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# COMPACTIFICATIONS OF $\mathbf{C}^n$

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ABSTRACT. Let $X$ be a compactification of $\mathbf{C}^n$. We assume that $X$ is a compact complex manifold and that $A = X - \mathbf{C}^n$ is a proper subvariety of $X$. If we suppose that $A$ is a Kähler manifold, then we prove that $X$ is projective algebraic, $H^*(A, \mathbf{Z}) \simeq H^*(\mathbf{P}^{n-1}, \mathbf{Z})$, and $H^*(X, \mathbf{Z}) \simeq H^*(\mathbf{P}^n, \mathbf{Z})$. Various additional conditions are shown to imply that $X = \mathbf{P}^n$. It is known that no additional conditions are needed to imply $X = \mathbf{P}^n$ in the cases $n = 1, 2$. In this paper we prove that if $n = 3$, $X = \mathbf{P}^3$.

**0. Introduction.** In this paper we continue our study of compactifications of $\mathbf{C}^n$ [2], [5], [19]. This introduction will give some definitions and state two of our most important results. More detailed statements will come in later sections. We will also state here some theorems which will be useful in our proofs.

0.1. DEFINITIONS. Let $X$ be a (connected) compact complex manifold and let $A$ be a (closed) subvariety of $X$. We say that $X$ is a *compactification of* $\mathbf{C}^n$ if $X - A$ is biholomorphic to $\mathbf{C}^n$–we will see later that $A$ is then necessarily a connected subvariety of pure dimension $n - 1$. By a *complex homology n-cell* we mean a noncompact complex manifold $M$ of complex dimension $n$ with $H_c^j(M, \mathbf{Z}) = 0$, for $j = 0, \ldots, 2n - 1$, where $H_c^j(M, \mathbf{Z})$ is integral cohomology with compact support.

First, let us make some remarks about dimensions $\leq 2$. For $n = 1$, we notice that the only complex homology 1-cell is the disk $D = \{z: |z| < 1\}$, or the line $\mathbf{C}^1$. However, $D$ is not compactifiable–if $X - A = D$, then any nonconstant, bounded, holomorphic function on $D$ would extend to a nonconstant (bounded) holomorphic function on the compact manifold $X$. Thus the only compactifiable homology 1-cell is $\mathbf{C}^1$. Now, we will see later that $A = X - \mathbf{C}^1$ is a point, where $X$ is a compactification of $\mathbf{C}^1$ ($A$ is a connected 0-dimensional set–see Theorem 1.1). We define a continuous map $\alpha: X \to \mathbf{P}^1 = \mathbf{C}^1 \cup \{\infty\}$ by $\alpha(z) = z$ if $z \in \mathbf{C}^1$, $\alpha(A) = \infty$. By Rado's theorem $\alpha$ is holomorphic. Since $\alpha$ is one-one, $\alpha^{-1}$ is holomorphic and $X$ is biholomorphic to $\mathbf{P}^1$. For $n = 2$ we have the results of [2], [19]. In particular, if $A = X - \mathbf{C}^2$ is a manifold, then $A = \mathbf{P}^1$ and $X = \mathbf{P}^2$. Already in the case $n = 2$, compacti-

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fiable complex homology cells can be different from $\mathbf{C}^2$ as the counterexample in [23] shows.

For $n = 3$, our best result is Theorem 2.4–if $X - A = \mathbf{C}^3$ and $A$ is nonsingular, then $X = \mathbf{P}^3$ and $A = \mathbf{P}^2$ which is linearly embedded in $\mathbf{P}^3$. Our best general result is given in Theorem 1.1. One of the consequences of this result is that if $X - A = \mathbf{C}^n$ and if $A$ is a Kähler manifold, then $X$ is algebraic, $A$ is positively embedded, and $H^*(X, \mathbf{Z}) \cong H^*(\mathbf{P}^n, \mathbf{Z})$, $H^*(A, \mathbf{Z}) \cong H^*(\mathbf{P}^{n-1}, \mathbf{Z})$ as rings. This leads us to expect that if $X - \mathbf{C}^n = A$ is nonsingular, then $X = \mathbf{P}^n$, $A = \mathbf{P}^{n-1}$, and $A$ is linearly embedded in $\mathbf{P}^n$.

We will want to use some criteria for a compact variety to be biholomorphic to $\mathbf{P}^n$. Such results are discussed in [11] and [18]. However, the paper of Kobayashi and Ochiai [13] gives the most convenient hypotheses. We will now state those results of theirs that we shall use.

0.2. THEOREM [13]. *Let $X$ be an $n$-dimensional compact complex manifold, and $L \to X$ a positive line bundle. Suppose either that*

(1) $(c(L))^n([X]) = 1$, *and* $\dim H^0(X, L) \geq n + 1$, *or that*

(2) $c_1(X) = c(L^k)$ *for some integer* $k \geq n + 1$.

*Then $X$ is biholomorphic to* $\mathbf{P}^n$.

In this statement $c(L) \in H^2(X, \mathbf{Z})$ is the (first) Chern class of $L$, and $c_1(X)$ is the first Chern class of $X$. $H^j(X, L)$ is the $j$th (sheaf) cohomology group with coefficients in $L$, and $L^k$ is the tensor product of $L$ with itself $k$ times. This notation will be in effect throughout this paper.

We have had helpful conversations with colleagues, too numerous to mention for thanks. However, we would like to thank David Lieberman for showing us the Euler sequence on $\mathbf{P}^n$.

