[K_1 + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right] \\
& \qquad + I_{\theta(\tau_2)=2}\left[K_2 + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right] \\
& \qquad + \cdots + I_{\theta(\tau_2)=n-1}\left[K_{n-1} + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right]\Bigg] \\
& + I_{\theta(\tau_1)=2}\Bigg[K_2 + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds \\
& \qquad + I_{\theta(\tau_2)=2}\left[K_2 + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right] \\
& \qquad + I_{\theta(\tau_2)=3}\left[K_3 + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right] + \cdots \\
& \qquad + I_{\theta(\tau_2)=n-1}\left[K_{n-1} + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right]\Bigg] \\
& + \cdots + I_{\theta(\tau_1)=n-1}\Bigg[K_{n-1} + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds \\
& \qquad + I_{\theta(\tau_2)=n-1}\left[K_{n-1} + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right]\Bigg]\Bigg].
\end{aligned} \tag{1.10}$$

Consider the special case where $c_0 < c_1 < \cdots < c_n$, $K_1 > K_2 > \cdots > K_{n-1}$ and $p_{i,j}(t) = 0$ if $j < i$. Then the cost $J_x^1(\tau)$ arises naturally in the following model:

A machine is manufacturing a product A_i when it is in position M_i, $1 \leqslant i \leqslant n$. The product A_i is preferable to the product A_{i+1}. The machine may shift at random from position M_i to position M_j (i.e., from $\theta(t) = i$ to $\theta(t) = j$) only if $j > i$. In particular, the state M_n is absorbing. The product A_i is a Brownian motion with drift λ_i, which we designate by $P_x^{\lambda_i}$. By looking at the product (i.e., at the $P_x^{\lambda_i}$) one cannot tell the position in which the machine is at present (i.e., the information in $\mathcal{F}_t$ is only a *partial* information on $\theta(t)$). In order to determine this position, one must actually check the machine itself; the cost thereby incurred is K_i if one has the knowledge that in the preceding inspection the machine was in position M_i.

Another cost incurred is due to the production of the less desirable products between the times of consecutive inspections; the cost, per product A_i, is c_i multiplied by the time of production. The functional $J_x^1(\tau)$ represents the total cost incurred during a sequence of inspections. It is assumed here that once the machine is found to be in state M_n we stop production.

Denote by $\mathcal{Q}$ the set of all sequences of inspections and set

$$V_i(x) = \inf_{\tau \in \mathcal{Q}} J_x^i(\tau).$$

We are interested, in this paper, in the problem of studying $V_i(x)$ and of finding $\bar{\tau}_i \in \mathcal{Q}$ which minimizes the cost, i.e.,

$$V_i(x) = J_x^i(\bar{\tau}_i).$$

This problem is called a *quality control problem* or a *quickest detection problem*.

2. **The** p**-process**. Let μ be any measure on the space Ω', i.e., $\mu = (p_1, \ldots, p_n)$ where $p_i \geqslant 0$, $\Sigma_{i=1}^n p_i = 1$. We set

$$P^{\mu,x} = \sum_{i=1}^n p_i P^{i,x}$$

and

$$p_j^\mu(t) = E^{P^{\mu,x}}\left[\theta(t) = j \mid \mathcal{F}_t\right] \qquad (1 \leqslant j \leqslant n), \tag{2.1}$$

where E^Q denotes the expectation with respect to the measure Q. For simplicity we write

$$p_j(t) = p_j^\mu(t).$$

Notice that $p_j(t) \geqslant 0$, $\Sigma_{j=1}^n p_j(t) = 1$.

THEOREM 2.1. *The process* $(p_1(t), \ldots, p_n(t))$ *is a continuous strong Markov process with respect to the* σ*-field* $\mathcal{F}_t$, *and*

$$dp_j(t) = \sum_{l=1}^{n} q_{l,j}p_l(t)\,dt + p_j\left(\lambda_j - \sum_{l=1}^{n} \lambda_l p_l(t)\right)\cdot d\bar{w}(t), \tag{2.2}$$

where $\bar{w}(t)$ is an m-dimensional Brownian motion adaptable to $\mathcal{F}_t$. If $p_i \neq 0$ for some i then $p_i(t) \neq 0$ for all $t > 0$, so that one can define $y_i(t) = p_j(t)/p_i(t)$ for $j = 1, \ldots, i-1, i+1, \ldots, n$; the process $(y_1(t), \ldots, y_{i-1}(t), y_{i+1}(t), \ldots, y_n(t))$ is a continuous strong Markov process with respect to $\mathcal{F}_t$, and

$$dy_j(t) = y_j(t)(\lambda_j - \lambda_i)\cdot(d\eta - \lambda_i dt) \;+ \sum_{l=1}^{n} y_l(t)\left[q_{l,j} - y_j(t)q_{l,i}\right]dt, \tag{2.3}$$

where

$$d\eta(t) = \sum_{l=1}^{n} \lambda_l p_l(t)\,dt + d\bar{w}(t), \qquad y_i(t) \equiv 1. \tag{2.4}$$

This theorem was proved by Shiryaev in [9]; see also [6]. It may be noted that the derivation of (2.3) from (2.2) and, vice versa, the derivation of (2.2) from (2.3), (2.4) can be carried out by straightforward calculation using Itô's calculus.

Later on we shall need an explicit formula for the $p_j(t)$. We set

$$p_j^i(t) = p_j^\mu(t) \quad \text{when } \mu = (0, \ldots, 0, 1, 0, \ldots, 0) \text{ with 1 in the } i\text{th component.} \tag{2.5}$$

Defining

$$z_{i,j}(s,t) = \exp\left[(\lambda_j - \lambda_i)\cdot(x(t) - x(s)) - \tfrac{1}{2}\left(|\lambda_j|^2 - |\lambda_i|^2\right)(t-s)\right], \tag{2.6}$$

we introduce the functions

$$\begin{aligned}
\bar{p}_{i,j}(t) = \sum_{\rho=0}^{\infty}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}{}' &\int_0^t du_{i,\gamma_1}\exp\{-q_i u_{i,\gamma_1}\}q_{i,\gamma_1} \\
&\cdot \int_0^{t-u_{i,\gamma_1}} du_{\gamma_1,\gamma_2}\exp\{-q_{\gamma_1}u_{\gamma_1,\gamma_2}\}q_{\gamma_1,\gamma_2} \\
&\cdots \int_0^{t-u_{i,\gamma_1}-u_{\gamma_1,\gamma_2}-\cdots-u_{\gamma_{\rho-1},\gamma_\rho}} du_{\gamma_\rho,j}\exp\{-q_{\gamma_\rho}u_{\gamma_\rho,j}\} \\
&\cdot q_{\gamma_\rho,j}\exp\{-q_j(t - u_{i,\gamma_1} - \cdots - u_{\gamma_{\rho-1},\gamma_\rho} - u_{\gamma_\rho,j})\} \\
&\cdot z_{i,\gamma}(u_{i,\gamma_1}, u_{i,\gamma_1} + u_{\gamma_1,\gamma_2})\; z_{i,\gamma_2}(u_{i,\gamma_1} + u_{\gamma_1,\gamma_2}, u_{i,\gamma_1} + u_{\gamma_1,\gamma_2} + u_{\gamma_2,\gamma_3}) \\
&\cdots z_{i,j}(u_{i,\gamma_1} + u_{\gamma_1,\gamma_2} + \cdots + u_{\gamma_{\rho-1},\gamma_\rho} + u_{\gamma_\rho,j}, t) \qquad (1 \leqslant i,j \leqslant n),
\end{aligned} \tag{2.7}$$

where $q_l = -q_{l,l}$.

Here we have used the following notation: $\sum'_{(i,\gamma_1,\ldots,\gamma_\rho,j)}$ indicates the sum taken over all $\gamma_1, \ldots, \gamma_\rho$ with each γ_l varying over $1, 2, \ldots, n$ in such a way that $i \neq \gamma_1 \neq \gamma_2 \neq \cdots \neq \gamma_{\rho-1} \neq \gamma_\rho \neq j$, i.e., $\gamma_l \neq \gamma_{l+1}$ for $l = 1, \ldots, \rho$ and $\gamma_1 \neq i$, $\gamma_\rho \neq j$. The notation

$$\sum_{\rho=0}^{\infty}{}^{*}$$

indicates that if $i \neq j$ then the summation is taken over $\rho = 0, 1, 2, 3, \ldots$; whereas if $i = j$, then the summation is over $\rho = -1, 1, 2, 3, \ldots$. When $\rho = 0$ there is only one integration on the right-hand side of (2.7) (generally for any ρ the number of integrations is $\rho + 1$); the term in (2.7) corresponding to $\rho = -1$ (when $i = j$) is understood to be $\exp(-q_i t)$.

THEOREM 2.2. *The following formula holds*:

$$p_j^i(t) = \frac{\bar{p}_{i,j}(t)}{\sum_{l=1}^{n} \bar{p}_{i,l}(t)}, \tag{2.8}$$

and, more generally,

$$p_j^\mu(t) = \sum_{i=1}^{n} p_i \bar{p}_{i,j}(t) \Big/ \left[\sum_{l=1}^{n} p_l z_{i,l}(0, t) \sum_{k=1}^{n} \bar{p}_{l,k}(t) \right] \tag{2.9}$$

if $\mu = p = (p_1, \ldots, p_n)$.

One can verify (2.8) (or (2.9)) by showing that the functions given by the right-hand side of (2.8) (or (2.9)) satisfy the stochastic differential system (2.2).

In the Appendix, however, we shall actually derive formulas (2.8) and (2.9). The method used will enable us to also give a new proof of Theorem 2.1. Unlike the method of Shiryaev, our method relies only marginally on the stochastic calculus, and it can be adopted to any Markov process or chain.

3. Auxiliary results. As in §2, let $p_i(t) = p_i^\mu(t)$.

LEMMA 3.1. *Let τ be an $\mathcal{F}_t$ stopping time and let $h(x)$ be a continuous bounded function in R^m. Then*

$$E^{\mu,x}\left[I_{\theta(\tau)=j} h(x(\tau)) \right] = E^{\mu,x}\left[p_j(\tau) h(x(\tau)) \right]. \tag{3.1}$$

PROOF. Suppose first that τ takes only countably many values r_k. Then

$$E^{\mu,x}\left[I_{\theta(\tau)=j} h(x(\tau)) \right] = \sum_k E^{\mu,x}\left[I_{\tau=r_k} I_{\theta(r_k)=j} h(x(r_k)) \right]$$

$$= \sum_k E^{\mu,x}\left[I_{\tau=r_k} p_j(r_k) h(x(r_k)) \right] = E^{\mu,x}\left[p_j(\tau) h(x(\tau)) \right].$$

Consider now the case of general τ. Let τ_m be a sequence of countably valued stopping times such that $\tau_m \downarrow \tau$ if $m \uparrow \infty$. Then (3.1) holds for $\tau = \tau_m$. Taking $m \to \infty$ and recalling that $p(t)$, $h(x(t))$ are continuous and $\theta(t)$ is right continuous, (3.1) follows.

LEMMA 3.2. *For any $\mathcal{F}_t$ stopping time τ,*

$$E^{\mu,x}\left[\int_0^\tau f(\theta(t))\,dt\right] = E^{\mu,x}\left[\int_0^\tau \sum_{j=1}^n c_j p_j(t)\,dt\right]. \tag{3.2}$$

PROOF. It is clearly sufficient to show that

$$E^{\mu,x}\left[\int_0^\tau I_{\theta(t)=j}\,dt\right] = E^{\mu,x}\left[\int_0^\tau p_j(t)\,dt\right]. \tag{3.3}$$

Consider first the case where τ is bounded and with countably many values r_k. If $\tau < T$ then

$$\begin{aligned} E^{\mu,x}\left[\int_\tau^T I_{\theta(t)=j}\,dt\right] &= \sum_k E^{\mu,x}\left[I_{\tau=r_k}\int_{r_k}^T I_{\theta(t)=j}\,dt\right] \\ &= \sum_k E^{\mu,x}\left[I_{\tau=r_k}\int_{r_k}^T E^{\mu,x}[I_{\theta(t)=j}|\mathcal{F}_t]\,dt\right] \\ &= \sum_k E^{\mu,x}\left[I_{\tau=r_k}\int_{r_k}^T p_j(t)\,dt\right] = E^{\mu,x}\left[\int_\tau^T p_j(t)\,dt\right]. \end{aligned}$$

Also,

$$E^{\mu,x}\left[\int_0^T I_{\theta(t)=j}\,dt\right] = E^{\mu,x}\left[\int_0^T p_j(t)\,dt\right],$$

so that (3.3) follows. The case of general τ follows by approximation.

COROLLARY 3.3. *Let τ and τ_m be stopping times such that $\tau_m \to \tau$ a.e. as $m \to \infty$. Then, for any continuous bounded function $h(x)$,*

$$\lim_{m\to\infty} E^{\mu,x}\left[I_{\theta(\tau_m)=j}h(x(\tau_m))\right] = E^{\mu,x}\left[I_{\theta(\tau)=j}h(x(\tau))\right]. \tag{3.4}$$

Indeed, a similar limit theorem is true for the right-hand side of (3.1) (since $p_j(t)$, $h(x(t))$ are continuous). Now use (3.1) to deduce (3.4).

COROLLARY 3.4. *Let τ be a predictable stopping time. Then*

$$P^{\mu,x}[\theta(\tau-0) = \theta(\tau)] = 1. \tag{3.5}$$

PROOF. The predictability assumption means that there is a sequence of stopping times τ_m such that $\tau_m < \tau$, $\tau_m \uparrow \tau$ as $m \uparrow \infty$. It follows that $\theta(\tau_m) \to \theta(\tau - 0)$ as $m \to \infty$. Applying (3.4) with $h \equiv 1$ we then get

$$P^{\mu,x}[\theta(\tau-0) = j] = P^{\mu,x}[\theta(\tau) = j]. \tag{3.6}$$

Since $\theta(\tau-0) \leq \theta(\tau)$, the set $(\theta(\tau-0) = n)$ is a subset of the set $(\theta(\tau) = n)$. Hence, by (3.6) with $j = n$, these sets are equal a.e.

Next,

$$(\theta(\tau - 0) = n - 1) \subset (\theta(\tau) = n) \cup (\theta(\tau) = n - 1).$$

Since the set $(\theta(\tau) = n)$ coincides a.e. with the set $(\theta(\tau - 0) = n)$, we conclude that $(\theta(\tau - 0) = n - 1)$ is contained in $(\theta(\tau) = n - 1)$, up to a set of measure 0. Applying (3.6) with $j = n - 1$ we find that the sets $(\theta(\tau - 0) = n - 1)$ and $(\theta(\tau) = n - 1)$ are equal a.e.

We can now proceed step by step in this manner and establish (3.5).

LEMMA 3.5. *Let $\tau_2 = \tau_1 + \sigma_2(\phi_{\tau_1})$ where τ_1 is a stopping time with respect to $\mathfrak{M}_t$ and σ_2 is a stopping time with respect to $\mathfrak{F}_t$. Then, for any bounded continuous function $h(x)$,*

$$E^{\mu,x}\left[I_{\theta(\tau_1)=i} I_{\theta(\tau_2)=j} h(x(\tau_2))\right]$$
$$= E^{\mu,x}\left[I_{\theta(\tau_1)=i} E^{i,x(\tau_1)}\left[I_{\theta(\sigma_2)=j} h(x(\sigma_2))\right]\right]$$
$$(1 \leqslant i,j \leqslant n). \qquad (3.7)$$

PROOF. Consider first the case where σ_2 has only countably many values r_k. Then

$$E^{\mu,x}\left[I_{\theta(\tau_1)=i} I_{\theta(\tau_1+\sigma_2(\phi_{\tau_1}))=j} h(x(\tau_1 + \sigma_2(\phi_\tau)))\right]$$
$$= \sum_k E^{\mu,x}\left[I_{\theta(\tau_1)=i} I_{\sigma_1(\phi_{\tau_1})=r_k} I_{\theta(\tau_1+r_k)=j} h(x(\tau_1 + r_k))\right]. \qquad (3.8)$$

As easily seen, the sets $(\sigma_2(\phi_{\tau_1}) = r_k)$ and $\phi_{\tau_1}^{-1}(\sigma_2 = r_k)$ are equal. By the strong Markov property (as stated in [4]) of the (x, θ) process we then have

$$E^{\mu,x}\left[I_{\theta(\tau_1)=i} I_{\phi_{\tau_1}^{-1}(\sigma_2=r_k)} I_{\theta(\tau_1+r_k)=j} h(x(\tau_1 + r_k))\right]$$
$$= E^{\mu,x}\left[I_{\theta(\tau_1)=i}\left[E^{i,x(\tau_1)}\left[I_{(\sigma_2=r_k)} I_{\theta(r_k)=j} h(x(r_k))\right]\right]\right]. \qquad (3.9)$$

Hence, the right-hand side of (3.8) is equal to the right-hand side of (3.7).