**1. General results on compactifications of $\mathbf{C}^n$.** Before we state the results of this section we make some remarks about $\mathbf{P}^n$. Let $T$ be the tangent bundle of $\mathbf{P}^n$, and $\Theta$ its sheaf of holomorphic sections. Let $\mathcal{O}(1)$ be the sheaf of holomorphic sections of the hyperplane bundle $H$ on $\mathbf{P}^n$, and let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbf{P}^n$. We note that the space of sections of $\mathcal{O}(1)$ over $\mathbf{P}^n$ can be identified with the homogeneous linear forms on $\mathbf{P}^n$. Then there is an exact sequence (the Euler sequence) of sheaves

$$0 \to \mathcal{O} \xrightarrow{\beta} (n + 1)\mathcal{O}(1) \xrightarrow{\alpha} \Theta \to 0, \tag{1}$$

and a corresponding exact sequence of bundles

$$0 \to \mathbf{C} \to (n + 1)H \to T \to 0. \tag{2}$$

In these sequences, $(n + 1)\mathcal{O}(1)$ and $(n + 1)H$ are $(n + 1)$-fold direct sums, and $\mathbf{C}$ is the trivial bundle. The maps $\alpha$ and $\beta$ are defined as follows. Let $[z_0, \ldots, z_n]$ be homogeneous coordinates on $\mathbf{P}^n$. Then vector fields of the form $\sum_{i=0}^n l_i(z)(\partial/\partial z_i)$ descend to $\mathbf{P}^n$ exactly when the $l_i(z)$ are homogeneous linear forms. The so defined vector field vanishes at $[\zeta_0, \ldots, \zeta_n]$ exactly when $(l_0(\zeta), \ldots, l_n(\zeta)) = (\lambda\zeta_0, \ldots, \lambda\zeta_n)$ for some $\lambda \in \mathbf{C}$. So the map $\alpha$ is defined by

$$\alpha(l_0(z), \ldots, l_n(z)) = \sum_{i=0}^{n} l_i(z)\left(\frac{\partial}{\partial z_i}\right) = Y$$

where we consider $Y$ as a vector field on $\mathbf{P}^n$. The map $\beta$ is defined by

$$\beta(1) = (z_0, \ldots, z_n) \in (n+1)\Gamma(\mathbf{P}^n, \mathcal{O}(1)).$$

The point of this discussion is that in (2) we have a surjection of a sum of $n + 1$ *line* bundles onto the tangent space of $\mathbf{P}^n$. This is our motivation for assumptions (h) and (i) in the theorem of this section.

This theorem summarizes our general knowledge about compactifications of $\mathbf{C}^n$.

1.1. THEOREM. *Let $X$ be a connected compact complex manifold with no exceptional subvarieties, and let $A$ be a Kähler submanifold such that $X - A$ is a complex homology cell. Then*:

(I) *$X$ is projective algebraic and $A$ is a positively embedded hypersurface.*

(II) *There is a continuous mapping $\psi$: $X \to \mathbf{P}^n$ taking $A$ into a (linearly embedded) hyperplane $H \cong \mathbf{P}^{n-1}$ which induces ring isomorphisms*

$$\psi^*: H^*(\mathbf{P}^n, \mathbf{Z}) \to H^*(X, \mathbf{Z}),$$

*and*

$$\psi_A^* = (\psi|_A)^*: H^*(\mathbf{P}^{n-1}) \to H^*(A, \mathbf{Z}).$$

(*Thus $A$ is a connected, $(n-1)$-dimensional complex manifold.*)

(III) *If any one of the following holds, then the map of* (II) *may be taken to be a biholomorphism*:

(a) $n \leqslant 5$.

(b) *$X$ is homeomorphic to $\mathbf{P}^n$.*

(c) *$A$ is homeomorphic to $\mathbf{P}^{n-1}$.*

(d) *The line bundle $[A]$ of the divisor of $A$ in $X$ admits at least $n+1$ linearly independent holomorphic sections.*

(e) *The normal bundle $N$ of $A$ for the embedding $A \subset X$ admits at least $n$ linearly independent holomorphic sections.*

(f) *$X$ admits a zero-free meromorphic $n$-form with a pole of order at least $n+1$ along some hypersurface, or with poles of any orders along any $n+1$ distinct hypersurfaces.*

(g) *$A$ admits a zero-free meromorphic $(n-1)$-form with a pole of order at least $n$ along some hypersurface, or with poles of any orders along any $n$ distinct hypersurfaces.*

(h) *There is an exact sequence*

$$0 \to L \to L_0 \oplus \cdots \oplus L_n \to T_X \to 0$$

*where the bundles $L, L_0, \ldots, L_n$ are holomorphic line bundles over $X$. ($T_X$ is the tangent bundle of $X$.)*

(i) *There is an exact sequence*

$$0 \to L \to L_0 \oplus \cdots \oplus L_{n-1} \to T_A \to 0$$

*where the bundles $L, L_0, \ldots, L_{n-1}$ are holomorphic line bundles over $A$. ($T_A$ is the tangent bundle of $A$.)*

(j) *$A$ admits a neighborhood $U$ in $X$ biholomorphic to a neighborhood of the zero section of the normal bundle $N$ of the embedding $A \subset X$.*

(k) $H^1(A, T_A \otimes N^{-\nu}) = 0$, *for all $\nu = 1, 2, \ldots$.*

Proof of (I). Consider the cohomology sequence (over $\mathbf{Z}$) of the pair $(X, A)$:

$$\ldots \to H^{k-1}(A) \to H^k(X, A) \to H^k(X) \xrightarrow{\phi} H^k(A) \to H^{k+1}(X, A) \to \ldots. \tag{3}$$

By assumption, $H^k(X, A) = H_c^k(X - A) = 0$, for $0 \leqslant k \leqslant 2n - 1$. Thus $\mathbf{Z} \cong H^0(X) \cong H^0(A)$, and $A$ is connected. Since $A$ is Kähler, $H^2(A, \mathbf{R}) \neq 0$. By (3),

$$0 \neq H^2(A, \mathbf{R}) \cong H^2(X, \mathbf{R}) \cong H^{2n-2}(X, \mathbf{R}) \cong H^{2n-2}(A, \mathbf{R}).$$

Thus $A$ is a connected complex manifold of dimension $n - 1$. Let $a \in H^2(X, \mathbf{Z})$ be the Poincaré dual of $A$. Let $[A]$ be the complex line bundle on $X$ associated to the divisor $A$. Then the normal bundle of $A$ in $X$ is $[A]|_A = N$. The Chern class of $N$ is $c(N) = \phi(A)$ where $\phi$ is defined by (3). Let $x \in H^{2n-2}(X)$. Then $(x \cdot a)(X) = \phi(x)(A)$ where "$\cdot$" means cup product. Since $\phi$ is an isomorphism, if $x \neq 0$ then $\phi(x) \neq 0$. Thus $x \cdot a \neq 0$ if $x \neq 0$. This proves that $c(N) \neq 0$, and hence $N$ is not topologically trivial. Now we also know that $N$ is not negative (in the sense of Grauert), because in that case $A$ would be exceptional and that is contrary to our assumption. In (II), we will show that $c(N) = \phi(a)$ generates $H^*(A, \mathbf{Z})$. In particular, $H^2(A, \mathbf{Z}) = \phi(a)\mathbf{Z}$. Since $A$ is Kähler and $\phi(a)$ is neither 0 nor negative, $\phi(a)$ must be positive. Accordingly, $N > 0$ and by Theorem 2.4 in Morrow and Rossi [20], $[A] > 0$ on $X$. This proves (I).

Proof of (II). We need to use the Thom-Gysin sequence

$$\ldots \to H^k(\sigma) \to H^{k-1}(A) \xrightarrow{\alpha} H^{k+1}(A) \to H^{k+1}(\sigma) \to \ldots, \tag{4}$$

where $\sigma$ is the boundary of a tubular neighborhood of $A$ (which is the same as the boundary of the normal disk bundle of $A$ in $X$). We claim that $H^k(\sigma) = 0$ for $0 < k < 2n - 1$, and $H^k(\sigma) = \mathbf{Z}$ for $k = 0$ or $2n - 1$. To prove this claim we proceed as follows. Let $D_\varepsilon$ be the $\varepsilon$-disk bundle of $A$ embedded as a tubular neighborhood of $A$ in $X$. Then we have the exact sequence (over $\mathbf{Z}$)

$$\ldots \to H^k(X - D_\varepsilon, \partial D_\varepsilon) \to H^k(X - D_\varepsilon)$$
$$\to H^k(\partial D_\varepsilon) \to H^{k+1}(X - D_\varepsilon, \partial D_\varepsilon) \to \ldots, \tag{5}$$

where $\partial D_\varepsilon$ is the boundary of $D_\varepsilon$, and thus $\sigma = \partial D_\varepsilon$. But $H^k(X, A) \cong H^k(X - D_\varepsilon, \partial D_\varepsilon)$. It follows then that $H^k(X - D_\varepsilon, \partial D_\varepsilon) = 0$ for $0 \leqslant k < 2n$. This implies by (5) that $H^k(X - D_\varepsilon) \cong H^k(\partial D_\varepsilon)$ for $0 \leqslant k < 2n - 1$. Now $\partial D_\varepsilon$ is a connected real $(2n - 1)$-manifold, so $H^k(\partial D_\varepsilon) = \mathbf{Z}$ for $k = 0$ or $2n - 1$. By Lefschetz duality $H_k(X - D_\varepsilon) \cong H^{2n-k}(X - D_\varepsilon, \partial D_\varepsilon)$. Consequently $H_k(X - D_\varepsilon) = 0$ for $0 < k \leqslant 2n$. By the universal coefficient theorem, $H^k(X - D_\varepsilon) = 0$ for $0 < k \leqslant 2n$. It follows then that $H^k(\partial D_\varepsilon) = 0$ if $0 < k < 2n - 1$, and $H^k(\partial D_\varepsilon) \cong Z$ if $k = 0$ or $2n - 1$. In (4), the map $\alpha$ is given by cupping with $\chi(\sigma)$, the Euler class of the circle bundle $\sigma$ over $A$. But $\chi(\sigma) = c(N)$. From what we have just proved, cupping with $c(N)$ defines an isomorphism $\alpha$: $H^k(A) \to H^{k+2}(A)$ as long as $0 \leqslant k \leqslant 2n - 3$. Now $H^0(A) \cong \mathbf{Z}$ by the proof of (I); and hence $H^{2k}(A) = [c(N)]^k\mathbf{Z}$, for $0 \leqslant k \leqslant n - 1$. In addition, in (4) if $k = 0$, we get

$$0 \to H^1(A) \to H^1(\sigma) \to \dots .$$

Accordingly, $H^1(A) = 0$. Using (4) again, we conclude that $H^{2k+1}(A) = 0$ for $0 \leqslant k \leqslant n - 2$. This proves that $H^*(A, \mathbf{Z}) \cong H^*(\mathbf{P}^{n-1}, \mathbf{Z})$ as rings where the generator of $H^*(A, \mathbf{Z})$ is $c(N) \in H^2(A, \mathbf{Z})$.

Next, if $a$ is the Poincaré dual of $A$ in $X$, then $c([A]) = a$; and by Poincaré duality,

$$(a^{n-1} \cdot a)(X) = (a^{n-1})(A) = 1.$$

Thus $H^*(X, \mathbf{Z}) \cong H^*(\mathbf{P}^n, \mathbf{Z})$ as rings, and the generator of this truncated polynomial ring is $c([A]) = a$. Let $\mathbf{P}^\infty$ be infinite projective space, which is the classifying space for complex line bundles. We think of $\mathbf{P}^\infty$ as $\bigcup_{n=0}^{\infty}\mathbf{P}^n$, with $\mathbf{P}^n \subset \mathbf{P}^{n+1}$. Let $\Gamma$ be the canonical line bundle on $\mathbf{P}^\infty$ and let $\hat{\psi}$: $X \to \mathbf{P}^\infty$ be a continuous map such that $\hat{\psi}^*\Gamma = [A]$. Then by the cellular approximation theorem there is a map $\psi$: $X \to \mathbf{P}^\infty$ which is homotopic to $\hat{\psi}$ and so that $\psi(X) \subset \mathbf{P}^n$ and $\psi(A) \subset \mathbf{P}^{n-1}$. Then $\psi$ and $\psi_A = \psi|_A$ are continuous maps inducing isomorphisms $\psi^*$: $H^*(\mathbf{P}^n, \mathbf{Z}) \cong H^*(X, \mathbf{Z})$ and $\psi_A^*$: $H^*(\mathbf{P}^{n-1}, \mathbf{Z}) \cong H^*(A, \mathbf{Z})$.

Before proving part (III) we note some further properties of $X$ and $A$:

($\alpha$) $H^i(X, \mathcal{O}_X) = H^i(A, \mathcal{O}_A) = 0 \; \forall i > 0$;

($\beta$) every holomorphic line bundle $L$ on $X$ (respectively on $A$) is analytically isomorphic to $[A]^k$ (respectively to $N^k$) for some integer $k$;

($\gamma$) in particular, for $K_X$, $K_A$ the canonical bundles we have isomorphisms $K_X \cong [A]^{-c_1}$, $K_A \cong N^{-c_1+1}$ for some integer $c_1 > 1$; and

($\delta$) $H^i(X, L) = 0 \; \forall i \neq 0, n$ for every line bundle $L$ on $X$, and $H^i(A, L) = 0 \; \forall i \neq 0, n - 1$ for every line bundle $L$ on $A$.