Having completed the proof of (3.7) in case σ_2 has countably many values, we now consider the case of a general σ_2 and let $\sigma_{2,m}$ be countably valued stopping times such that $\sigma_{2,m} \downarrow \sigma_2$ if $m \uparrow \infty$. Writing (3.7) for $\sigma = \sigma_{2,m}$, $\tau_2 = \tau_1 + \sigma_{2,m}(\phi_{\tau_1})$, taking $m \uparrow \infty$ and using the continuity of $h(x)$ and the right continuity of $\theta(t)$, assertion (3.7) follows.

LEMMA 3.6. *Let $\tau_2 = \tau_1 + \sigma_2(\phi_{\tau_1})$, where τ_1 is a stopping time with respect to $\mathfrak{M}_t$ and σ_2 is a stopping time with respect to $\mathfrak{F}_t$. Then*

$$E^{\mu,x}\left[I_{\theta(\tau_1)=i} \int_{\tau_1}^{\tau_2} f(\theta(s))\, ds \right]$$

$$= E^{\mu,x}\left[I_{\theta(\tau_1)=i} E^{i,x(\tau_1)}\left[\int_0^{\sigma_2} f(\theta(s))\, ds \right]\right]. \tag{3.10}$$

Proof. Suppose first that $\sigma_2 = T$. Then, by the Markov property,

$$E^{\mu,x}\left[I_{\theta(\tau_1)=i} \int_{\tau_1}^{\tau_1+T} f(\theta(s))\, ds \right]$$

$$= E^{\mu,x}\left[I_{\theta(\tau_1)=i} \int_0^{T} f(\theta(\tau_1+s))\, ds \right]$$

$$= \int_0^T E^{\mu,x}\left[I_{\theta(\tau_1)=i} f(\theta(\tau_1+s))\, ds \right]$$

$$= \int_0^T E^{\mu,x}\left[I_{\theta(\tau_1)=i} E^{i,x(\tau_1)}\left[f(\theta(s)) \right]\right] ds$$

$$= E^{\mu,x}\left[I_{\theta(\tau_1)=i} E^{i,x(\tau_1)}\left[\int_0^{T} f(\theta(s))\, ds \right]\right].$$

We can next establish (3.10) when σ_2 takes only countably many values and, by approximation, by any σ_2.

4. Reduction to quasi variational inequality. Any probability measure μ on the space Ω' is determined by numbers $p_i \geqslant 0$, $1 \leqslant i \leqslant n$, such that $\sum p_i = 1$, i.e., $p_i = \mu\{i\}$. We shall write $p = (p_1, \ldots, p_n)$ and, for simplicity, identify μ with p.

We shall extend the cost function (1.9) to the case where $E^{i,x}$ is replaced by $E^{\mu,x}$ (or $E^{p,x}$ with $\mu = p$). But first we have to define a function $K(p)$.

Consider the case where the Markov process $\theta(t)$ goes only to the right, i.e.,

$$p_{i,j}(t) = 0 \quad \text{if } j < i,\ t > 0. \tag{4.1}$$

In this case we define

$$K(p) = K_{i_0+1} \quad \text{where } p = (p_1, \ldots, p_{i_0}, p_{i_0+1}, \ldots, p_n)$$

$$\text{and } p_1 = \cdots = p_{i_0} = 0,\ p_{i_0+1} \neq 0. \tag{4.2}$$

In the case where the Markov process can go both to the left and to the right, we assume that $K_1 = K_2 = \cdots = K_{n-1} = K$ and define

$$K(p) = K. \tag{4.3}$$

These definitions make good sense from the motivation of the quality control problem given in §1 (following (1.10)), and they are also needed for mathematical reasons.

We now extend the cost (1.9) to the case where the distribution of $\theta(0)$ is the measure $\mu = p$:

$$J_x^p(\tau) = E^{p,x}\left[K(p) + \int_0^{\tau_1} f(\theta(s))\,ds \right.$$
$$\left. + \sum_{l=1}^{\infty} I_{\theta(\tau_l)\neq n}\left[K(\theta(\tau_l)) + \int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\,ds\right]\right]. \quad (4.4)$$

We also define

$$V(x,p) = \inf_{\tau\in\mathcal{Q}} J_x^p(\tau).$$

Proceeding heuristically, we shall derive a quasi variational inequality for V.

Suppose we start from a point (x,p). We can make one of two choices:

(i) apply a stopping time $\tau_1 = 0$, or

(ii) apply a stopping time $\tau_1 > 0$.

The first choice entails an immediate cost $K(p)$, and, if we proceed optimally thereafter we incur an additional cost, $\sum_{j=1}^{n-1} p_j V(x, e_j)$, where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the jth component. Thus choice (i) gives

$$V(x,p) \leqslant K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j). \quad (4.5)$$

In case (ii), if we proceed optimally subsequent to τ_1, we get

$$V(x,p) \leqslant E^{p,x}\left[\int_0^{\tau_1}\left[\sum_{j=1}^{n} c_i p_i(t)\right] dt\right] + E^{p,x}\left[V(x(\tau_1), p(\tau_1))\right],$$

where the $p_i(t) = p_i^\mu(t)$ are defined in §2, with $\mu = p$. Proceeding as in **[1]**, we deduce the inequality

$$M_{x,p}V(x,p) + \sum_{j=1}^{n} c_j p_j \geqslant 0, \quad (4.6)$$

where $M_{x,p}$ is the infinitesimal generator of the (x,p) process.

Finally, at each point (x,p), equality should hold either in (4.5) or in (4.6), that is,

$$\left[K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) - V(x,p)\right]\left[M_{x,p}V(x,p) + \sum_{j=1}^{n} c_j p_j\right] = 0. \quad (4.7)$$

The system (4.5)–(4.7) is a quasi variational inequality.

The operator $M_{x,p}$ is given by

$$M_{x,p}u(x,p) = \frac{1}{2}\sum_{l=1}^{n}\frac{\partial^2 u}{\partial x_l^2} + \frac{1}{2}\sum_{j,k=1}^{n} p_j p_j\left(\lambda_j - \sum_{l=1}^{n}\lambda_l p_l\right)$$

$$\cdot\left(\lambda_k - \sum_{l=1}^{n}\lambda_l p_l\right)\frac{\partial^2 u}{\partial p_j \partial p_k}$$

$$+ \sum_{j=1}^{n} p_j\left(\lambda_j - \sum_{l=1}^{n}\lambda_l p_l\right)\cdot\frac{\partial}{\partial p_j}\nabla_x u$$

$$+ \sum_{l=1}^{n}\lambda_l p_l\cdot\nabla_x u + \sum_{j,k=1}^{n} q_{j,k}p_j\frac{\partial u}{\partial p_k}. \tag{4.8}$$

This formula, for a function $u = u(p)$, follows from (2.2). For a function $u = u(x,p)$, formula (4.8) follows by noting (from the proof of Theorem 2.1) that

$$\bar{w}(t) = x(t) - \int_0^t \sum_{l=1}^{n}\lambda_l p_l(s)\,ds.$$

Formula (4.8) is actually valid also when the given drifts λ_j are functions of x (since Theorem 2.1 extends to this case; the (x,p)-process is a Markov process in this case, but not the p-process alone).

Later on we shall apply Itô's formula

$$E^x\left[V(X(\tau))\right] - V(X) = E^x\left[\int_0^\tau MV(X(s))\,ds\right], \tag{4.9}$$

where $X(t)$ is the process $(x(t), p(t))$, $M = M_{x,p}$ and τ is an $\mathcal{F}_t$ stopping time with finite expectation. The standard assumptions on V are

$$V \in C^2(A), \qquad MV \text{ is bounded in } A, \tag{4.10}$$

where A is the phase space.

If $X(t)$ is a continuous Markov process with nondegenerate generator M, then (4.9) is actually valid under the weaker assumptions:

$$V \text{ continuous in } A, \quad V \in W^{2,2}_{\mathrm{loc}}(A), \quad MV \in L^\infty(A) \tag{4.11}$$

(see [2]). In our case, however, M is degenerate. We claim:

LEMMA 4.1. *Formula* (4.9) *is valid if* $V \in C^1(A)$, $D^2V \in L^\infty(A)$, $MV \in L^\infty(A)$.

PROOF. Suppose first that $\tau \leqslant m$ where m is a positive integer. Let $M_\varepsilon = M + \varepsilon\Delta$ (Δ = Laplacian, $\varepsilon > 0$) and denote the corresponding solution of the stochastic differential system by $X^\varepsilon(t)$. The coefficients of the Brownian differentials dw_i, in the equations for X^ε, converge to the corresponding coefficients of dw_i in the stochastic differential system for X, uniformly in bounded sets (this follows, for instance, from [5, p. 129]). Employing the martingale inequality we find that, for any $\delta > 0$,

$$P^x\left[\sup_{0 \leqslant t \leqslant m}|X_\varepsilon(t) - X(t)| > \delta\right] \to 0 \quad \text{if } \varepsilon \to 0.$$

Hence, for a subsequence $\varepsilon = \varepsilon_i \downarrow 0$,

$$\sup_{0 < t < m} |X_\varepsilon(t) - X(t)| \to 0 \quad \text{a.e.} \tag{4.12}$$

Since M_ε is nondegenerate and V satisfies (4.11), we have

$$E^x[V(X^\varepsilon(\tau))] - V(X) = E^x\left[\int_0^\tau M_\varepsilon V(X^\varepsilon(s))\,ds\right].$$

Noting that $M_\varepsilon V \to MV$ uniformly as $\varepsilon \to 0$, and using (4.12), assertion (4.9) follows.

So far we have assumed that $\tau \leqslant m$. Now take any τ, apply (4.9) to $\tau \wedge m$, and take $m \uparrow \infty$.

When the λ_j are constants the p-process alone is Markovian and its infinitesimal generator M_p is given by

$$M_p u(p) = \frac{1}{2}\sum_{j,k=1}^n p_j p_k\left(\lambda_j - \sum_{l=1}^n \lambda_l p_l\right)\cdot\left(\lambda_k - \sum_{l=1}^n \lambda_l p_l\right)\frac{\partial^2 u}{\partial p_j \partial p_k}$$
$$+ \sum_{j,k=1}^n q_{j,k} p_j \frac{\partial u}{\partial p_k}. \tag{4.13}$$

Let

$$A_j = \left\{(0,\ldots,0,p_j,p_{j+1},\ldots,p_n) \in R^n;\right.$$
$$\left. p_j > 0, p_l \geqslant 0, \sum_{l=j}^n p_l = 1\right\}. \tag{4.14}$$

If (4.1) holds and if the $p(t)$ process starts at A_j then it remains in A_j for all t.

Observe that

$$\text{if } p_i \neq 0 \text{ then } p_i(t) \neq 0 \text{ for all } t > 0. \tag{4.15}$$

Indeed, this is stated in Theorem 2.1 (and it is also a consequence of Theorem 2.2). It follows that if (4.1) holds, then

$$K(p(t)) = K(p) \text{ a.s. } P^{p,x} \qquad (t > 0). \tag{4.16}$$

Also, if $p \in A_j$ then the process $(x(t), p(t))$ remains in $R^m \times A_j$ for all $t > 0$ (if (4.1) holds). Thus, Lemma 4.1 is valid, in this case, with $A = R^m \times A_j$.

In case $K(p)$ is defined by (4.2) (rather than by (4.3)), the function $K(p)$ is generally discontinuous in $\bar{A}_1$. Thus, we cannot expect $V(x,p)$ to be continuous in $R^m \times \bar{A}_1$. However, since $K(p)$ is continuous when restricted to each A_j, we may expect $V(x,p)$ likewise to be continuous when restricted to each $R^m \times A_j$.

Consider now the case where (4.3) holds and let

$$A = \left\{(x,p);\, x \in R^m, p_j \geqslant 0, \sum_{j=1}^n p_j = 1\right\}.$$

With a solution $V(x,p)$ of the q.v.i. (4.5)–(4.7) we associate a set

$$S = \left\{ (x,p) \in A;\ V(x,p) = K(p) + \sum_{i=1}^{n-1} p_i V(x, e_i) \right\}.$$

When V is a continuous function, S is a closed set.

DEFINITION. We denote by σ_*^p the hitting time of the set S by the (continuous) process $(x(t), p(t))$ with $p(0) = p$ and let $\sigma_*^l = \sigma_*^p$ when $p = e_l$. Set

$$\bar{\tau}_1^p = \sigma_*^p,\ \bar{\tau}_{m+1}^p = \bar{\tau}_m^p + \sum_{l=1}^{n-1} I_{(\bar{\tau}_m^p) = l}\, \sigma_*^l(\phi_{\bar{\tau}_m^p}) \qquad (1 \leqslant m < \infty),$$

$$\bar{\tau}^p = (\bar{\tau}_1^p, \bar{\tau}_2^p, \dots). \tag{4.17}$$

THEOREM 4.2. *Let $V(x,p)$ be a bounded function in $C^1(A)$ with D^2V in $L^\infty(A)$ and with MV in $L^\infty(A)$. Assume that V satisfies the q.v.i.* (4.5)–(4.7) *a.e. and that the σ_*^p are finite valued for all $p \neq e_n$. Then*

$$V(x,p) = \min_{\tau \in \mathcal{Q}} J_x^p(\tau) = J_x^p(\bar{\tau}^p), \tag{4.18}$$

where $\bar{\tau}^p$ is given in (4.17).

PROOF. We first show that

$$J_x^p(\tau) \geqslant V(x,p). \tag{4.19}$$

By (4.6), Lemma 4.1 and (4.5),

$$E^{p,x}\left[\int_0^{\sigma_1} \sum_{i=1}^{n} c_i p_i(t)\, dt\right] \geqslant E^{p,x}\left[\int_0^{\sigma_1} MV(x(t), p(t))\, dt\right]$$

$$= V(x,p) - E^{p,x}\left[V(x(\sigma_1), p(\sigma_1))\right]$$

$$\geqslant V(x,p) - E^{p,x}\left[K(p(\sigma_1)) + \sum_{i=1}^{n-1} p_i(\sigma_1) V(x(\sigma_1), e_i)\right].$$

Using Lemmas 3.1, 3.2 and (4.3), we conclude that

$$V(x,p) \leqslant E^{p,x}\left[\int_0^{\tau_1} f(\theta(t))\, dt + K(p)\right] + E^{p,x}\left[\sum_{i=1}^{n-1} I_{\theta(\tau_1) = i} V(x(\tau_1), e_i)\right]. \tag{4.20}$$

Of course, $K(p)$ is the constant K, but we prefer to write it as $K(p)$ in order to make it clear later on how to extend the proof of Theorem 4.2 to the case where $K(p)$ is defined by (4.2).