Proof of ($\alpha$)–($\delta$). Since $X$ is projective algebraic (in particular, Kähler) by part (1), ($\alpha$) for $X$ follows from the Hodge-Kähler bidegree decomposition

$$H^i(X, \mathbf{C}) \cong \bigoplus_{p+q=i} H^p(X, \Omega^q),$$

Hodge-Kodaira duality

$$H^i(X, \mathcal{O}_X) = H^0(X, \Omega^i),$$

and the conclusion $H^*(X, \mathbf{Z}) = H^*(\mathbf{P}^n, \mathbf{Z})$ of part (II); and similarly for $A$. Also from part (II) we have that $H^2(X, \mathbf{Z})$ (respectively $H^2(A, \mathbf{Z})$) is isomorphic to the integers and generated by the Chern class of the bundle $[A]$ (respectively $N$). Thus ($\beta$) for $X$ follows from ($\alpha$) via the exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X),$$

and similarly for $A$. For ($\gamma$), we have in any case $K_X \cong [A]^{-c_1}$ for some integer $c_1$ by ($\beta$), whence $K_A \cong K_X|_A \otimes [A]|_A \cong N^{-c_1+1}$ by adjunction. Now if $c_1 \leqslant 0$ then $K_X$ would have a nontrivial section (namely $\sigma^{c_1}$ for $\sigma$ the canonical section of the positive divisor $A$); but this contradicts, via Serre duality, the conclusion $H^n(X, \mathcal{O}_X) = 0$ of ($\alpha$). Similarly $c_1 = 1$ implies that $K_A$ is trivial, contradicting $H^{n-1}(A, \mathcal{O}_A) = 0$. Thus $c_1 > 1$ as claimed.

Finally, if $L \cong [A]^k$ is any line bundle on $X$ then, since $[A]$ is positive, ($\gamma$) shows that either $L$ is negative (if $k < 0$) or else $K_X \otimes L^{-1}$ is negative (if $k > -c_1$). Thus ($\delta$) for $X$ follows from Kodaira vanishing and Serre duality, and similarly for $A$.

Proof of (III). Given part (I), (a) is proved in Van de Ven [**25**, Theorem 4.3]. Since $K_X$ is negative by ($\gamma$), (b) is proved in Hirzebruch and Kodaira [**11**] under the slightly stronger hypothesis that $X$ is *diffeomorphic* to $\mathbf{P}^2$. The proof, however, uses only the fact that $X$ has the right rational Pontrjagin classes, and these are shown to be only topological invariants by Novikov [**22**] (see also [**18**]). Similarly, (c) implies at least that $A$ is biholomorphic to $\mathbf{P}^{n-1}$, so that (c) implies, for instance, (i), which will be dealt with shortly.

(d) and the implication (e) $\Rightarrow A \cong \mathbf{P}^{n-1}$ are (given (I) and (II)) direct applications of Theorem 0.2(1), while (f) and (g) follow similarly from 0.2(2). (To see this last fact, let $\omega^n$ be a zero-free meromorphic $n$-form with pole set $P = \sum_{i=1}^k P_i$, $P_i$ irreducible. Then if $m_i$ is the order of the pole along $P_i$, we have, by definition,

$$K_X \cong \bigotimes_{i=1}^{k} [P_i]^{-m_i}.$$

But from ($\beta$), each of the bundles $[P_i]$ is isomorphic to $[A]^{l_i}$ for some integer $l_i$, and $l_i > 0$ lest the bundle $[P_i]$ be a negative or trivial line bundle on a compact manifold admitting a nontrivial analytic section. Thus

$$K_X \cong [A]^{-\sum_{i=1}^k l_i m_i}.$$

Now the two alternate hypotheses of (f) imply, respectively, that one of the $m_i \geqslant n+1$ or that $k \geqslant n+1$. In either case we have $K_X \cong [A]^{-c_1}$ for $c_1 \geqslant n+1$. Now we can apply Theorem 0.2(1). The proof of (g) $\Rightarrow A \cong \mathbf{P}^{n-1}$ is identical, substituting the normal bundle $N$ for the bundle $[A]$.)

We shall now show (h) $\Rightarrow$ (j), (i) $\Rightarrow$ (k) $\Rightarrow$ (j), and finally (j) $\Rightarrow$ the conclusion $X \cong \mathbf{P}^n$.

(h) $\Rightarrow$ (j). Consider the mapping $\phi: A \to N$ given by the zero section. We want to extend $\phi$ to a mapping of a neighborhood of $A$ in $X$ isomorphically onto a neighborhood of $\phi(A)$ in $N$. By Griffiths [**8**, Proposition 1.3], the obstructions for a *formal* extension are elements of $H^1(A, T_X|_A \otimes N^{-\nu})$, $\nu = 1, 2, \ldots,$ for $T_X$ the tangent bundle to $X$. But since $A$ is *positively* embedded, such a formal extension always converges in a neighborhood [**8**, Theorem II(i)]. Notice we may assume $n > 5$ by (a). Thus it suffices to show

$$H^1(A, T_X|_A \otimes N^{-\nu}) = 0, \quad \forall \nu > 0. \tag{6}$$

Let

$$0 \to L \to L_0 \oplus \cdots \oplus L_n \to T_X \to 0$$

be the exact sequence of (h). Now restrict this sequence to $A$ and tensor with $N^{-\nu}$, where $N$ is the normal bundle of $A$ in $X$. Taking cohomology, we get

$$\ldots \to H^1(A, L_0 \otimes N^{-\nu}) \oplus \cdots \oplus H^1(A, L_n \otimes N^{-\nu})$$
$$\to H^1(A, T_X|_A \otimes N^{-\nu}) \to H^2(A, L \otimes N^{-\nu}) \to \ldots.$$

By property ($\delta$) above, the first and last groups vanish provided $\dim(X) = n > 3$. As remarked above we may assume $n > 5$ by (a). Consequently, (6) is proved and, accordingly, (h) implies (j).

Next we show (i) $\Rightarrow$ (k) $\Rightarrow$ (j). By (i) we have the exact sequence

$$0 \to L \to L_0 \oplus \cdots \oplus L_{n-1} \to T_A \to 0.$$

As above we can conclude that (k) is true, i.e., $H^1(A, T_A \otimes N^{-\nu}) = 0$ for $\nu > 0$. (In fact we notice that we have no need to restrict $\nu$ in either case.) Now we have

$$0 \to T_A \to T_X|_A \to N \to 0.$$

Tensoring with $N^{-\nu}$ and using (k) and ($\delta$) yields $H^1(A, T_X|_A \otimes N^{-\nu}) = 0$. We have already seen that this implies (j).

(j) $\Rightarrow X \cong \mathbf{P}^n$. Assume (j): $\exists$ a neighborhood $U$ of $A$ in $X$ and a mapping $\phi$: $U \to N$, $N$ the normal bundle, which is a biholomorphism onto its image $\phi(U)$ and which maps $A$ onto the zero section $A_0$. Let $\tilde{N} \supset N$ be the associated projectivized bundle and denote by $A_\infty = \tilde{N} - N$ the infinity section. Since $N$ is positive the dual bundle $\tilde{N} - A_0 \to A_\infty$ is negative, so $A_\infty$ is exceptional in $\tilde{N}$. Let $\pi$: $\tilde{N} \to Y$ blow down $A_\infty$. That is, $Y$ is the topological space $\tilde{N}/A_\infty$ together with the unique normal analytic structure for which the quotient map $\pi$ is analytic (Narasimhan [**21**]).

Consider the two irreducible normal spaces $X - A$ and $Y - \pi(A_0)$. These are both obviously Stein, since $A$, respectively $\pi(A_0)$, is positively embedded respectively in $X$ and $Y$ and neither $X - A$ nor $Y - \pi(A_0)$ has positive dimensional compact subvarieties. Furthermore these spaces are "isomorphic at infinity"; namely, the map $\pi \circ \phi$: $U - A \to Y - \pi(A_0)$ is a biholomorphism of $U - A$ onto its image, an open set in $Y - \pi(A_0)$ whose complement is compact. By [9, Theorems VII C10 and V1I D4], this map extends to a biholomorphic mapping from $X$ to $Y$.

We conclude from this argument that $Y$ is biholomorphic to the complex *manifold* $X$. In particular, the point $y = \pi(A_0)$ is nonsingular. But Moisheson [17, Theorem 1, p. 139] shows that the only exceptional submanifold which collapses to a *regular* point is $\mathbf{P}^{n-1}$, and that the normal bundle to the embedding must be the tautological (Hopf) bundle. We have by duality then that $A \cong A_0 \cong \mathbf{P}^{n-1}$ and that $N \to A$ is isomorphic to the hyperplane bundle $H \to \mathbf{P}^{n-1}$.

Finally, consider the sheaf sequence

$$0 \to \mathcal{O}_X \xrightarrow{\chi} [\mathbf{A}] \to [\mathbf{A}]|_A \to 0$$

where $\chi$ is multiplication by the canonical section of the divisor $A$. Passing to cohomology this gives

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, [A]) \to H^0(A, N) \to H^1(X, \mathcal{O}_X) \to \cdots.$$

But $H^1(X, \mathcal{O}_X) = 0$ by ($\alpha$), dim $H^0(A, N) = n$ by the conclusion $(N \to A) \cong (H \to \mathbf{P}^{n-1})$, and dim $H^0(X, \mathcal{O}_X) = 1$ by compactness. We conclude that we are back in case (d)–dim $H^0(X, [A]) = n + 1$–and the desired result $X \cong \mathbf{P}^n$ follows as before from Kobayashi and Ochiai. Indeed, it is easy to check that the $n + 1$ sections $\sigma_0, \ldots, \sigma_n$ of the bundle $[A]$ map $X$ biholomorphically onto $\mathbf{P}^n$, $A$ onto a hyperplane $\Pi \cong \mathbf{P}^{n-1}$, and $X - A$ onto $\mathbf{P}^n - \Pi \cong \mathbf{C}^n$, via $x \to$ the point in $\mathbf{P}^n$ with homogeneous coordinates $[\sigma_0(X), \ldots, \sigma_n(x)]$.

**2. The case $n = 3$.** In this section we give our best results on compactifications of $\mathbf{C}^3$. Below, a *surface* is any reduced irreducible two-dimensional analytic space, singularities allowed. An algebraic surface (possibly singular) is *rational* if it is birationally equivalent to $\mathbf{P}^2$. A nonsingular surface $S$ is *ruled* over a curve $C$ if there is a holomorphic mapping $\pi$: $S \to C$ whose generic fibre is a nonsingular rational curve. If $S$ is a (possibly singular) surface we will use the traditional notation for its numerical invariants: $q =$ dim $H^1(S, \mathcal{O}_S)$, the irregularity; $p_g =$ dim $H^2(S, \mathcal{O}_S)$, the geometric genus; $b_i =$ dim $H^i(S, \mathbf{R})$, the $i$th Betti number; $b^+, b^-$, respectively, the dimensions of the positive and negative eigenspaces of the cup product pairing $H^2(S, \mathbf{R}) \times H^2(S, \mathbf{R}) \to \mathbf{R}$; and if $S$ is nonsingular, $P_m =$ dim $H^0(S, K_S^m)$, the $m$th plurigenus, $K_S$ the canonical bundle. For $F$ and $G$ line bundles on $S$ we will write $(F \cdot G)$ for the integer $[c(F) \cup c(G)][(S)]$, $c$ the Chern class map, $(S)$ the fundamental oriented 4-cycle of $S$, and similarly $(C \cdot D)$ for two divisors $C$ and $D$, etc.

2.1. THEOREM. *Let $X$ be a nonsingular analytic compactification of $\mathbf{C}^3$ such that the analytic set $A = X - \mathbf{C}^3$ has only isolated singularities. Suppose*

*$b_3(X) \neq 1$. Then $X$ is projective algebraic and $A$ is birationally equivalent to a ruled surface over a curve of genus $g = \frac{1}{2}b_3(X)$.*