Applying the inequality (4.20) with $x = x(\tau_1)$, $p = e_i$, $\tau_1 = \sigma_{2,i}$ and inserting the result into (4.20), we get

$$V(x,p) \leqslant E^{p,x}\left[\int_0^{\tau_1} f(\theta(t))\, dt + K(p) + \sum_{i=1}^{n-1} I_{\theta(\tau_1) = i} E^{i, x(\tau_1)} \cdot \left[\int_0^{\sigma_{2,i}} f(\theta(t))\, dt + K_i + \sum_{j=1}^{n-1} I_{\theta(\sigma_{2,i}) = j} V(x(\sigma_{2,i}), e_j)\right]\right].$$

Applying Lemmas 3.5 and 3.6, we obtain

$$V(x,p) \leqslant E^{p,x}\left[\int_0^{\tau_1} f(\theta(t))\,dt + K(p) + \sum_{i=1}^{n-1} I_{\theta(\tau_1)=i}\right.$$

$$\left.\cdot\left(\int_{\tau_1}^{\tau_2} f(\theta(t))\,dt + K_i + \sum_{j=1}^{n-1} I_{\theta(\tau_2)=j} V\big(x(\tau_2), e_j\big)\right)\right],$$

which can be written in the form

$$\begin{aligned} V(x,p) \leqslant{} & E^{p,x}\left[\int_0^{\tau_1} f(\theta(t))\,dt + K(p) + \sum_{i=1}^{n-1} I_{\theta(\tau_1)=i}\left(\int_{\tau_1}^{\tau_2} f(\theta(t))\,dt + K_i\right)\right] \\ & + \sum_{j=1}^{n-1} E^{p,x}\big[I_{\theta(\tau_2)=j} V\big(x(\tau_2), e_j\big)\big] \\ ={} & I + J. \end{aligned} \tag{4.21}$$

This inequality is analogous to (4.20).

We proceed to evaluate J by using (4.20) with $x = x(\tau_2)$, $p = e_j$ and $\tau_1 = \sigma_{3,j}$. We then get an inequality analogous to (4.21), but the new term I contains more of the terms which appear in $J_x^p(\tau)$. Proceeding step by step, we arrive at the inequality

$$\begin{aligned} V(x,p) \leqslant E^{p,x}\Bigg[& K(p) + \int_0^{\tau_1} f(\theta(t))\,dt \\ & + \sum_{l=1}^{m}\sum_{j=1}^{n-1} I_{\theta(\tau_l)=j}\left(K_j + \int_{\tau_l}^{\tau_{l+1}} f(\theta(t))\,dt\right)\Bigg] + R_m, \end{aligned} \tag{4.22}$$

where

$$R_m = E^{p,x}\left[\sum_{j=1}^{n-1} I_{\theta(\tau_{m+1})=j} V\big(x(\tau_{m+1}), e_j\big)\right]. \tag{4.23}$$

Now, we have to prove (4.19) only for such τ for which $J_x^p(\tau) < \infty$. Since $J_x^p(\tau)$ consists of a finite number of infinite series (with positive terms), the remainder must converge to zero. This implies that

$$E^{p,x}\left[\sum_{p=m}^{\infty}\sum_{j=1}^{n-1} I_{\theta(\tau_p)=j}\right] \to 0 \quad \text{if } m \to \infty. \tag{4.24}$$

Since V is a bounded function, we conclude from (4.23) that $R_m \to 0$. Taking $m \to \infty$ in (4.22) and using the last remark, assertion (4.19) follows.

We next have to prove that

$$J_x^p(\bar{\tau}^p) = V(x,p). \tag{4.25}$$

Taking $\sigma_1 = \sigma_*^p$ in (4.20) we obtain an equality. Similarly, we get the equality in (4.21) (here we take $\sigma_{2,l} = \sigma_*^l$). More generally we get the equality in (4.22) (when $\tau_l = \bar{\tau}_l^p$). Since $R_m \geqslant 0$, it follows that the partial sums of the series (4.4) (when $\tau = \bar{\tau}^p$) are bounded above by the quantity $V(x, p)$. Since the terms are all nonnegative, the infinite series is convergent. Consequently, $R_m \to 0$, and relation (4.25) follows upon taking $m \to \infty$.

REMARK 1. Theorem 4.2 means that the optimal stopping rule is to stop as soon as $(x(t), p(t))$ hits the set S, given that in the previous inspection θ was at state $p(0)$ and x at $x(0)$. In order to render this theorem useful, one should express the optimal stopping rule in terms of the observable process $x(t)$. This can be done by means of Theorem 2.2. We can now state:

The optimal stopping rule is to stop as soon as $(x(t), p(t))$, *with* $p(t)$ *defined by* (2.9), *hits the set* S, *given that at the preceding inspection the distribution of* θ *is* $\mu = (p_1, \ldots, p_n)$ *and* $x(0) = x$. Notice that, after the first inspection, θ is at a particular state, say i, and then $p_j(t)$ is given simply by $p_j^i(t)$.

REMARK 2. If (4.1) holds and $K(p)$ is defined by (4.2), then the proof of Theorem 4.2 remains valid provided we make use of (4.16), instead of (4.3). Further, if initially p is in A_j (i.e., $p = (0, \ldots, 0, p_j, p_{j+1}, \ldots, p_n)$ and $p_j > 0$), then we have to impose the assumptions of Theorem 4.2 with $A = R^m \times A_j$. The stopping set S is defined accordingly.

REMARK 3. In order that there exists at least one τ in $\mathcal{Q}$ with $J_x^p(\tau) < \infty$, it suffices to assume that the state n is absorbing, i.e.,

$$P[\theta(t) < n] \to 0 \quad \text{if } t \to \infty. \tag{4.26}$$

(We then have, by the Markov property, $P[\theta(t) < n] \leqslant e^{-\alpha t}$ for some $\alpha > 0$, so that $J_x^p(\tau) < \infty$ if $\tau = (\tau_1, \tau_2, \ldots)$ with $\tau_j = j$.) Condition (4.26) is satisfied in case (4.1), and also in the case where

$$q_{n,n} = 0, \qquad q_{i,n} > 0 \quad \text{for } i = 1, \ldots, n-1. \tag{4.27}$$

In fact one can prove, in both cases, the stronger result

$$P^{i,x}[\theta(\tau_m) = n] \to 1 \quad \text{if } \tau_m \to \infty \text{ a.s as } m \to \infty, \tag{4.28}$$

where $\{\tau_m\}$ is any sequence of inspections. Let us prove (4.28) in case (4.27) holds (the proof in case (4.1) holds is a consequence of (4.26)).

Consider the function

$$f(t) = P^i[\theta(s) \neq n \text{ for all } 0 \leqslant s \leqslant t].$$

It satisfies $f'(t) = -q_{i,n} f(t)$, so that

$$f(t) = e^{-q_{i,n} t} \to 0 \quad \text{if } t \to \infty. \tag{4.29}$$

For any inspection time τ_m we then have

$$\begin{aligned} P^{i,x}[\theta(t) = n] &= P^{i,x}[\theta(t) = n, \tau_m > t] + P^{i,x}[\theta(t), \tau_m \leqslant t] \\ &\leqslant P^{i,x}[\theta(\tau_m) = n] + P^{i,x}[\tau_m \leqslant t], \end{aligned} \tag{4.30}$$

since $P^{n,x}[\theta(t) = n] = 1$, by (4.27). Taking $m \to \infty$ in (4.30) and using (4.29), assertion (4.28) follows by letting $t \to \infty$.

REMARK 4. Theorem 4.2 and the previous remarks hold (with the same proofs) also when the drifts λ_j are functions of x. However, when the λ_j are constants, $V(x, p)$ is actually independent of x (see §5), and the q.v.i. for $V = V(p)$ becomes

$$V(p) \leqslant K(p) + \sum_{j=1}^{n-1} p_j V(e_j), \tag{4.31}$$

$$M_p V(p) + \sum_{j=1}^{n} c_j p_j \geqslant 0, \tag{4.32}$$

where M_p is defined in (4.9), and

$$\left[K(p) + \sum_{j=1}^{n-1} p_j V(e_j) - V(p)\right]\left[M_p V(p) + \sum_{j=1}^{n} c_j p_j\right] = 0. \tag{4.33}$$

In case (4.1) holds (and the λ_j are constants), set

$$V_i(p_i, p_{i+1}, \ldots, p_n) = V(0, \ldots, 0, p_i, p_{i+1}, \ldots, p_n). \tag{4.34}$$

Then (4.31)–(4.33) reduce to a successive sequence of simpler q.v.i. for $V_{n-1}(p_{n-1}, p_n)$, $V_{n-2}(p_{n-2}, p_{n-1}, p_n), \ldots, V_1(p_1, p_2, \ldots, p_n)$:

$$V_{n-i}(p_{n-i}, \ldots, p_n) \leqslant K_{n-i} + \sum_{j=n-i}^{n-1} p_j V_j(e_j), \tag{4.35}$$

$$M_{n-i} V_{n-i}(p_{n-i}, \ldots, p_n) + \sum_{j=n-i}^{n} c_j p_j \geqslant 0, \tag{4.36}$$

$$\left[K_{n-i} + \sum_{j=n-i}^{n-1} p_j V_j(e_j) - V_{n-i}(p_{n-i}, \ldots, p_n)\right] \cdot \left[M_{n-i} V_{n-i}(p_{n-i}, \ldots, p_n) - \sum_{j=n-i}^{n} c_j p_j\right] = 0, \tag{4.37}$$

where

$$M_{n-i} u(p_{n-i}, \ldots, p_n) = \frac{1}{2} \sum_{j,k=n-i}^{n} p_j p_k \left(\lambda_j - \sum_{l=n-i}^{n} \lambda_l p_l\right) \cdot \left(\lambda_k - \sum_{l=n-i}^{n} \lambda_l p_l\right) \frac{\partial^2 u}{\partial p_j \partial p_k} + \sum_{j,k=n-i}^{n} q_{j,k} p_j \frac{\partial u}{\partial p_k}. \tag{4.38}$$

REMARK 5. The process $(x(t), y_j(t);\ 2 \leqslant j \leqslant n)$ where $y_j(t) = p_j(t)/p_1(t)$ $(2 \leqslant j \leqslant n)$ is a Markov process and its generator $A_{x,y}$ is given by

$$A_{x,y}v(x,y) = \frac{1}{2}\sum_{l=1}^{n}\frac{\partial^2 v}{\partial x_l^2} + \frac{1}{2}\sum_{j,k=2}^{n} y_j y_k(\lambda_j - \lambda_1)\cdot(\lambda_k - \lambda_1)\frac{\partial^2 v}{\partial y_j \partial y_k}$$

$$+ \sum_{j=2}^{n} y_j(\lambda_j - \lambda_1)\cdot\frac{\partial}{\partial y_j}\nabla_x v$$

$$+ \sum_{j=2}^{n}\left[(\lambda_1 - \lambda_j)\cdot\lambda_1 y_j + \sum_{k=1}^{n}(q_{k,j} - q_{k,1}y_j)y_k\right.$$

$$\left. + (\lambda_j - \lambda_1)y_j\cdot\frac{\lambda_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n}{1 + y_2 + \cdots + y_n}\right]\frac{\partial v}{\partial y_j}\cdot \quad (4.39)$$

REMARK 6. The elliptic operator $M_{x,p}$ is degenerate throughout the (x, p) region A. Let

$$B = \left\{(p_1, \ldots, p_n),\ \ p_i > 0,\ \ \sum_{i=1}^{n} p_i = 1\right\}.$$

In case the λ_j are constants, the elliptic operator M_p is nondegenerate in B if and only if the vectors $\lambda_j - \lambda_1$ $(2 \leqslant j \leqslant n)$ are linearly independent; this can easily be seen from (4.39). Thus if $m \geqslant n - 1$ and the λ_j are in "general position", then M_p is nondegenerate in B. It does degenerate, however, on the boundary of B.

REMARK 7. When M_p is nondegenerate in B, one can slightly improve the assertion of Theorem 4.2 by requiring weaker regularity conditions on $V(p)$, namely:

$$V \in C(\bar{B}),\quad V \in W^{2,2}(B),\quad M_p V \in L^2(B). \tag{4.40}$$

Indeed, these conditions already insure that Itô's formula can be applied for bounded stopping times (see [2]). By approximation one can then justify the use of Itô's formula for any stopping time. A similar improvement is possible in the case (4.1), (4.2).

REMARK 8. If for the solution $V(x, p)$ as in Theorem 4.2 the closure of the set $A \setminus S$ does not intersect the set $p_1 = 0$, and if $p_{i,1} = 0$ when $i > 1$, then the σ_*^p are finite valued a.s. Indeed, since

$$M_{x,p}p_1 = \sum_{i=1}^{n} q_{i,1}p_i = q_{1,1}p_1, \qquad q_{1,1} > 0,$$

Itô's formula applied to p_1, integrated from 0 to σ_*^p, gives $E^{p,x}\sigma_*^p < \infty$.

5. Properties of the optimal cost.

5.1. *Other cost functions.* Suppose we modify the cost function $J_x^i(\tau)$ by including a discount factor α, $\alpha > 0$, i.e.

$$J_x^i(\tau) = E^{i,x}\left[K_i e^{-\alpha\tau_1} + \sum_{j=1}^{n-1} K_j \sum_{l=1}^{\infty} e^{-\alpha\tau_{l+1}} I_{\theta(\tau_l)=j}\right]$$
$$+ E^{i,x}\left[\int_0^{\tau_1} e^{-\alpha s} f(\theta(s))\,ds + \sum_{j=1}^{n-1}\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\int_{\tau_l}^{\tau_{l+1}} e^{-\alpha s} f(\theta(s))\,ds\right]. \quad (5.1)$$

Then the corresponding q.v.i. are the same as before, except that Mu is replaced by $Mu - \alpha u$. The proof is essentially the same as in the case $\alpha = 0$.

Another generalization of the cost functions is when K_i and $f(i) = c_i$ are replaced by functions $K_i(x)$, $g_i(x)c_i$ (so that $f(\theta(s))$ is replaced by $g_{\theta(s)}(x(s))f(\theta(s))$). In this case the q.v.i. remains unchanged except that K_i is replaced by $K_i(x)$ and c_i by $g_i(x)c_i$.

Consider next a quality control problem with option of repairs, that is, we allow repairs immediately after inspection. Thus, if $\theta(\tau_i)$ is at state l ($l = 2, \ldots, n$) one is allowed to repair the machine by restarting at state 1, with cost N_l. (We assume, for simplicity, that no "partial" repairs are allowed.) We shall write down the new J_x^1 just in the case corresponding to (1.10) with $n = 3$:

$$J_x^1(\tau, \alpha) = E^{1,x}\left[K_1 + \int_0^{\tau_1} f(\theta(s))\,ds\right.$$
$$+ I_{\theta(\tau_1)=1}\left[K_1 + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds\right.$$
$$\left.\left. + I_{\theta(\tau_2)=1}\left[K_1 + \int_{\tau_2}^{\tau_3} f(\theta(s))\,ds + \ldots\right]\right]\right.$$
$$+ I_{\theta(\tau_2)=2}\left[\alpha_2\left(K_2 + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds\right.\right.$$
$$\left. + I_{\theta(\tau_3)=2}[\ \cdots\ ] + I_{\theta(\tau_3)=2}[\ \cdots\ ]\right) + (1 - \alpha_2)$$
$$\left.\cdot\left(N_2 + E^{1,x(\tau_2)}\left[K_1 + \int_0^{\hat\tau_3} f(\theta(s))\,ds + \ldots\right]\right)\right]$$
$$+ I_{\theta(\tau_1)=2}\left[\alpha_3\left(K_2 + \int_{\tau_1}^{\tau_2} f(\theta(s))\,ds + \ldots\right) + (1 - \alpha_3)\right.$$
$$\left.\cdot\left(N_2 + E^{1,x(\tau_1)}\left[K_1 + \int_0^{\hat\tau_2} f(\theta(s))\,ds + \ldots\right]\right)\right]$$
$$\left. + I_{\theta(\tau_1)=3}\left[N_3 + E^{1,x(\tau_1)}\left[K_1 + \int_0^{\hat\tau_2} f(\theta(s))\,ds + \ldots\right]\right]\right], \quad (5.2)$$

where $0 \leqslant \alpha_i \leqslant 1$ and $\hat{\tau}_2 = \sigma_2$, $\hat{\tau}_3 = \sigma_3$. Similarly, one defines $J_x^2(\tau, \alpha)$. We are assuming here that if the machine is at state 3 then we must repair it. The differential inequality (4.6) for $V(x, p) = \inf_{\tau,\alpha} J_x^p(\tau, \alpha)$ remains the same, but (4.5) is replaced by

$$V(x, p_1, p_2) \leqslant K(p_1, p_2) + p_1 \min_{0 \leqslant \mu \leqslant 1} \left[\mu V(x, e_1) + (1 - \mu)(N_2 + V(x, e_2)) \right]$$
$$+ p_2 \left[N_3 + V(x, e_2) \right]. \tag{5.3}$$

5.2. *$J_x^p(\tau)$ in terms of the (p, x)-process.* We shall need the following formula:

$$J_x^p(\tau) = E^{p,x}\left[K(p) + \int_0^{\tau_1} \sum_{j=1}^n c_j p_j(s)\, ds \right.$$
$$\left. + \sum_{l=1}^{\infty} \sum_{i=1}^{n-1} p_i(\tau_l) \left[E^{i,x(\tau_l)} \left[K_i + \int_0^{\sigma_{l,i}} \sum_{j=1}^n c_j p_j(s)\, ds \right] \right] \right]. \tag{5.4}$$

PROOF. By (4.4),

$$J_x^p(\tau) = E^{p,x}\left[K(p) + \int_0^{\tau_1} f(\theta(s))\, ds \right.$$
$$\left. + \sum_{l=1}^{\infty} \sum_{i=1}^{n-1} I_{\theta(\tau_l) = i} \left[K_i + \int_{\tau_l}^{\tau_{l+1}} f(\theta(s))\, ds \right] \right]$$

(by Lemma 3.6)

$$= E^{p,x}\left[K(p) + \int_0^{\tau_1} f(\theta(s))\, ds \right.$$
$$\left. + \sum_{l=1}^{\infty} \sum_{i=1}^{n-1} I_{\theta(\tau_l) = i} E^{i,x(\tau_l)} \left[K_i + \int_0^{\sigma_{l,i}} f(\theta(s))\, ds \right] \right].$$

Applying Lemmas 3.1 and 3.2 to the right-hand side, assertion (5.4) follows.