PROOF. Since $\mathbf{C}^3$ is connected at infinity, $A = X - \mathbf{C}^3$ is connected. By holomorphic convexity of $\mathbf{C}^3$ and Hartogs' theorem it is clear that $A$ has pure dimension 2. Since the singularities of $A$ are isolated and $A$ is contained in a 3-fold, then $A$ is normal, hence locally irreducible, hence irreducible. As in the general case the standard sequence

$$\cdots \to H_c^i(\mathbf{C}^3, \mathbf{Z}) \to H^i(X, \mathbf{Z}) \xrightarrow{\phi} H^i(A, \mathbf{Z}) \to H_c^{i+1}(\mathbf{C}^3, \mathbf{Z}) \to \ldots$$

shows that the natural map $\phi$: $H^i(X, \mathbf{Z}) \to H^i(A, \mathbf{Z})$ induced by the inclusion of $A$ into $X$ is an isomorphism for all $i \leqslant 4$. When $i = 5$, we find $H^5(X, \mathbf{Z}) \subset H^5(A, \mathbf{Z}) = 0$. Thus $H^5(X, \mathbf{Z}) = H^5(A, \mathbf{Z}) = 0$. By Poincaré duality, $H_1(X, \mathbf{Z}) = 0$. Now the universal coefficient theorem says that $H^1(X, \mathbf{Z})$ is free and is isomorphic to the free part of $H_1(X, \mathbf{Z})$. Thus $H^1(X, \mathbf{Z}) = 0$. Next we claim that $H^2(X, \mathbf{Z}) \cong H^2(A, \mathbf{Z}) \cong \mathbf{Z}$ and that $p(A)$, the Poincaré dual of $A$ considered as an element of $H_4(X, \mathbf{Z})$, generates $H^2(X, \mathbf{Z})$. For, $\mathbf{Z} \cong H^4(A, \mathbf{Z}) \cong H^4(X, \mathbf{Z}) \cong H_2(X, \mathbf{Z})$, where the last isomorphism is by Poincaré duality. But the universal coefficient theorem implies that $H^2(X, \mathbf{Z}) \cong H_2(X, \mathbf{Z}) \oplus T_1$ where $T_1$ is the torsion part of $H_1(X, \mathbf{Z}) = 0$. Thus $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$. To see that $p(A)$ generates $H^2(X, \mathbf{Z})$, notice that $[p(A) \cdot \omega](X) = \omega(A)$ where $\omega \in H^4(X, \mathbf{Z})$ and "$\cdot$" means cup product. Choose $\omega$ so that $\omega(A) = 1$. If $p(A) = kg$, where $g$ generates $H^2(X, \mathbf{Z})$, then $[g \cdot \omega](X) = 1/k$. Hence $k = \pm 1$ and $p(A)$ generates $H^2(X, \mathbf{Z})$. Now we have $b_2(A) = b_2(X) = 1$, and $b_3(A) = b_3(M) \neq 1$. Let $N = [A]|_A$ be the normal bundle of $A$. Then the Chern class $c([A]) = p(A)$, and $\phi(p(A)) = p(A)|_A = c(N)$. Thus $c(N)$ generates $H^2(A, \mathbf{Z})$. In [3] the proof of Proposition 6 shows that these conditions imply that $A$ is algebraic and that $N$ is either positive or negative on $A$ (in that proof we use only that no tensor power of $L$ is topologically trivial). Now, as in Theorem 1.1, the fact that $\mathbf{C}^3$ is Stein implies that $N$ is positive; indeed, $[A]$ is positive on $[X]$. Thus $X$ is projective algebraic.

Since $X$ is algebraic and $b_1(X) = 0$ we again use the Hodge-Kähler theory to conclude that $H^1(X, \mathcal{O}) = 0$. We know also that $H^2(X, \mathbf{Z})$ is generated by $c([A])$ and thus that $H^2(X, \mathcal{O}) = 0$. Now by a result of Kodaira [**15**, Theorem 3], for any compactification $Y$ of $\mathbf{C}^n$, $H^0(Y, K_Y) = 0$. By duality $H^3(X, \mathcal{O}) = 0$ (in our case). We now conclude as we did in ($\delta$) of §1 that $K_X \cong [A]^{-c_1}$ for some $c_1 > 0$.

We want to examine the structure of $A$. Let $\pi$: $\tilde{A} \to A$ be a resolution of the singularities of $A$ with exceptional curve $C = \pi^{-1}$ (singular points) $= \bigcup_{\alpha=1}^{s} C_\alpha$, $C_\alpha$ irreducible. Since $A$ is algebraic, so is $\tilde{A}$. It is easy to show from properties of the maps $\pi_i^*$: $H^i(A, \mathbf{Z}) \to H^i(\tilde{A}, \mathbf{Z})$ that $b^+(\tilde{A}) = b^+(A) = 1$, that $b_3(\tilde{A}) = b_3(A)$, and that the canonical bundle on $\tilde{A}$ is given by

$$K_{\tilde{A}} = \pi^*(N)^{-c_1+1} \otimes \bigotimes_{\alpha=1}^{s} [C_\alpha]^{-n_\alpha}$$

for some nonnegative integers $n_\alpha$ [**3**, Corollary 3 and the concluding section]. Hence from Kodaira's equation $b^+ = 2p_g + 1$ [**14**, I, Theorem 3], we have that $p_g(\tilde{A}) = \dim H^0(\tilde{A}, K_{\tilde{A}}) = 0$.

We claim that indeed $P_m(\tilde{A}) = 0\ \forall m > 0$. To see this note that

$$\begin{aligned}\dim H^0(\tilde{A}, K_{\tilde{A}}^m) &= \dim H^0\Big(\tilde{A}, \pi^*(N)^{-m(c_1-1)} \otimes \bigotimes_{\alpha=1}^{s} [C_\alpha]^{-mn_\alpha}\Big)\\ &\leqslant \dim H^0(\tilde{A}, \pi^*(N)^{-m(c_1-1)})\\ &= \dim H^0(A, N^{-m(c_1-1)}),\end{aligned}$$

the last equality holding by normality of $A$. Since $N$ is positive and $c_1 > 0$ this last group vanishes unless $c_1 = 1$. Then surely $K_{\tilde{A}}^m = \otimes_{\alpha=1}^{s}[C_\alpha]^{-mn_\alpha}$ has no section unless all the $n_\alpha = 0$, and in this case $K_{\tilde{A}}$ is trivial, contradicting $P_g(\tilde{A}) = 0$. Thus $P_m(\tilde{A}) = 0$ as claimed.

By the criterion of Enriques (see, e.g., [**14**, IV, Theorem 52] or [**1**, Theorems 4.1 and 5.2]) the only *minimal* algebraic surfaces with all $P_m = 0$ are the projective ruled surfaces (or $\mathbf{P}^2$). Furthermore, any such surface $A'$ has irregularity $q(A')$ equal to the genus of its base. But

$$q(\tilde{A}) = \tfrac{1}{2}b_1(\tilde{A}) = \tfrac{1}{2}b_3(\tilde{A}) = \tfrac{1}{2}b_3(A) = \tfrac{1}{2}b_3(X).$$

Since $q$ is a birational invariant of nonsingular surfaces we conclude that $\tilde{A}$ is obtained, by blowing up points, from a surface $A'$ ruled over a curve of genus $g = \frac{1}{2}b_3(X)$, and hence that $A$ is birationally equivalent to such an $A'$. This completes the proof.

DEFINITION. A normal isolated singular point $x$ of a complex surface $A$ is called *rational* if the stalk at $x$ of the first right derived sheaf $R^1\pi^*\mathcal{O}_{\tilde{A}}$ vanishes for some (hence any) resolution $\pi: \tilde{A} \to A$ of the singularity of $A$ at $x$.

2.2. COROLLARY. *Let $X$ be a nonsingular analytic compactification of $\mathbf{C}^3$ such that the analytic set $A = X - \mathbf{C}^3$ has only isolated rational singular points. Then $X$ is projective algebraic and $A$ is birationally equivalent to the complex projective plane $\mathbf{P}^2$.*

PROOF. Let $\pi: \tilde{A} \to A$ resolve the singularities of $A$, with $\pi^{-1}$ (singular points) $= C$. In general (see [**3**, Lemma 1]) we have a natural exact sequence

$$\begin{aligned}0 \to H^1(A, \mathbf{Z}) &\xrightarrow{\pi_1^*} H^1(\tilde{A}, \mathbf{Z}) \xrightarrow{\mathrm{incl}_1^*} H^1(C, \mathbf{Z}) \xrightarrow{\delta_1} H^2(A, \mathbf{Z})\\ &\xrightarrow{\pi_2^*} H^2(\tilde{A}, \mathbf{Z}) \xrightarrow{\mathrm{incl}_2^*} H^2(C, \mathbf{Z}) \xrightarrow{\delta_2} \mathrm{Tor}(H^3(A, \mathbf{Z})).\end{aligned}$$

Since the singularities of $A$ are rational, $H^1(C, \mathbf{Z}) = 0$. Since also $H^1(A, \mathbf{Z}) = 0$ we conclude that $b_1(\tilde{A}) = b_3(\tilde{A}) = b_3(X) = 0$. Theorem 2.1 now supplies the desired conclusion, since a surface ruled over a rational curve is birationally equivalent to $\mathbf{P}^2$.

REMARK. It will be noted that in the proof we used only a small part ($H^1(C, \mathbf{Z}) = 0$) of the hypothesis of rationality of the singularities. Actually much more can be said about the surface $A$. In particular it can be shown that either $A = \tilde{A} = \mathbf{P}^2$, or $\tilde{A}$ = the rational ruled surface $S_2$, or $\tilde{A}$ is derived from $\mathbf{P}^2$ by blowing up 8 or fewer points and $H^2 \cong (A, \mathbf{Z}) \simeq \mathbf{Z}$ is generated by the Chern class of a nonsingular elliptic divisor. There are only a few such surfaces $A$. They are of interest because among them are examples of singular surfaces $A$ which are cohomology projective planes ($H^*(A, \mathbf{Z}) = H^*(\mathbf{P}^2, \mathbf{Z})$ as graded rings). An example of such a surface is given in [**5**], and they are discussed in detail in [**4**]. It should be emphasized, however, that no compactification of $\mathbf{C}^3$ is known with one of these last-mentioned surfaces at infinity.

2.3. COROLLARY. *Let $X$ be a nonsingular analytic compactification of $\mathbf{C}^3$ with $b_3(X) \neq 1$. Suppose that $A = X - \mathbf{C}^3$ has only isolated singularities and that $A$ admits a nonsingular rational divisor $D$. (That is, $D$ is a Cartier divisor–a locally principal ideal subsheaf $I_D \subset \mathcal{O}_A$–such that* $(\operatorname{supp}(\mathcal{O}_A/I_D), \mathcal{O}_A/I_D) \approx (\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$.) *Then $X$ is biholomorphic either to complex projective space $\mathbf{P}^3$ or to the nonsingular quadric hypersurface $Q_3 \subset \mathbf{P}^4$.*

PROOF. By the theorem and its proof $X$ is algebraic, the line bundle $[A]$ is positive on $X$, and $K_X \cong [A]^{-c_1}$ for some positive integer $c_1$. Since $D$ is nonsingular and locally principle the implicit function theorem shows that the support of $D$ is contained in the set $A_0$ of regular points of $A$. We have by iterated adjunction

$$K_{A_0} = [A]^{-c_1+1}|_{A_0} = N^{-c_1+1}|_{A_0},$$

and

$$-2 = (K_{A_0} + D) \cdot D = (-c_1 + 1 + d)d(N^2), \tag{1}$$

where $d$, the "degree" of $D$, is the unique integer for which $c([D]) = d \cdot c(N) \in H^2(A, \mathbf{Z})$. By (1) $d \neq 0$. Since $[D]$ has a section it cannot be negative. Thus $d > 0$. Next, let $\omega$ be a Kähler form which represents $c([A])$. Since $[A]$ is positive, $\omega^2$ is a volume form for complex 2-dimensional subvarieties of $X$. Thus $\int_A \omega^2 > 0$. But this implies that $(N^2) > 0$. Taken together, (1), $d > 0$, $c_1 > 0$, and $(N^2) > 0$ have only the following solutions:

(i) $(N^2) = d = 1$, $c_1 = 4$,
(ii) $(N^2) = 1$, $d = 2$, $c_1 = 4$,
(iii) $(N^2) = 2$, $d = 1$, $c_1 = 3$.

In cases (i) and (ii), Theorem 0.2(2) implies that $X \cong \mathbf{P}^3$. In case (iii) Kobayashi and Ochiai [**13**] have a similar result for quadrics which allows us to conclude that $X \cong Q_3$.

REMARK. If $X$ above is isomorphic to $\mathbf{P}^3$ then necessarily the isomorphism maps $A$ isomorphically onto a hyperplane $H$, for only such hypersurfaces generate $H^2(\mathbf{P}^3, \mathbf{Z})$. In case (i) $D$ is a projective line in $H$ and in case (ii) $D$ is a nonsingular curve of degree 2 in $H \cong \mathbf{P}^2$. If $\mathbf{X} \cong Q_3$, then $A$ is the intersection of $X$ with a tangent hyperplane $H$ in $\mathbf{P}^4$ and is biholomorphic to the space obtained from the rational ruled surface $S_2$ by blowing down the zero section, and $D$ is the intersection of $A$ with a projective plane $P \subset H$ missing the singular point.