Similarly to (5.4) we can write the cost function associated with (5.1) in the form

$$J_x^p(\tau) = E^{p,x}\left[K(p) e^{-\alpha\tau_1} + \int_0^{\tau_1} e^{-\alpha s} \sum_{j=1}^n c_j p_j(s)\, ds \right.$$
$$+ \sum_{l=1}^{\infty} \sum_{i=1}^{n-1} e^{-\alpha\tau_l} p_i(\tau_l) E^{i,x(\tau_l)}$$
$$\left. \cdot \left[K_i e^{-\alpha\sigma_{l,i}} + \int_0^{\sigma_{l,i}} e^{-\alpha s} \sum_{j=1}^n c_j p_j(s)\, ds \right] \right]. \tag{5.5}$$

Let

$$V_\alpha(x,p) = \inf_{\tau\in\mathfrak{A}} J_x^p(\tau), \qquad J_x^p(\tau) \text{ defined by (5.5).} \tag{5.6}$$

5.3. $V = V(p)$. We shall prove

THEOREM 5.1. *If the λ_j do not depend on x then $V(x,p)$ and $V_\alpha(x,p)$ do not depend on x.*

PROOF. It is enough to prove the assertion for V.

Define γ_x: $\Omega \to \Omega$ by $\gamma_x(\omega)(t) = x + \omega(t)$. Then γ_x: $\mathcal{F}_t \to \mathcal{F}_t$ is a σ-isomorphism and

$$P^{p,x+y}(\Lambda) = P^{p,y}(\gamma_x(\Lambda)) \quad \text{for any } \Lambda \in \mathcal{F}. \tag{5.7}$$

Indeed, it suffices to verify (5.7) for cylinder sets, and the verifications in this case follows from (1.3).

From (5.7) we conclude that

$$E^{p,x+y}[F(\cdot)] = E^{p,y}[F(\gamma_x\cdot)] \tag{5.8}$$

for any integrable and $\mathcal{F}$-measurable function F.

We note, by Theorem 2.2 and (2.9), that $p_j(t)$ depends only on the differences $x(v) - x(u)$ $(0 \leqslant u \leqslant v \leqslant t)$, so that, writing $p_j(t) = p_j(t,\omega)$, we have

$$p_j(t,\gamma_x\omega) = p_j(t,\omega). \tag{5.9}$$

From (5.4), (5.8) (with $y = 0$) and (5.9) we see that

$$\begin{aligned} J_x^p(\tau) = E^{p,0}\Bigg[K(p) &+ \int_0^{\tau_1\gamma_x} \sum_{j=1}^n c_j p_j(s)(\gamma_x)\,ds \\ &+ \sum_{l=1}^\infty \sum_{i=1}^{n-1} p_i(\tau_l\gamma_x)(\gamma_x) E^{i,x(\tau_l\gamma_x)(\gamma_x)} \\ &\qquad\qquad \cdot\left[K_i + \int_0^{\sigma_{l,i}} \sum_{j=1}^n c_j p_j(s)\,ds\right]\Bigg] \\ = E^{p,0}\Bigg[K(p) &+ \int_0^{\tilde{\tau}_1} \sum_{j=1}^n c_j p_j(s)\,ds \\ &+ \sum_{l=1}^\infty \sum_{i=1}^{n-1} p_i(\tilde{\tau}_l) E^{i,x(\tilde{\tau}_l)+x}\left[K_i + \int_0^{\sigma_{l,i}} \sum_{j=1}^n c_j p_j(s)\,ds\right]\Bigg], \end{aligned}$$

where

$$\tilde{\tau}_l(\omega) = \tau_l(\gamma_x\omega);$$

here we have also used the relation

$$x(\tau_l\gamma_x)(\gamma_x\omega) = \omega(\tau_l(\gamma_x\omega)) + x.$$

Letting $\tilde{\sigma}_{l,i} = \sigma_{l,i}\gamma_x$, and using (5.8) (with $y = x(\tilde{\tau}_l)$), we find that

$$J_x^p(\tau) = E^{p,0}\Bigg[K(p) + \int_0^{\tilde{\tau}_1} \sum_{j=1}^n c_j p_j(s)\,ds$$
$$+ \sum_{l=1}^{\infty} \sum_{i=1}^{n-1} p_i(\tilde{\tau}_l) E^{i,x(\tilde{\tau}_l)}\Bigg[K_i + \int_0^{\tilde{\sigma}_{l,i}} \sum_{j=1}^n c_j p_j(s)\,ds\Bigg]\Bigg],$$

which, by (5.4), is equal to $J^p(\tilde{\tau})$, where $\tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2, \dots)$. It is easy to check that $\tilde{\tau}$ is a sequence of inspection times, i.e., $\tilde{\tau} \in \mathcal{Q}$. Hence

$$J_x^p(\tau) = J_0^p(\tilde{\tau}) \geqslant \inf_{\hat{\tau} \in \mathcal{Q}} J_0^p(\hat{\tau}) = V(0,p).$$

Since τ is arbitrary, we get $V(x,p) \geqslant V(0,p)$. Similarly, one can show that $V(0,p) \geqslant V(x,p)$, and the proof of the theorem is complete.

Set $E^p = E^{p,0}$, $E^i = E^{i,0}$.

THEOREM 5.2. *If the λ_j do not depend on x then*

$$V_\alpha(0,p) = \inf_\theta E^p\Bigg[K(p)e^{-\alpha\theta} + \int_0^\theta e^{-\alpha s} \sum_{j=1}^n c_j p_j(s)\,ds$$
$$+ \sum_{j=1}^{n-1} e^{-\alpha\theta} p_i(\theta) V_\alpha(0, e_i)\Bigg], \qquad (5.10)$$

where θ varies over all $\mathcal{F}_t$ stopping times.

PROOF. From (5.5),

$$V_\alpha(0,p) = \inf_{\tau_1} E^{p,0}\Bigg[K(p)e^{-\alpha\tau_1} + \int_0^{\tau_1} e^{-\alpha s} \sum_{j=1}^n c_j p_j(s)\,ds$$
$$+ \sum_{j=1}^{n-1} e^{-\alpha\tau_1} p_i(\tau_1) \inf_{\tau_i'} J^i_{x(\tau_1)}(\tau_i')\Bigg], \quad (5.11)$$

where τ_i' is the sequence of inspection times beginning with $\sigma_{i,2}$. It is easy to see that, as τ varies in $\mathcal{Q}$, $(\tau_1', \dots, \tau_{n-1}')$ varies over the entire product space $\mathcal{Q} \times \cdots \times \mathcal{Q}$. By Theorem 5.1, we may replace $\inf_{\tau_i'} J^i_{x(\tau_1)}(\tau_i')$ by $\inf_{\tau_i'} J_0^i(\tau_i')$ in (5.11), and (5.10) then follows.

Formula (5.10) shows that $V_\alpha(0,p)$ has the form

$$V_\alpha(0,p) = \inf_\theta E^p\Bigg[\int_0^\theta e^{-\alpha s} f(p(s))\,ds + e^{-\alpha\theta}\psi(p(\theta))\Bigg], \qquad (5.12)$$

where θ is any $\mathcal{F}_t$ stopping time. Recall that $p(t)$ satisfies a system of stochastic differential equations with a Brownian motion that is nonanticipative with respect to $\mathcal{F}_t$. The functions f, ψ in (5.12) are continuous and bounded.

Let

$$A_0 = \left\{ p = (p_1, \ldots, p_n); p_i \geqslant 0, \sum_{i=1}^{n} p_i = 1 \right\}. \tag{5.13}$$

THEOREM 5.3. *If* $K(p) \equiv \text{const}$ *and* $\alpha > 0$ *then* $V_\alpha(0, p)$ *is continuous in* p, $p \in A_0$.

This follows from (5.12) by a standard argument, using the fact that the $p_j(t)$, as solutions of a stochastic differential system, are continuous functions in bounded times (in suitable norm) of the initial values.

REMARK. If the λ_j depend on x then one can prove the continuity of $V_\alpha(x, p)$ in (x, p).

We set

$$V_\alpha(p) = V_\alpha(0, p). \tag{5.14}$$

By iterating (5.11) we can express $V_\alpha(p)$ as the infimum of a cost function expressed only in terms of the p-process and the τ_j, but this cost function is rather complicated and will not be needed later on.

If we restrict the stopping times θ in (5.10) to be stopping times which are invariant with respect to translations of $x(t)$, then by iterating (5.10) any number of times l and letting $l \to \infty$ we obtain

$$V_\alpha(p) \leqslant \tilde{V}_\alpha(p) \equiv \inf_{\tau \in \mathcal{Q}_0} J^p(\tau), \tag{5.15}$$

where

$$J^p(\tau) = E^p\Bigg[K(p) e^{-\alpha\tau_1} + \int_0^{\tau_1} e^{-\alpha s} \sum_{j=1}^{n} c_j p_j(s)\, ds + \sum_{l=1}^{\infty} \sum_{j=1}^{n-1} e^{-\alpha\tau_l} p_i(\tau_l) \cdot E^i\Bigg[K_i e^{-\alpha\sigma_{l,i}} + \int_0^{\sigma_{l,i}} e^{-\alpha s} \sum_{j=1}^{n} c_j p_j(s)\, ds \Bigg]\Bigg], \tag{5.16}$$

and $\mathcal{Q}_0$ is the subset of $\mathcal{Q}$ consisting of all inspections with $\sigma_{m,l}$ which are invariant with respect to translations of $x(t)$.

6. Existence and uniqueness of solution of the q.v.i. In this section we assume that $\alpha > 0$. We also assume that the λ_j do not depend on x and

$$\text{the vectors } \lambda_j - \lambda_1 \ (2 \leqslant j \leqslant n) \text{ span the entire space } R^m. \tag{6.1}$$

We shall denote by $-A$ the infinitesimal generator of the Markov process $(p_1(t), \ldots, p_{n-1}(t))$. For simplicity of notation we identify $p = (p_1, \ldots, p_{n-1}, p_n)$ $(\sum_{i=1}^{n} p_i = 1)$ with $(p_1, \ldots, p_{n-1})$. Let

$$\Omega = \left\{ (p_1, \dots, p_{n-1}); p_i > 0 \; (1 \leqslant i \leqslant n-1), \; \sum_{i=1}^{n-1} p_i < 1 \right\}. \tag{6.2}$$

THEOREM 6.1. *Assume that* (6.1) *holds and that* $K(p) \equiv K$ *(constant). Then:*

(i) *There exists a unique solution* $u = u(p)$ *in* Ω *of the q.v.i.* (4.5)–(4.7) *with* M *replaced by* $M - \alpha$, *such that* $u \in C(\bar{\Omega})$ *and* $u \in W^{2,p}_{\text{loc}}(\Omega)$ *for any* $1 < p < \infty$.

(ii) *This solution coincides with* $V_\alpha(p)$, *i.e.,* $u(p) = V_\alpha(p)$.

(iii) *Let* $S = \{p \in A_0;\ V_\alpha(p) = K + \sum_{j=1}^{n-1} p_j V(e_j)\}$, *and let* $\sigma^p_* =$ *hitting time of the set* S *by the process* $p(t)$ *with* $p(0) = p$. *Assume that* $\sigma^p_* < \infty$ *a.s. for all* $p \in A_0, p \neq e_n$. *Define* $\bar{\tau}^p$ *by* (4.17). *Then*

$$V_\alpha(p) = J^p_0(\bar{\tau}^p) \qquad (p \in A_0, p \neq e_n). \tag{6.3}$$

If the $\theta(t)$ process goes only to the right then the proof of Theorem 6.1 immediately extends to the case where $K(p)$ is not a constant, but is defined as in (4.2). In this case the sequence corresponding to q.v.i. (4.35)–(4.37), with M_{n-i} replaced by $M_{n-i} - \alpha$ $(\alpha > 0)$, has a unique regular solution $(V_{n-1}, \dots, V_1)$.

PROOF. For any $\delta > 0$, let

$$\Omega_\delta = \left\{ (p_1, \dots, p_{n-1}); p_j > \delta, \; \sum_{j=1}^{n-1} p_j < 1 - \delta \right\}.$$

Given continuous $f \geqslant 0$ and $\psi \geqslant 0$ in $\bar{\Omega}$, consider the Dirichlet problem

$$Au + \alpha u + \varepsilon^{-1}(u - \psi)^+ = f \quad \text{in } \Omega_\delta \qquad (\varepsilon > 0), \tag{6.4}$$

$$u = 0 \quad \text{on } \partial\Omega_\delta. \tag{6.5}$$

Since A does not degenerate in $\bar{\Omega}_\delta$, this problem has a solution $u = u_{\varepsilon,\delta}$. The solution is unique and is given probabilistically (see [3]) by

$$u_{\varepsilon,\delta}(p) = \inf_{v \in \mathfrak{B}} E^p\left[\int_0^{T_\delta} \left(f(p(t)) + \frac{v(t)}{\varepsilon} \psi(t) \right) \cdot \exp\left\{ - \int_0^t \frac{v(x)}{\varepsilon} \, ds \right\} e^{-\alpha t} \, dt \right], \tag{6.6}$$

where $\mathfrak{B}$ is the class of all $\mathfrak{F}_t$ nonanticipating functions $v(t)$ with $0 \leqslant v(t) \leqslant 1$, and T_δ is the exit time of $p(t)$ from Ω_δ. In fact, the proof of (6.6) is easily obtained after writing down Itô's formula for $u_{\varepsilon,\delta} \exp[-\alpha t - \int_0^t (v/\varepsilon)]$, integrated from $t = 0$ to $t = T_\delta$.

Recall that $p(t) \in \Omega$ for all $t > 0$ if $p(0) \in \Omega$ (since $p_i(0) > 0$ implies $p_i(t) > 0$ for all $t > 0$). Hence, if $p(0) \in \Omega$ then $T_\delta \to \infty$ as $\delta \to 0$. It follows that the cost $E^p[\cdots]$ on the right-hand side of (6.6) converges, uniformly in $v \in \mathfrak{B}$, to the same cost with T_δ replaced by ∞. Thus,

$$\lim_{\delta \to 0} u_{\varepsilon,\delta}(p) = u_\varepsilon(p) \equiv \inf_{v \in \mathfrak{B}} J^{p,\varepsilon}(v) \qquad (p \in \Omega), \tag{6.7}$$

where

$$J^{p,\varepsilon}(v) = E^p\left[\int_0^\infty \left(f + \frac{v}{\varepsilon}\psi\right)\exp\left\{-\int_0^t \frac{v}{\varepsilon}\right\}e^{-\alpha t}\,dt\right]. \tag{6.8}$$

The function $u_\varepsilon(p)$ can be extended continuously into $\bar{\Omega}$ by the right-hand side of (6.7). (This follows using the continuity, in bounded times, of the solution of the stochastic differential system (2.2).)

Notice that no boundary conditions are prescribed for u_ε, and the convergence in (6.7) is generally not uniform in Ω (although it is uniform in compact subsets of Ω).

Now let $\varepsilon \to 0$. Setting, for any $p \in \bar{\Omega}$,

$$u(p) = \inf_\theta \tilde{J}^p(\theta), \tilde{J}^p(\theta) = E^p\left[\int_0^\theta f(p(s))e^{-\alpha s}\,ds + \psi(p(\theta))e^{-\alpha\theta}\right], \tag{6.9}$$

where θ is any $\mathcal{F}_t$ stopping time, we claim that

$$\lim_{\varepsilon\to 0} u_\varepsilon(p) = u(p) \quad \text{uniformly in } \bar{\Omega}. \tag{6.10}$$

By the standard elliptic theory, the function u_ε satisfies the elliptic equation in (6.4) in the interior of Ω. Let θ_ε = exit time from the set $\{u_\varepsilon < \psi\}$ (θ_ε may be infinite valued). Applying Itô's formula to $u_\varepsilon(p(t))$ between $t = 0$ and $t = \theta_\varepsilon \wedge T_\delta \wedge T$ $(T < \infty)$ we get

$$u_\varepsilon(p) \geqslant \tilde{J}^p(\theta_\varepsilon \wedge T_\delta \wedge T) - CE^p\left[e^{-\alpha(T_\delta\wedge T)}\right],$$

where $C = \sup \psi$. Since $\tilde{J}^p(\theta_\varepsilon \wedge T_\delta \wedge T) \geqslant u(p)$, taking $\delta \to 0$, $T \to \infty$ we conclude that $u_\varepsilon(p) \geqslant u(p)$.

Next, for any stopping time θ, take $v = v_\theta = 0$ if $s < \theta$, $v = v_\theta = 1$ if $s > \theta$. For any $h > 0$ we can then estimate, analogously to [3, Chapter III],

$$\left|J^{p,\varepsilon}(v_\theta) - \tilde{J}^p(\theta)\right| \leqslant \eta(\varepsilon, h),$$

where

$$\overline{\lim_{\varepsilon\to 0}}\ \eta(\varepsilon, h) = \eta_0(h), \quad \lim_{h\to 0} \eta_0(h) = 0.$$

Since $u_\varepsilon(p) \leqslant \tilde{J}^{p,\varepsilon}(v_\theta)$, we deduce that $\overline{\lim}(u_\varepsilon - u) \leqslant 0$. Together with $u_\varepsilon \geqslant u$, assertion (6.10) follows.

LEMMA 6.2. *If the first two derivatives of ψ are bounded functions then there exists a constant C independent of δ, ε such that, setting*

$$Au_{\varepsilon,\delta} + \alpha u_{\varepsilon,\delta} \equiv f_{\varepsilon,\delta} \quad \text{in } \Omega_\delta, \tag{6.11}$$

we have

$$|f_{\varepsilon,\delta}| \leqslant C, \qquad |u_{\varepsilon,\delta}| \leqslant C \quad \text{in } \Omega_\delta. \tag{6.12}$$

PROOF. Let $w = (u_{\varepsilon,\delta} - \psi)^+/\varepsilon$. Then w takes its positive maximum in $\bar{\Omega}_\delta$ at some interior point $\bar{p}$. At that point also $u_{\varepsilon,\delta} - \psi$ takes its positive maximum.

Consequently,

$$A(u_{\varepsilon,\delta} - \psi) + \alpha(u_{\varepsilon,\delta} - \psi) \geqslant 0 \quad \text{at } \bar{p}.$$

From (6.4) we then obtain the inequality

$$w(\bar{p}) \leqslant f(\bar{p}) - A\psi(\bar{p}) - \alpha\psi(\bar{p}) \leqslant C_0,$$

where C_0 is a constant independent of δ, ε. This implies the first inequality in (6.12). The second inequality follows by applying the maximum principle to $u_{\varepsilon,\delta}$; at the maximum point $\tilde{p}$ we have $Au_{\varepsilon,\delta} \geqslant 0$, $(u_{\varepsilon,\delta} - \psi)^+ \geqslant 0$, so that $u(\tilde{p}) \leqslant f(\tilde{p})/\alpha$.

If (6.11), (6.12) hold then we can use the elliptic L^p estimates in any compact subset of Ω, and conclude, after taking $\delta \to 0$ and $\varepsilon \to 0$ and using (6.7), (6.10), that

$$u \in W^{2,p}_{\text{loc}}(\Omega). \tag{6.13}$$

Next, (6.4) implies that $Au_{\varepsilon,\delta} + \alpha u_{\varepsilon,\delta} \leqslant f$ so that, as $\delta \to 0$, $\varepsilon \to 0$,

$$Au + \alpha u \leqslant f \quad \text{a.e. in } \Omega. \tag{6.14}$$

On any compact subset of the open set $\{p \in \Omega,\ u(p) < \psi(p)\}$ we have $u_{\varepsilon,\delta} < \psi$ if δ, ε are sufficiently small. Hence, by (6.4),

$$Au_{\varepsilon,\delta} + \alpha u_{\varepsilon,\delta} = f$$

on such a set. Taking $\delta \to 0$, $\varepsilon \to 0$ we conclude that $Au + \alpha u = f$ on this set. Thus we have proved that

$$(u - \psi)(Au + \alpha u - f) = 0 \quad \text{a.e. in } \Omega. \tag{6.15}$$

We also have

$$u \leqslant \psi \quad \text{in } \Omega. \tag{6.16}$$

We sum up: *if ψ is as in Lemma* 6.2 *then u is a regular solution, in the sense of* (6.13), *of the variational inequality given by* (6.14)–(6.16).

Let

$$f(p) = \sum_{i=1}^{n} c_i p_i, \qquad \psi(p) = K + \sum_{j=1}^{n-1} p_j V_\alpha(e_j). \tag{6.17}$$

From (5.10) we see that

$$V_\alpha(p) = u(p), \tag{6.18}$$

where $u(p)$ is defined by (6.9) with f, ψ given by (6.17). Applying the last italicized statement we obtain assertion (i) of Theorem 6.1 with $u = V_\alpha$.

In order to prove assertion (iii) of Theorem 6.1 it suffices to show that the proof of Theorem 4.2 can be carried out with $V(x, p)$ replaced by $V_\alpha(p)$. We already know that $V_\alpha(p)$ belongs to $C(\bar{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$. This is actually sufficient for establishing (6.3). Indeed, we just have to be a bit careful in applying Itô's formula. We illustrate it in the derivation of (4.20). Here we apply Itô's formula (see Remark 7 at the end of §4), integrated from $t = 0$ to

$t = \sigma_1 \wedge T_\delta$, where T_δ = exit time from Ω_δ, provided p is in Ω_δ. We then take $\delta \to 0$ and obtain (4.20), provided $p \in \Omega$. By continuity, this relation holds for $p \in \bar{\Omega}$.

It is now clear how to complete the proof that

$$V_\alpha(p) \leq J_0^p(\tau), \quad V_\alpha(p) = J_0^p(\bar{\tau})$$

(the first inequality is, of course, already known); the second assertion coincides with (iii). If σ_*^p are not assumed to be finite valued then one can still establish (6.3) by the above proof, provided in applying Itô's formula as above we replace $\sigma_1 \wedge T_\delta$ by $\sigma_1 \wedge T_\delta \wedge N$ and then take $\delta \to 0$, $N \to \infty$, and continue in this manner step by step. Thus, if u is any regular solution of the q.v.i. as in (i) then $u(p) = J_0^p(\bar{\tau})$ even when σ_*^p may not be finite valued. This yields the uniqueness of u, and completes the proof of Theorem 6.1. As a by-product we have obtained

COROLLARY 6.3. *Assertion* (6.3) *is valid even if the* σ_*^p *are not finite valued a.s.*

It follows that the minimal sequence of inspections exists and the corresponding stopping times $\sigma_{m,l}$ are invariant under translation of $x(t)$. Consequently:

COROLLARY 6.4. *If* (6.1) *is valid then, in* (5.15), $V_\alpha(p) = \tilde{V}_\alpha(p)$.

REMARK. We give another proof of (6.3) which is valid also for more general ψ (of p and V_α) than in (6.17). In fact, ψ will only be required to be continuous. We return to the function u defined in (6.9) and let

$$S_0 = \{p \in \bar{\Omega}, u(p) = \psi(p)\},$$

$$\hat{\theta}_p = \text{hitting time of the set } S_0, \text{ given } p(0) = p.$$

Assume that

$$\hat{\theta}_p \text{ is finite valued a.e.} \quad (\text{for any } p \neq e_n). \tag{6.19}$$

Then

$$u(p) = E^p\left[\int_0^{\hat{\theta}_p} f(p(s))e^{-\alpha s}\,ds + \psi\big(p(\hat{\theta}_p)\big)e^{-\alpha\hat{\theta}_p}\right]. \tag{6.20}$$

The proof is as in [3, end of Chapter III] with minor obvious changes.

Applying result (6.20) with f, ψ as in (6.17), we get

$$V_\alpha(p) = E^p\left[Ke^{-\alpha\bar{\tau}_1} + \int_0^{\bar{\tau}_1} e^{-\alpha s}\sum_{j=1}^{n} c_j p_j(s)\,ds + \sum_{j=1}^{n-1} e^{-\alpha\bar{\tau}_1} p_i(\bar{\tau}_1)V_\alpha(e_i)\right]. \tag{6.21}$$

We now iterate this relation any number of times l; taking $l \to \infty$, assertion (6.3) follows.

Approximation of V_α. Define successively functions u^k as follows: $u^k(p) = u(p)$, where u is given by (6.9) with

$$f(p) = \sum_{j=1}^{n} c_j p_j, \quad \psi(p) = K + \sum_{j=1}^{n-1} p_j u^{k-1}(e_j),$$

if $k \geqslant 1$, and

$$u^0(p) = E^p\left[\int_0^\infty f(p(s))e^{-\alpha s}\, ds\right].$$

THEOREM 6.5. *As $k \uparrow \infty$,*

$$u^k(p) \downarrow V_\alpha(p) \quad \textit{uniformly in } p \in \bar{\Omega}. \tag{6.22}$$

PROOF. Taking $\theta \to \infty$ in (6.9) for $u = u^1$ gives $u^1 \leqslant u^0$. Next, by induction,

$$u^{k+1} \leqslant u^k. \tag{6.23}$$

We have

$$\overline{\lim_{k\to\infty}}\; u^k(p) \leqslant J_0^p(\tau) \quad \text{for any } \tau \in \mathcal{Q}_0. \tag{6.24}$$

The proof is similar to the proof of the corresponding result in [3, Chapter V, cf. (3.19)].

We also have

$$\underline{\lim_{k\to\infty}}\; u^k(p) \geqslant V_\alpha(p). \tag{6.25}$$

Indeed, from the definition of u^k and from (6.23),

$$u^k(p) \geqslant \inf_\theta E^p\left[Ke^{-\alpha\theta} + \int_0^\theta f(p(s))e^{-\alpha s}\, ds + \sum_{j=1}^{n-1} p_j(\theta)e^{-\alpha\theta}u^k(e_j)\right].$$

Iterating this inequality any number l of times and taking $l \to \infty$, we obtain

$$u^k(p) \geqslant \inf_{\tau\in\mathcal{Q}_0} J_0^p(\tau) = V_\alpha(p) \quad \text{(by Corollary 6.4)},$$

which, of course, implies (6.25).

From (6.23)–(6.25) we obtain assertion (6.22) for each $p \in \bar{\Omega}$. Since u^k and V_α are continuous in $\bar{\Omega}$ and since $\{u^k\}$ is monotone decreasing to V_α, Dini's theorem implies that the convergence in (6.22) is uniform in $\bar{\Omega}$; this completes the proof of the theorem.

Theorem 6.5 shows that $V_\alpha(p)$ can be obtained as $\lim u^k$; each u^k is $\lim u_\varepsilon^k$, and u_ε^k is a limit of solutions $u_{\varepsilon,\delta}^k$ of a Dirichlet problem of the form (6.4), (6.5).

7. Explicit solution of the q.v.i. in one case. We shall consider the special case $n = 2$, with $c_1 = 0$, $c_2 > 0$, and general $q_{i,j}$, namely: $q_{1,1} = -\alpha'$, $q_{1,2} = \alpha'$, $q_{2,1} = \beta'$, $q_{2,2} = -\beta'$, where $\alpha' > 0$, $\beta' \geqslant 0$; the special case $\beta' = 0$ was considered in [1]. Set

$$K = K_1, \quad c = \frac{c_2}{(\lambda_2 - \lambda_1)^2}, \quad \alpha = \frac{\alpha'}{(\lambda_2 - \lambda_1)^2}, \quad \beta = \frac{\beta'}{(\lambda_2 - \lambda_1)^2}.$$

Setting $y = p_2/p_1$ and

$$Lu \equiv \tfrac{1}{2} y^2 u'' + \left[\alpha(1+y) - \beta y(1+y) + \frac{y^2}{1+y}\right] u', \tag{7.1}$$

the q.v.i. (4.5)–(4.7) reduces to

$$LV(y) + cy/(1+y) \geqslant 0 \quad \text{if } 0 < y < \infty, \tag{7.2}$$

$$V(y) \leqslant K + V(0)/(1+y) \quad \text{if } 0 < y < \infty, \tag{7.3}$$

$$\left(LV(y) + \frac{cy}{(1+y)}\right)\left(V(y) - K - \frac{V(0)}{1+y}\right) = 0 \quad \text{if } 0 < y < \infty. \tag{7.4}$$

THEOREM 7.1. *There exists a function $V(y)$ in $C^1[0, \infty)$ and $b > 0$ such that*

$$LV(y) + cy/(1+y) = 0 \quad \textit{if } 0 < y < b, \tag{7.5}$$

$$V(y) = K + V(0)/(1+y) \quad \textit{if } b \leqslant y < \infty. \tag{7.6}$$

The pair $V(y)$, b is unique and, further,

$$LV(y) + cy/(1+y) > 0 \quad \textit{if } b < y < \infty, \tag{7.7}$$

$$V(y) < K + K(0)/(1+y) \quad \textit{if } 0 \leqslant y < b. \tag{7.8}$$

Notice that V is a solution of the q.v.i. (7.2)–(7.4) ($V''(y)$ has a jump discontinuity at $y = b$). Setting $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \dots) = \bar{\tau}^i$ for $i = 1$, where $\bar{\tau}_j^i$ is defined in (4.17), with $p = e_i$, and using Lemmas 3.1, 3.5, we see that

$$R_m \leqslant CE^{1,x}\left[I_{\theta(\bar{\tau}_1)=1} I_{\theta(\bar{\tau}_2)=1} \cdots I_{\theta(\bar{\tau}_m)=1}\right] = C\left(\frac{1}{1+b}\right)^m \to 0$$

if $m \to \infty$. Consequently, $J_x^1(\bar{\tau}) < \infty$.

PROOF. We begin by solving (7.5) with b undetermined as yet. Dividing both sides by $y^2/2$ and multiplying by

$$\exp\left\{\int\left[\frac{2(\alpha - \beta y)(1+y)}{y^2} + \frac{2}{1+y}\right] dy\right\}$$
$$= \exp\{2[(\alpha - \beta)\ln y + \ln(1+y) - (\alpha/y + \beta y)]\}$$
$$= y^{2\alpha - 2\beta}(1+y)^2 e^{-2(\alpha/y + \beta y)},$$

we get

$$V'(y) y^{2\alpha - 2\beta}(1+y)^2 e^{-2(\alpha/y + \beta y)}$$
$$= -\int_0^y 2cz^{2\alpha - 2\beta - 1}(1+z)e^{-2(\alpha/z + \beta z)}\, dz, \tag{7.9}$$

where we have assumed that

$$V'(y)y^{2\alpha-2\beta}(1+y)^2 e^{-2(\alpha/y+\beta y)} \to 0 \quad \text{if } y \to 0. \tag{7.10}$$

Hence

$$V(b) - V(0) = -\int_0^b dy \frac{1}{y^{2\alpha-2\beta}} \frac{1}{(1+y)^2} e^{2(\alpha/y+\beta y)}$$

$$\cdot \int_0^y 2cz^{2\alpha-2\beta-1}(1+z)e^{-2(\alpha/z+\beta z)}\,dz. \tag{7.11}$$

We now impose two conditions on $V(y)$ at $y = b$:

$$V(b) = K + V(0)/(1+b), \tag{7.12}$$

$$V'(b) = -V(0)/(1+b)^2. \tag{7.13}$$

In view of (7.9), the second condition reduces to

$$\frac{V(0)}{c} = \frac{1}{b^{2\alpha-2\beta}} e^{2(\alpha/b+\beta b)} \cdot \int_0^b dz 2z^{2\alpha-2\beta-1}(1+z)e^{-2(\alpha/z-\beta z)}, \tag{7.14}$$

and, in view of (7.11), the first condition is equivalent to

$$K = \frac{b}{1+b} V(0) - c\int_0^b dy \frac{1}{y^{2\alpha-2\beta}} \frac{1}{(1+y)^2} e^{2(\alpha/y+\beta y)}$$

$$\cdot \int_0^y 2cz^{2\alpha-2\beta-1}(1+z)e^{2(\alpha/z+\beta z)}\,dz. \tag{7.15}$$

Substituting $V(0)$ from (7.14) into (7.15) we get

$$\frac{K}{c} = \frac{1}{b^{2\alpha-2\beta-1}} \frac{1}{1+b} e^{2(\alpha/b+\beta b)}$$

$$\cdot \int_0^b dz\, 2z^{2\alpha-2\beta-1}(1+z)e^{-2(\alpha/z+\beta z)}$$

$$-\int_0^b dy \frac{1}{y^{2\alpha-2\beta}} \frac{1}{(1+y)^2} e^{2(\alpha/y+\beta y)}$$

$$\cdot \int_0^y dz\, 2z^{2\alpha-2\beta-1}(1+z)e^{-2(\alpha/z+\beta z)} \equiv H(b). \tag{7.16}$$

If we solve (7.16) and then define V in $0 \leqslant y \leqslant b$ by (7.9), (7.14) and in $b < y < \infty$ by (7.6), then (since (7.12), (7.13) hold) V is a solution in $C^1[0, \infty)$ of (7.5), (7.6). Conversely, any solution in $C^1[0, \infty)$ of (7.5), (7.6) must satisfy (7.12), (7.13) and, consequently, it is given, for $0 \leqslant y \leqslant b$, by (7.9), (7.14) with b satisfying (7.16). We now claim:

$$\text{there exists a unique solution } b \text{ of (7.16).} \tag{7.17}$$

Once (7.17) is proved, the uniqueness assertion of Theorem 7.1 follows.

In order to establish (7.17), we introduce the function

$$q(y) = \frac{1}{y^{2\alpha-2\beta}} e^{2(\alpha/y+\beta y)} \int_0^y dz\, z^{2\alpha-2\beta-1}(1+z)e^{-2(\alpha/z+\beta z)}. \tag{7.18}$$

Then

$$H(b) = 2\left[\frac{b}{1+b} q(b) - \int_0^b dy\, \frac{1}{(1+y)^2} q(y)\right]. \tag{7.19}$$

Clearly

$$q'(b) = \frac{1+b}{b} - \left(\frac{2\alpha-2\beta}{b} + 2\left(\frac{\alpha}{b^2} - \beta\right)\right) q(b),$$

so that

$$\frac{b^2}{1+b} q'(b) = b + 2\beta b q(b) - 2\alpha q(b). \tag{7.20}$$

Now,

$$\int_0^b dz\, z^{2\alpha-2\beta-1}(1+z)e^{-2\beta z}e^{-2\alpha/z}$$

$$= \frac{1}{2\alpha}\int_0^b dz\, z^{2\alpha-2\beta+1}e^{-2\beta z}\frac{2\alpha}{z^2}e^{-2\alpha/z} + \int_0^z dz\, z^{2\alpha-2\beta}e^{-2\beta z}e^{-2\alpha/z}$$

$$= \text{(by integration by parts)}\quad \frac{1}{2\alpha} b^{2\alpha-2\beta+1}e^{-2(\alpha/b+\beta b)}$$

$$- \frac{1}{2\alpha}\int_0^b dz\,(2\alpha-2\beta+1-2\beta z)z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}$$

$$+ \int_0^b dz\, z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}$$

$$= \frac{1}{2\alpha}\left[b^{2\alpha-2\beta+1}e^{-2(\alpha/b+\beta b)} - \int_0^b dz\, z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}\right.$$

$$\left. + 2\beta\int_0^b dz\,(1+z)z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}\right].$$

Hence

$$q(b) = \frac{1}{2\alpha}\left[b - \frac{1}{b^{2\alpha-2\beta}} e^{2(\alpha/b+\beta b)}\int_0^b dz\, z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}\right.$$

$$\left. + \frac{2\beta}{b^{2\alpha-2\beta}} e^{2(\alpha/b+\beta b)}\int_0^b dz\,(1+z)z^{2\alpha-2\beta}e^{-2(\alpha/z+\beta z)}\right]. \tag{7.21}$$

We now substitute $q(b)$ from (7.21) into $-2\alpha q(b)$ on the right-hand side of (7.20), and $q(b)$ from (7.18) (with $y = b$) into $2\beta q(b)$ on the right-hand side of (7.20), and obtain

$$\frac{b^2}{1+b} q'(b) = 2\beta\Bigg[\frac{1}{b^{2\alpha-2\beta}} e^{2(\alpha/b+\beta b)} \cdot \int_0^b dz\, (1+z) z^{2\alpha-2\beta} e^{-2(\alpha/z+\beta z)} \left(\frac{b}{z} - 1 \right) + \frac{1}{b^{2\alpha-2\beta}} e^{2(\alpha/b+\beta b)} \int_0^b dz\, z^{2\alpha-2\beta} e^{-2(\alpha/z+\beta z)} \Bigg] > 0.$$

Hence $q'(b) > 0$. Next, from (7.19),

$$H'(b) = 2\left[\frac{1}{(1+b)^2} q(b) + \frac{b}{1+b} q'(b) - \frac{1}{(1+b)^2} q(b) \right] = (2b/(1+b))q'(b) > 0.$$

Thus $H(y)$ is strictly monotone increasing. Since $H(0) = 0$ and, as easily seen, $H(b) \to \infty$ if $b \to \infty$, assertion (7.17) follows. It remains to prove (7.7), (7.8).

Set

$$l(y) = K + V(0)/(1+y) - V(y).$$

Then

$$l'(y) = -\frac{V(0)}{(1+y)^2} - V'(y) = c\left[-\frac{2q(b)}{(1+y)^2} + \frac{2q(y)}{(1+y)^2} \right] = \left(2c/(1+y)^2\right)\left(q(y) - q(b)\right) < 0,$$

since $q' > 0$. Noting that $l(0) = K$, $l(b) = 0$, we conclude that $l(y) > 0$ if $0 \leqslant y < b$, i.e., (7.8) holds.

To derive (7.7), note that

$$L(1/(1+y)) = -(\alpha - \beta y)/(1+y).$$

Therefore

$$L\left(K + \frac{V(0)}{1+y} \right) + \frac{cy}{1+y} = \frac{c}{1+y}\left[y - (\alpha - \beta y) q(b) \right] > 0$$

if $y > b$, since

$$b - (\alpha - \beta b) q(b) = \tfrac{1}{2} b + \tfrac{1}{2}\left(b + 2\beta b q(b) - 2\alpha q(b)\right) = \tfrac{1}{2} b + \left(b^2/(1+b)\right) q'(b) > 0.$$

Appendix: Proofs of Theorems 2.1 and 2.2. If P_x^1 and P_x^2 are Markov processes then the tensor product $P_x^1 \otimes_u^0 P_{x(u)}^2$ is defined by the two conditions: (1) it agrees with P_x^1 on $\mathcal{F}_u$; (2) its regular conditional probability distribution, given $\mathcal{F}_u$, coincides with $P_{x(u)}^2$ a.e. These two conditions determine $P_x^1 \otimes_u^0 P_{x(u)}^2$ uniquely (see [9]). We have

$$P_x^1 \otimes_u^0 P_{x(u)}^2(A \cap B) = \int_A P_{x(u)}^2\big(\phi_u^{-1}(B)\big)\, dP_x^1 \tag{A.1}$$

if $A \in \mathcal{F}_u$, $B \in \mathcal{F}_t^u$. Using this rule one can prove the associate law for the tensor product.

We also have, for any Markov process P_x,

$$P_x \otimes_u^0 P_{x(s)} = P_x. \tag{A.2}$$

We denote by P_x^λ the Markov process on $(\Omega, \mathcal{F}_t)$ corresponding to m-dimensional Brownian motion with drift λ.

We now define functions $P_{i,x}$:

$$\begin{aligned}
&P_{i,x}(\theta(t) = j, x(t) \in B) \\
&\quad = \sum_{\rho=0}^{\infty}{}^{*} \sum_{(i, \gamma_1, \ldots, \gamma_\rho, j)}{}' \int_0^t du_{i,\gamma_1} \exp\{-q_i u_{i,\gamma_1}\} \\
&\quad \cdot \int_0^{t-u_{i,\gamma_1}} du_{\gamma_1,\gamma_2} \exp\{-q_{\gamma_1} u_{\gamma_1,\gamma_2}\} q_{\gamma_1,\gamma_2} \\
&\quad \cdots \int_0^{t-u_{i,\gamma_1}-u_{\gamma_1,\gamma_2}-\cdots-u_{\gamma_{l-1},\gamma_l}} du_{\gamma_l,\gamma_{l+1}} \exp\{-q_{\gamma_l} u_{\gamma_l,\gamma_{l+1}}\} q_{\gamma_l,\gamma_{l+1}} \\
&\quad \cdots \int_0^{t-u_{i,\gamma_1}-u_{\gamma_1,\gamma_2}-\cdots-u_{\gamma_{\rho-1},\gamma_\rho}} du_{\gamma_\rho,j} \exp\{-q_{\gamma_\rho} u_{\gamma_\rho,j}\} q_{\gamma_\rho,j} \\
&\quad \cdot \exp\big\{-q_j\big(t - u_{i,\gamma_1} - u_{\gamma_1,\gamma_2} - \cdots - u_{\gamma_{\rho-1},\gamma_\rho} - u_{\gamma_\rho,j}\big)\big\} \\
&\quad \cdot P_x^{\lambda_i} \otimes_{u_{i,\gamma_1}}^0 P_{x(u_{i,\gamma_1})}^{\lambda_{\gamma_1}} \otimes_{u_{i,\gamma_1}+u_{\gamma_1,\gamma_2}}^{u_{i,\gamma_1}} P_{x(u_{i,\gamma_1}+u_{\gamma_1,\gamma_2})}^{\lambda_{\gamma_2}} \\
&\quad \otimes \cdots \otimes_{u_{i,\gamma_1}+u_{\gamma_1,\gamma_2}+\cdots+u_{\gamma_l,\gamma_{l+1}}}^{u_{i,\gamma_1}+u_{\gamma_1,\gamma_2}+\cdots+u_{\gamma_{l-1},\gamma_l}} P_{x(u_{i,\gamma_1}+\cdots+u_{\gamma_{l-1},\gamma_l})}^{\gamma_{l+1}} \\
&\quad \otimes \cdots \otimes_{u_{i,\gamma_1}+\cdots+u_{\gamma_\rho,j}}^{u_{i,\gamma_1}+\cdots+u_{\gamma_{\rho-1},\gamma_\rho}} P_{x(u_{i,\gamma_1}+\cdots+u_{\gamma_\rho,j})}^{\lambda_j}(x(t) \in B),
\end{aligned} \tag{A.3}$$

where $q_l = -q_{l,l}$ and B is any set in $\mathcal{B}(R^m)$. Here we used the same summation notation as in §2. However, the term in (A.3) corresponding to $\rho = 1$ (when $i = j$) is understood to be $e^{-q_i t} P_x^{\lambda_i}(x(t) \in B)$.

We shall show later on that the $P_{i,x}$ are the transition probabilities of the random evolution given in Lemma 1.1.

To motivate definition (A.3) we observe that the probability of jumping from state i to state j in one jump in time t has the density $e^{-q_i t} q_{i,j}$.

Theorem A.1. *The functions $P_{\theta,x}$ define a strong Markov process with respect to the σ-fields $\mathfrak{M}_t$. The finite dimensional distributions of this process coincide with the finite dimensional distributions of the process constructed in Lemma* 1.1.

Proof. We first verify the Chapman-Kolmogorov equation, i.e.,

$$
\begin{aligned}
&P_{i,x}(\theta(t) = j, x(t) \in B)\\
&\quad = \Sigma\Bigg[\int_0^s du_{i,\alpha_1}\exp\{-q_i u_{i,\alpha_1}\}q_{i,\alpha_1}\\
&\qquad \cdot \int_0^{s-u_{i,\alpha_1}} du_{\alpha_1,\alpha_2}\exp\{-q_{\alpha_1}u_{\alpha_1,\alpha_2}\}q_{\alpha_1,\alpha_2}\\
&\qquad \cdots \int_0^{s-u_{i,\alpha_1}-\cdots-u_{\alpha_{\mu-1},\alpha_\mu}} du_{\alpha_\mu,l}\exp\{-q_{\alpha_\mu}u_{\alpha_\mu,l}\}q_{\alpha_\mu,l}\\
&\qquad \cdot \exp\{-q_l(s-u_{i,\alpha_1}-\cdots-u_{\alpha_\mu,l})\}P_x^{\lambda_i}\otimes^0_{u_{i,\alpha_1}}P^{\lambda_{\alpha_1}}_{x(u_{i,\alpha_1})}\otimes\cdots\\
&\qquad\qquad \otimes^{u_{i,\alpha_1}+\cdots+u_{\alpha_{\mu-1},\alpha_\mu}}_{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}}P^{\lambda_l}_{x(u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l})}(x(s)\in dy)\Bigg]\\
&\qquad \cdot\Bigg[\int_0^{t-s} du_{l,\beta_1}\exp\{-q_l u_{l,\beta_1}\}q_{l,\beta_1}\\
&\qquad\quad \cdot\int_0^{t-s-u_{l,\beta_1}} du_{\beta_1,\beta_2}\exp\{-q_{\beta_1}u_{\beta_1,\beta_2}\}q_{\beta_1,\beta_2}\\
&\qquad\quad \cdots\int_0^{t-s-u_{l,\beta_1}-\cdots-u_{\beta_{\nu-1},\beta_\nu}} du_{\beta_\nu,j}\exp\{-q_{\beta_\nu}u_{\beta_\nu,j}\}\\
&\qquad\quad \cdot q_{\beta_\nu,j}\exp\{-q_j(t-s-u_{l,\beta_1}-\cdots-u_{\beta_\nu,j})\}\\
&\qquad\quad \cdot P_y^{\lambda_l}\otimes^0_{u_{l,\beta_1}}P^{\lambda_{\beta_1}}_{x(u_{l,\beta_1})}\otimes\cdots\otimes^{u_{l,\beta_1}+\cdots+u_{\beta_{\nu-1},\beta_\nu}}_{u_{l,\beta_1}+\cdots+u_{\beta_\nu,j}}\\
&\qquad\qquad \cdot P^{\lambda_j}_{x(u_{l,\beta_1}+\cdots+u_{\beta_\nu,j})}(x(t-s)\in B)\Bigg], \qquad (A.4)
\end{aligned}
$$

where

$$\Sigma = \sum_{l=1}^{n}\sideset{}{^*}\sum_{\mu=0}^{\infty}\sideset{}{'}\sum_{(i,\alpha_1,\ldots,\alpha_\mu,l)}\sideset{}{^*}\sum_{\nu=0}^{\infty}\sum_{(l,\beta_1,\ldots,\beta_\nu,j)}.$$

We bring together all the P's and use (A.1), (A.2). Next, make a substitution $u_{l,\beta_1} + s \to u_{l,\beta_1}$ and then another one,

$$u_{l,\beta_1} \to u_{i,\alpha_1} + u_{\alpha_1,\alpha_2} + \cdots + u_{\alpha_\mu,l} + u_{l,\beta_1}.$$

The total effect is to transform (A.4) into

$$
\begin{aligned}
&P_{i,x}(\theta(t) = j, x(t) \in B) \\
&\quad = \sum \int_0^s du_{i,\alpha_1} \int_0^{s-u_{i,\alpha_1}} du_{\alpha_1,\alpha_2} \cdots \int_0^{s-u_{i,\alpha_1}-\cdots-u_{\alpha_{\mu-1},\alpha_\mu}} du_{\alpha_\mu,l} \\
&\quad \cdot \int_{s-u_{i,\alpha_1}-\cdots-u_{\alpha_\mu,l}}^{t-u_{i,\alpha_1}-\cdots-u_{\alpha_\mu,l}} du_{l,\beta_1} \int_0^{t-u_{i,\alpha_1}-\cdots-u_{\alpha_\mu,l}-u_{l,\beta_1}} du_{\beta_1,\beta_2} \\
&\quad \cdots \int_0^{t-u_{i,\alpha_1}-\cdots-u_{\alpha_\mu,l}-u_{\beta_1,\beta_2}-\cdots-u_{\beta_{\nu-1},\beta_\nu}} du_{\beta_\nu,j} \\
&\quad \cdot \exp\{-q_i u_{i,\alpha_1}\} q_{i,\alpha_1} \exp\{-q_{\alpha_1} u_{\alpha_1,\alpha_2}\} q_{\alpha_1,\alpha_2} \\
&\quad \cdots \exp\{-q_{\alpha_\mu} u_{\alpha_\mu,l}\} q_{\alpha_\mu,l} \exp\{-q_l u_{l,\beta_1}\} q_{l,\beta_1} \\
&\quad \cdot \exp\{-q_{\beta_1} u_{\beta_1,\beta_2}\} q_{\beta_1,\beta_2} \cdots \exp\{-q_{\beta_\nu} u_{\beta_\nu,j}\} q_{\beta_\nu,j} \\
&\quad \cdot \exp\{-q_j(t - u_{i,\alpha_1} - \cdots - u_{\beta_\nu,j})\} P_x^{\lambda_i} \otimes_{u_{i,\alpha_1}}^{0} P_{x(u_{i,\alpha_1})}^{\lambda_{\alpha_1}} \\
&\quad \otimes \cdots \otimes_{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}}^{u_{i,\alpha_1}+\cdots+u_{\alpha_{\mu-1},\alpha_\mu}} P_{x(u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l})}^{\lambda_l} \\
&\quad \otimes_{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}}^{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}} P_{x(u_{i,\alpha_1}+\cdots+u_{l,\beta_1})}^{\lambda_{\beta_1}} \\
&\quad \otimes_{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}+u_{\beta_1,\beta_2}}^{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}} P_{x(u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}+u_{\beta_1,\beta_2})}^{\lambda_{\beta_2}} \otimes \cdots \\
&\quad \otimes_{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}+\cdots+u_{\beta_{\nu-1},\beta_\nu}+u_{\beta_\nu,j}}^{u_{i,\alpha_1}+\cdots+u_{\alpha_\mu,l}+u_{l,\beta_1}+\cdots+u_{\beta_{\nu-1},\beta_\nu}} P_{x(u_{i,\alpha_1}+\cdots+u_{\beta_\nu,j})}^{\lambda_j}(x(t) \in B).
\end{aligned}
\tag{A.5}
$$

We shall denote the general term on the right-hand side of (A.5) by

$$I_{l,j}^{i,l}(0, s, \alpha_1, \ldots, \alpha_\mu; s - \bar{u}, t - \bar{u}, \beta_1, \ldots, \beta_\nu),$$

and the general term on the right-hand side of (A.3) by

$$I^{i,j}(0, t, \gamma_1, \ldots, \gamma_\rho).$$

Then assertion (A.5) reduces to the following combinatorial lemma.

LEMMA A.2. *The following formula holds*:

$$\sum I_{l,j}^{i,l}(0, s, \alpha_1, \ldots, \alpha_\mu; s - \bar{u}, t - \bar{u}, \beta_1, \ldots, \beta_\nu) = \sum_{\rho=0}^{\infty} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}' I^{i,j}(0, t, \gamma_1, \ldots, \gamma_\rho). \tag{A.6}$$

PROOF. Consider first the case where $i \neq j$. Then the left-hand side of (A.6) is equal to

$$\sum_{\rho=1}^{\infty}\left[\sum_{\nu=0}^{\rho-1}\sum_{l=1}^{n}\sum''_{(i,\alpha_1,\ldots,\alpha_{\rho-\nu-1},l)}\sum''_{(l,\beta_1,\ldots,\beta_\nu,j)}\right.$$

$$\left.\cdot\, I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-\nu-1}; s-\bar{u}, t-\bar{u}, \beta_1, \ldots, \beta_\nu\right)\right]$$

$$+\sum_{\rho=0}^{\infty}\left[\sum'_{(i,\alpha_1,\ldots,\alpha_\rho,j)} I^{i,j}\left(0, s, \alpha_1, \ldots, \alpha_\rho\right)\right]$$

$$+\sum_{\rho=0}^{\infty}\left[\sum'_{(i,\beta_1,\ldots,\beta_\rho,j)} I^{i,j}\left(s, t, \beta_1, \ldots, \beta_\rho\right)\right]$$

$$\equiv\sum_{\rho=1}^{\infty}A_\rho+\sum_{\rho=0}^{\infty}B_\rho+\sum_{\rho=0}^{\infty}C_\rho; \tag{A.7}$$

here the first Σ'' indicates the usual restrictions in the definition of Σ' plus the additional restriction that when $\nu = 0$ we must take $l \neq j$. Similarly, in the second Σ'' we have the additional restriction that when $\nu = \rho - 1$ we must take $l \neq i$. Note that the terms B_ρ and C_ρ come from those terms on the right-hand side of (A.6) corresponding to $\nu = -1$, $l = j$ and $\mu = -1$, $l = i$, respectively.

Take $\rho > 0$. Combining B_ρ with the terms in A_ρ for which $\nu = 0$, we obtain

$$\sum'_{(i,\alpha_1,\ldots,\alpha_\rho,j)} I^{i,j}\left(0, s, \alpha_1, \ldots, \alpha_\rho\right)$$

$$+\sum_{\substack{l=1\\ l\neq j}}^{n}\sum'_{(i,\alpha_1,\ldots,\alpha_{\rho-1},l)} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-1}; s-\bar{u}, t-\bar{u}\right)$$

$$=\sum_{\substack{l=1\\ l\neq j}}^{n} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-1}; 0, t-\bar{u}\right), \tag{A.8}$$

as easily seen by writing down explicitly the corresponding integrals.

We now add the right-hand side of (A.8) the terms in A_ρ with $\nu = 1$:

$$\sum_{l=1}^{n}\sum'_{(i,\alpha_1,\ldots,\alpha_{\rho-2},l)}\sum'_{(l,\beta_1,j)} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-2}; s-\bar{u}, t-\bar{u}, \beta_1\right)$$

$$+\sum_{\substack{l=1\\ l\neq j}}^{n} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-1}; 0, t-\bar{u}\right)$$

$$=\sum_{l=1}^{n}\sum'_{(i,\alpha_1,\ldots,\alpha_{\rho-2},l)}\sum'_{(l,\beta_1,j)} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-2}; 0, t-\bar{u}, \beta_1\right), \tag{A.9}$$

as easily seen by writing down explicitly the corresponding integrals.

Next we add to the right-hand side of (A.9) the terms in A_ρ corresponding to $\nu = 2$ and obtain

$$\sum_{l=1}^{n} \sum_{(i,\alpha_1,\ldots,\alpha_{\rho-3},l)}' \sum_{(l,\beta_1,\beta_2,j)} I_{l,j}^{i,l}\left(0, s, \alpha_1, \ldots, \alpha_{\rho-3}; 0, t-\bar{u}, \beta_1, \beta_2\right).$$

Proceeding this way step by step, we obtain, after adding the terms in A_ρ with $\nu = \rho - 2$,

$$\sum_{l=1}^{n} \sum_{(i,\alpha_1,l)}' \sum_{(l,\beta_1,\ldots,\beta_{\rho-2},j)}' I_{l,j}^{i,l}\left(0, s, \alpha_1; 0, t-\bar{u}, \beta_1, \ldots, \beta_{\rho-2}\right).$$

Adding the terms in A_ρ with $\nu = \rho - 1$, i.e., the sum

$$\sum_{\substack{l=1\\ l\neq i}}^{n} \sum_{(l,\beta_1,\ldots,\beta_{\rho-1},j)} I_{l,j}^{i,l}\left(0, s; s-\bar{u}, \beta_1, \ldots, \beta_{\rho-1}\right),$$

we obtain

$$\sum_{\substack{l=1\\ l\neq i}}^{n} \sum_{(l,\beta_1,\ldots,\beta_{\rho-1},j)} I_{l,j}^{i,l}\left(0, s; 0, t-\bar{u}, \beta_1, \ldots, \beta_{\rho-1}\right).$$

We finally add the terms in C_ρ and obtain

$$A_\rho + B_\rho + C_\rho = \sum_{(i,\beta_1,\ldots,\beta_\rho,j)}' I^{i,j}(0, t, \beta_1, \ldots, \beta_\rho) \qquad (\rho > 0).$$

Since, obviously, $B_0 + A_0 = I^{i,j}(0, t)$, assertion (A.6) follows in case $i \neq j$. The proof for $i = j$ is similar.

Having proved the Chapman-Kolmogorov equation, it follows that the $P_{\theta,x}$ define a Markov process with respect to $\hat{\mathfrak{M}}_t$. Since the Feller property can easily be verified, the process satisfies the strong Markov property not only with respect to $\hat{\mathfrak{M}}_t$ but also with respect to $\mathfrak{M}_t$.

It remains to prove that the one dimensional distributions of this process and of the process constructed in Lemma 1.1 are the same. For this we simply compute the expectation of

$$\exp\left[\mu_1\left[w(t) + \int_0^t f(\theta(s))\,ds + x\right] + \mu_2\theta(t)\right]$$

(for any $\mu_1 \in R^m$, $\mu_2 \in R^1$) using the respective probabilities, and check that the results are the same.

We shall henceforth denote by $P^{\theta,x}$ the measures of the Markov process constructed in Theorem A.1.

LEMMA A.3. *For any $A \in \mathcal{F}_t$,*

$$\int_A I_{\theta(t)=j} dP^{i,x} = \sum_{\rho=0}^{\infty}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}' \int_0^t du_{i,\gamma_1} \exp\{-q_i u_{i,\gamma_1}\}$$

$$\cdot \int_0^{t-u_{i,\gamma_1}} du_{\gamma_1,\gamma_2} \exp\{-q_{\gamma_1} u_{\gamma_1,\gamma_2}\} q_{\gamma_1,\gamma_2}$$

$$\cdots \int_0^{t-u_{i,\gamma_1}-\cdots-u_{\gamma_{\rho-1},\gamma_\rho}} du_{\gamma_{\rho-1},j} \exp\{-q_{\gamma_\rho} u_{\gamma_\rho,j}\} q_{\gamma_\rho,j}$$

$$\cdot \exp\{-q_j(t-u_{i,\gamma_1}-\cdots-u_{\gamma_{\rho-1},\gamma_\rho}-u_{\gamma_\rho,j})\} P_x^{\lambda_i} \otimes_{u_{i,\gamma_1}}^{0} P_{x(u_{i,\gamma_1})}^{\lambda_{\gamma_1}}$$

$$\otimes \cdots \otimes_{u_{i,\gamma_1}+\cdots+u_{\gamma_\rho,j}}^{u_{i,\gamma_1}+\cdots+u_{\gamma_{\rho-1},\gamma_\rho}} P_{x(u_{i,\gamma_1}+\cdots+u_{\gamma_\rho,j})}^{\lambda_j}(A). \tag{A.10}$$

PROOF. It suffices to establish (A.10) for a cylinder set

$$A = (x(t_1) \in B_1, \ldots, x(t_m \in B_m)),$$
$$0 \leq t_1 < t_2 < \cdots < t_{m-1} < t_m = t.$$

Set

$$A = (x(t_1) \in B_1, \ldots, x(t_i) \in B_i),$$
$$C_i = (x(t_{i+1}-t_i) \in B_{i+1}, \ldots, x(t_n-t_i) \in B_n).$$

Using (A.3) and the Markov property, and then substituting $u_{l,\beta_1} + t_{m-1} \to u_{l,\beta_1}$, we find that

$$\int_A I_{\theta(t)=j} dP^{i,x} = \sum_{l=1}^{n} \int_{A_{m-1}} I_{\theta(t_{m-1})=l} \sum_{\nu=0}^{\infty}{}^{*} \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}' \exp\{\alpha_l t_{m-1}\}$$

$$\cdot \int_{t_{m-1}}^{t_m} du_{l,\beta_1} \exp\{-q_l u_{l,\beta_1}\} q_{l,\beta_1}$$

$$\cdot \int_0^{t_m-u_{l,\beta_1}} du_{\beta_1,\beta_2} \exp\{-q_{\beta_1} u_{\beta_1,\beta_2}\} q_{\beta_1,\beta_2}$$

$$\cdots \int_0^{t_m-u_{l,\beta_1}-\cdots-u_{\beta_{\nu-1},\beta_\nu}} du_{\beta_\nu,j} \exp\{-\beta_\nu u_{\beta_\nu,j}\} q_{\beta_\nu,j}$$

$$\cdot \exp\{-q_j(t_m-u_{l,\beta_1}-\cdots-u_{\beta_\nu,j})\}$$

$$\cdot P_{x(t_{m-1})}^{\lambda_l} \otimes_{u_{l,\beta_1}-t_{m-1}}^{0} P_{x(u_{l,\beta_1}-t_{m-1})}^{\lambda_{\beta_1}} \otimes \cdots$$

$$\otimes_{u_{l,\beta_1}+\cdots+u_{\beta_\nu,j}-t_{m-1}}^{u_{l,\beta_1}+\cdots+u_{\beta_{\nu-1},\beta_\nu}-t_{m-1}} P_{x(u_{l,\beta_1}+\cdots+u_{\beta_\nu,j}-t_{m-1})}^{\lambda_j}(C_{m-1}) dP^{i,x}. \tag{A.11}$$

Using the Markov property we obtain, after employing (A.3),

$$\int_A I_{\theta(t)=j}\, dP^{i,x} = \sum_{k=1}^{n} \int_{A_{m-2}} I_{\theta(t_{m-2})=k}$$

$$\cdot \sum_{l=1}^{n} \sum_{\mu=0}^{\infty}{}^{*} \sum_{(k,\alpha_1,\ldots,\alpha_\mu,l)}{}' \sum_{\nu=0}^{\infty}{}^{*} \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}{}'$$

$$\cdot \int_0^{t_{m-1}-t_{m-2}} du_{k,\alpha_1} \exp\{-q_k u_{k,\alpha_1}\} q_{k,\alpha_1}$$

$$\cdot \int_0^{t_{m-1}-t_{m-2}-u_{k,\alpha_1}} du_{\alpha_1,\alpha_2} \exp\{-q_{\alpha_1} u_{\alpha_1,\alpha_2}\} q_{\alpha_1,\alpha_2}$$

$$\cdots \int_0^{t_{m-1}-t_{m-2}-u_{k,\alpha_1}-\cdots-u_{\alpha_{\mu-1},\alpha_\mu}} du_{\alpha_\mu,l} \exp\{-q_{\alpha_\mu} u_{\alpha_\mu,l}\} q_{\alpha_\mu,l}$$

$$\cdot \exp\{-q_l(t_{m-1}-t_{m-2}-u_{k,\alpha_1}-\cdots-u_{\alpha_\mu,l})\}$$

$$\cdot P^{\lambda_k}_{x(t_{m-2})} \otimes^0_{u_{k,\alpha_1}} P^{\lambda_{\alpha_1}}_{x(u_{k,\alpha_1})} \otimes \cdots \otimes^{u_{k,\alpha_1}+u_{\alpha_1,\alpha_2}+\cdots+u_{\alpha_{\mu-1},\alpha_\mu}}_{u_{k,\alpha_1}+u_{\alpha_1,\alpha_2}+\cdots+u_{\alpha_\mu,l}}$$

$$\cdot P^{\lambda_l}_{x(u_{k,\alpha_1}+\cdots+u_{\alpha_\mu,l})}\left[(x(t_{m-1}-t_{m-2}) \in B_{m-1}) \cdots\right].$$

Now substitute $u_{k,\alpha_1} + t_{m-2} \to u_{k,\alpha_1}$ and then

$$u_{l,\beta_1} \to u_{k,\alpha_1} + u_{\alpha_1,\alpha_2} + \cdots + u_{\alpha_{\mu-1},\alpha_\mu} + u_{\alpha_\mu,l} + u_{l,\beta_1}.$$

Using the rule

$$P^{\lambda_1}_x\left[(x(t) \in A) P^{\lambda_1}_{x(t)} \otimes^0_u P^{\lambda_2}_{x(u)}(x(T_1) \in B_1, \ldots, x(T_h) \in B_h)\right]$$
$$= P^{\lambda_1}_x \otimes^0_{u+t} P^{\lambda_2}_{x(u+t)}\left[x(t) \in A, x(t+T_1) \in B_1, \ldots, x(T+T_h) \in B_h\right]$$
$$(u < T_1 < \cdots < T_h), \tag{A.12}$$

$$P^{\lambda_1}_x \otimes^0_u P^{\lambda_2}_{x(u)}\left[(x(t) \in A) P^{\lambda_2}_{x(t)}(x(T_1) \in B_1, \ldots, x(T_h) \in B_h)\right]$$
$$= P^{\lambda_1}_x \otimes^0_u P^{\lambda_2}_{x(u)}\left[x(t) \in A, x(t+T_1) \in B_1, \ldots, x(t+T_h) \in B_h\right]$$
$$(u < t) \tag{A.13}$$

(whose proof for $h = 1$ is given in [**1**]; for general h the proof is similar), and then applying Lemma A.2 (with a slightly different notation), we obtain relation (A.11) with $m - 1$ replaced by $m - 2$. Proceeding in this way step by step and setting $t_0 = 0$, $B_0 = R^1$, $A_0 = (x(t_0) \in B_0)$ and $C_0 = A$, we finally arrive at the expression (A.11) with $m - 1$ and t_{m-1} replaced by 0 and $t_0 = 0$, i.e., (A.10) holds.

PROOF OF THEOREM 2.2. Set

$$\bar{p}_{i,j}(t) = P^{i,x}\left[\theta(t) = j \mid \mathfrak{F}_t\right] \frac{dP^{i,x}}{P^{\lambda_i}_x} \Big| \mathfrak{F}_t. \tag{A.14}$$

Proceeding analogously to [**1**, Proof of (2.7)], but applying Lemma A.3, we find that $\bar{p}_{i,j}(t)$ is given by (2.7). Summing on j in (A.14), we get

$$\sum_{l=1}^{m} \bar{p}_{i,l}(t) = \frac{dP^{i,x}}{dP_x^{\lambda_i}} \,|\mathfrak{F}_t. \tag{A.15}$$

Substituting $dP^{i,x}/dP_x^{\lambda_i}$ from (A.15) into (A.14), assertion (2.8) follows.

To prove (2.9), we write

$$\begin{aligned} p_j(t) &= E^{p,x}[\theta(t) = j|\mathfrak{F}_t] \\ &= \sum_{i=1}^{n} p_i E^{i,x}[\theta(t) = j|\mathfrak{F}_t] \frac{dP^{i,x}}{dP^{x,p}} \,|\mathfrak{F}_t. \end{aligned} \tag{A.16}$$

Now

$$\frac{dP^{p,x}}{dP^{i,x}} \,|\mathfrak{F}_t = \sum_{l=1}^{n} p_l \frac{dP^{l,x}}{dP^{i,x}} \,|\mathfrak{F}_t,$$

and

$$\frac{dP^{l,x}}{dP^{i,x}} \,|\mathfrak{F}_t = \frac{dP^{l,x}}{dP_x^{\lambda_l}} \,|\mathfrak{F}_t \frac{dP_x^{\lambda_l}}{dP_x^{\lambda_i}} \,|\mathfrak{F}_t \frac{dP_x^{\lambda_i}}{dP^{i,x}} \,|\mathfrak{F}_t.$$

Noting, by (A.15) and by (2.6) (plus Girsanov's formula), that

$$\frac{dP^{h,x}}{dP_x^{\lambda_h}} \,|\mathfrak{F}_t = \sum_{k=1}^{n} \bar{p}_{h,k}(t), \qquad \frac{dP_x^{\lambda_l}}{dP_x^{\lambda_i}} \,|\mathfrak{F}_t = z_{i,l}(0, t),$$

we obtain an expression for $(dP^{p,x}/dP^{i,x})|\mathfrak{F}_t$. Substituting it into (A.16) and expressing also $p_j^i(t) = E^{i,x}[\theta(t) = j|\mathfrak{F}_t]$ from (2.8), the right-hand side of (A.16) reduces to the right-hand side of (2.9).

Proof of Theorem 2.1. For $p = (p_1, p_2, \ldots, p_n)$ with $p_i > 0$, $\Sigma p_i = 1$, we write

$$p_j(p_1, p_2, \ldots, p_n, t) \equiv p_j(t) \equiv E^{p,x}[\theta(t) = j|\mathfrak{F}_t].$$

We shall need the following "randomized" Chapman-Kolmogorov equation:

Lemma A.4. *The following relations hold*:

$$\bar{p}_{ij}(t) = \sum_{l=1}^{n} \bar{p}_{i,l}(s)\bar{\bar{p}}_{l,j}(t-s) z_{i,l}(s, t), \tag{A.17}$$

where $\bar{\bar{p}}_{l,j}(t-s)$ *has the same meaning as* $p_{l,j}(t-s)$ *except that* $(0, x(0))$ *has been shifted to* $(s, x(s))$.

The proof is similar to the proof of (A.4).

Lemma A.5. *The following relations hold*:

$$p_j(p_1, \ldots, p_n, t) = \sum_{i=1}^{n} p_i(p_1, \ldots, p_n, s) \cdot \frac{\bar{\bar{p}}_{i,j}(t-s)}{\Sigma_{l=1}^{n} p_l(p_1, \ldots, p_n, s) z_{i,l}(s, t) \Sigma_{k=1}^{n} \bar{\bar{p}}_{l,k}(t-s)} \tag{A.18}$$

PROOF. By (2.9) and Lemma A.4,

$$p_j(p_1, \ldots, p_n, t)$$
$$= \frac{\sum_{i=1}^n p_i \sum_{q=1}^n \bar{p}_{i,q}(s) \bar{\bar{p}}_{q,j}(t-s) z_{i,q}(s,t)}{\sum_{l=1}^n p_l z_{i,l}(0,t) \sum_{k=1}^n \sum_{r=1}^n \bar{p}_{l,r}(s) \bar{\bar{p}}_{r,k}(t-s) z_{l,r}(s,t)}. \tag{A.19}$$

Thus it remains to verify that the right-hand sides of (A.18) and (A.19) are equal. This is achieved by multiplying both expressions by the product of the denominators and verifying that the coefficients of $\bar{\bar{p}}_{i,j}(t-s)\bar{\bar{p}}_{r,k}(t-s)$ on both sides are equal.

Let

$$\Gamma = \{x_x(t) \in A, p_j(p_1, \ldots, p_n, t) \in B_j, 1 \leqslant j \leqslant n\},$$
$$\tilde{\Gamma} = \{x_{x(s)}t \in A, p_j(p_1(p_1, \ldots, p_n, s), \ldots,$$
$$p_n(p_1, \ldots, p_n, s), t-s) \in B_j, 1 \leqslant j \leqslant n\}.$$

Lemma A.5 implies that

$$\Gamma = \tilde{\Gamma}. \tag{A.20}$$

Now,

$$P^{p,x}[\Gamma|\mathcal{F}_s] = \sum_{i=1}^n p_i P^{i,x}[\Gamma|\mathcal{F}_s] \frac{dP^{i,x}}{dP^{p,x}}\bigg|\mathcal{F}_s,$$

and, since

$$\frac{dP^{i,x}}{dP^{p,x}}\bigg|\mathcal{F}_s = 1 \bigg/ \left[\sum_{l=1}^n p_l z_{i,l}(s) \frac{\sum_{k=1}^n \bar{p}_{l,k}(s)}{\sum_{q=1}^n \bar{p}_{i,q}(s)}\right],$$

we have

$$P^{p,x}[\Gamma|\mathcal{F}_s] = \sum_{i=1}^n p_i P^{i,x}[\Gamma|\mathcal{F}_s] \frac{\sum_{q=1}^n \bar{p}_{i,q}(s)}{\sum_{l=1}^n p_l z_{i,l}(s) \sum_{k=1}^n \bar{p}_{l,k}(s)}.$$

Using (A.20) and the Markov property of the (x, θ) process, we get

$$P^{p,x}[\Gamma|\mathcal{F}_s] = \sum_{i=1}^n p_i \frac{\sum_{q=1}^n \bar{p}_{i,q}(s)}{\sum_{l=1}^n p_l z_{i,l}(0,s) \sum_{k=1}^n \bar{p}_{l,k}(s)}$$
$$\cdot \sum_{r=1}^n E^{i,x}\left[I_{\theta(s)=r} P^{r,x(s)}[\tilde{\Gamma}|\mathcal{F}]\right].$$

The inner sum is equal to $\sum_{r=1}^n P_r^i(s) P^{r,x(s)}(\tilde{\Gamma})$. Substituting $p_r^i(s)$ from (2.8), we obtain, after using (2.9),

$$P^{p,x}[\Gamma|\mathcal{F}_s] = \sum_{r=1}^n \left[\sum_{i=1}^n \frac{p_i \bar{p}_{i,r}(s)}{\sum_{l=1}^n p_l z_{i,l}(0,s) \sum_{k=1}^n \bar{p}_{l,k}(s)}\right] P^{r,x(s)}(\tilde{\Gamma})$$
$$= \sum_{r=1}^n p_r(p_1, \ldots, p_n, s) P^{r,x(s)}(\tilde{\Gamma})$$
$$= P^{p(p_1, \ldots, p_n, s), x(s)}(\tilde{\Gamma}),$$

which completes the proof that $(p_j(p_1, \ldots, p_n, t), 1 \leq j \leq n)$ is a Markov process.

We proceed to compute the differentials of $p_j(p_1, \ldots, p_n, t)$. We shall use here the model given in Lemma 1.1. Accordingly, we may resort to Itô's calculus. Using the rule

$$\frac{d}{dt}\int_0^t du_{l,\gamma_1}\int_0^{t-u_{l,\gamma_1}} du_{\gamma_1,\gamma_2}\int_0^{t-u_{l,\gamma_1}-u_{\gamma_1,\gamma_2}} du_{\gamma_2,\gamma_3}$$

$$\cdots\int_0^{t-u_{l,\gamma_1}-u_{\gamma_1,\gamma_2}-\cdots-u_{\gamma_{p-1},\gamma_p}} du_{\gamma_p,j}$$

$$\cdot\int_0^{t-u_{l,\gamma_1}-\cdots-u_{\gamma_{p-1},\gamma_p}-u_{\gamma_p,j}} I\left(u_{j,\gamma_1}, u_{\gamma_1,\gamma_2}, \ldots, u_{\gamma_{p-1},\gamma_p}, u_{\gamma_p,j}\right)$$

$$= \int_0^t du_{l,\gamma_1}\int_0^{t-u_{l,\gamma_1}} du_{\gamma_1,\gamma_2}$$

$$\cdots\int_0^{t-u_{l,\gamma_1}-u_{\gamma_1,\gamma_2}-\cdots-u_{\gamma_{p-1},\gamma_p}} I\Big(u_{l,\gamma_1}, u_{\gamma_1,\gamma_2}, \ldots, u_{\gamma_{p-1},\gamma_p},$$

$$t - u_{l,\gamma_1} - u_{\gamma_1,\gamma_2} - \cdots - u_{\gamma_{p-1},\gamma_p}\Big),$$

we find that

$$d\bar{p}_{l,j}(t) = (\lambda_j - \lambda_l)\bar{p}_{l,j}(t)\cdot dx(t)$$

$$+ \left\{\tfrac{1}{2}|\lambda_j - \lambda_l|^2 - \tfrac{1}{2}\left(|\lambda_j|^2 - |\lambda_l|^2\right)\bar{p}_{l,j}(t) - q_j\bar{p}_{l,j}(t) + \sum_{i\neq j} q_{i,j}\bar{p}_{l,i}(t)\right\} dt$$

$$= (\lambda_j - \lambda_l)\bar{p}_{l,j}(t)\cdot dx(t) + \left[\lambda_l(\lambda_l - \lambda_j)\bar{p}_{l,j} + \sum_{i=1}^n q_{i,j}\bar{p}_{l,i}(t)\right] dt. \quad \text{(A.21)}$$

Using (A.21) and Itô's formula we obtain, from (2.9), after some tedious calculation,

$$dp_j(t) = \sum_{l=1}^n q_{l,j}p_l(t)\,dt + p_j(t)\left(\lambda_j - \sum_{l=1}^n \lambda_l p_l(t)\right)$$

$$\cdot\left(dx(t) - \sum_{l=1}^n \lambda_l p_l(t)\,dt\right), \quad \text{(A.22)}$$

which agrees with (2.2) if

$$y(t) \equiv x(t) - \int_0^t \sum_{l=1}^n \lambda_l p_l(s)\,ds \quad \text{(A.23)}$$

is a Brownian motion. In order to complete the proof of Theorem 2.1 it remains to prove that $y(t)$ is an m-dimensional Brownian motion. For simplicity we shall take $m = 1$.

Now, from the martingale formulation of [8] we have that

$$\exp\left[\xi x(t) - \tfrac{1}{2}\xi^2 t - \xi\int_0^t f(\theta(s))\,ds\right] \quad (\text{for any } \xi \in R^1)$$

is a martingale with respect to $\mathfrak{M}_t$, $P^{\mu,x}$. Differentiating once and twice with respect to ξ and substituting $\xi = 0$, we deduce, respectively, that

$$x(t) - \int_0^t f(\theta(u))\,du, \quad \left(x(t) - \int_0^t f(\theta(u))\,du\right)^2 - t,$$

are martingales with respect to $\mathfrak{M}_t$. Conditioning the martingale relations with respect to $\mathfrak{F}_s$ and using the relation $P^{\mu,x}[\theta(u) = j] = p_j(u)$ and the Markov property of both the (x, θ) and the (x, p) processes, we find that $y(t)$ and $y^2(t) - t$ are martingales with respect to $\mathfrak{F}_t$; hence $y(t)$ is a Brownian motion.

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