2.4. THEOREM. *Let $X$ be a nonsingular analytic compactification of $\mathbf{C}^3$ such that $X - \mathbf{C}^3$ is nonsingular. Then $X$ is biholomorphic to $\mathbf{P}^3$, and $X - \mathbf{C}^3 = \mathbf{P}^2$ is linear subspace.*

PROOF. By Corollary 2.2, $A$ is birationally equivalent to $\mathbf{P}^2$. Since $A$ is nonsingular and $b_2(A) = 1$, $A$ must *be* $\mathbf{P}^2$, and the positive generator $N = [A]|_A$ of $H^1(A, \mathcal{O}^*) \cong \mathbf{Z}$ must be the standard hyperplane bundle. But then $K_A = N^{-3}$, and from the relation

$$K_A = (K_X \otimes [A])|_A$$

and the fact that restriction of $[A]$ to $A$ induces an isomorphism $H^1(X, \mathcal{O}^*) \cong H^1(A, \mathcal{O}^*)$, we conclude that

$$K_X = [A]^{-4}.$$

The desired result now follows from the theorem of Kobayashi and Ochiai as before.

2.5. REMARK. Theorem 2.4 also follows from Theorem 1.1(III)(a), and Corollary 2.2.

**3. Open questions.** We close by listing some of the open questions which the foregoing suggests. These questions are "open" in the sense that we do not know the answers to them, and are listed more or less in order of apparent difficulty. First the case $n = 3$.

1. Is the condition $b_3(X) \neq 1$ in Theorem 2.1 redundant? This would fail to be the case only if $\mathbf{C}^3$ could be compactified by a singular surface derived from an elliptic Hopf surface, or from a surface of type $\mathrm{VII}_0$, for instance one of the kinds recently discovered by Inoue, Hirzebruch, and Bombieri **[12]**, **[1**, part 6**]**. Of course if $X$ is assumed Kähler, then $b_3(X)$ is even.

2. If the surface $A$ at infinity has only rational singularities (which must be rational double points by local embedding dimension) then Corollary 2.2 shows that $b_3(X) = 0$ and that $A$ is derived from a rational ruled surface. Conversely, does $b_3(X) = 0$ imply that $A$ has only rational singularities? More generally, is it true that if $X$ is any compactification of $\mathbf{C}^3$ by a normal surface $A$ then the sheaf $R^1\pi_* I_C$ vanishes identically, where $I_C \subset \mathcal{O}_{\tilde{A}}$ is the ideal sheaf of the exceptional curve $C$ in a resolution $\pi: \tilde{A} \to A$ of singularities? (It is easy to show that at worst $R^1\pi_* I_C$ is supported on a single point and has complex dimension 1 there. It follows that if $b_3(X) = 0$ and $A$ has only isolated singularities, then $X$ is algebraic, $A$ is rational, and each singular point except at most one is a rational double point. That one exception, if it occurs, is a topologically rational ($H^1(C, \mathbf{R}) = 0$ for $C \subset \tilde{A}$ as above) mini-

mally elliptic double or triple point; these are enumerated in [**16**].)

3. Throughout §2 can $\mathbf{C}^3$ be replaced by any Stein homology cell? (The answer is yes in Corollaries 2.2 and 2.4, but in Theorem 2.1 it is not clear that the canonical bundle on $X$ is negative under weaker hypotheses on $X - A$.)

4. We have considered compactifications of $\mathbf{C}^3$ for which the surface at infinity has only isolated singularities. The classical statement of the compactification problem [**10**, problem 27] inquires more generally about compactifications $X$ for which $A = X - \mathbf{C}^3$ is *irreducible* (i.e., $b_2(X) = 1$). Are these equivalent? That is, does $A$ irreducible imply $A$ normal? (Cf. the result in dimension 2: If $X$ is a compactification of $\mathbf{C}^2$ with $b_2(X) = 1$ then $A$ is nonsingular–and hence $X \cong \mathbf{P}^2$ (Remmert and Van de Ven [**24**]).)

5. In fact only two compactifications $X$ of $\mathbf{C}^3$ with $b_2(X) = 1$ are known: $\mathbf{P}^3$ and $Q_3$ (cf. [**25**, Theorems 4.3 and 4.4] as well as Corollary 2.3 above). If these are the only ones, Questions 1–4 above become moot. Is this the case?

6. Finally, suppose we remove all restrictions on the analytic set $A$ at infinity. In dimension 2 we know that every compactification $X$ of $\mathbf{C}^2$ is a rational projective algebraic surface, and each component of $A = X - \mathbf{C}^2$ is a locally irreducible rational curve. The appropriate analogue in dimension 3 seems to be the following:

3.1. Conjecture. Let $X$ be a compactification of $\mathbf{C}^3$. Put $A = X - \mathbf{C}^3$. Then

(i) $X$ is a projective algebraic 3-fold with $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$ and $H^0(X, K_X^m) = 0 \ \forall m > 0$, and

(ii) each component of $A$ is a (possibly singular) locally irreducible surface birationally equivalent to a projective ruled surface.

The status of this conjecture is this: If $A$ is assumed normal and if $b_3(X) \neq 1$, this is precisely our Theorem 2.1. $H^0(X, K_X^m) = 0$ is always true by Kodaira's result [**15**, Theorem 3], while $H^i(X, \mathcal{O}_X) = 0$ is true at least in the algebraic (equivalently, Kähler) case. The question of algebraicity in the general case is completely open. As for the structure of $A$, little is known except conversely that projective surfaces ruled over curves of any genus can occur, as is seen by modifying $\mathbf{P}^3$ by monoidal transformations centered at points and on nonsingular curves in (the proper transforms of) a hyperplane.

We proceed now to compactifications of $\mathbf{C}^n$, $n > 3$. We are guided by the conjecture mentioned in the introduction and suggested by our Theorem 1.1.

3.2. Conjecture. $\mathbf{P}^n$ is the only compactification of $\mathbf{C}^n$ by a submanifold.

A proof of this conjecture would appear to have two steps, which we list as separate questions.

7. Is every such compactification of $\mathbf{C}^n$ projective algebraic? (If not, is it at least Moisheson?)

8. Is Conjecture 3.2 true with the additional assumption of algebraicity?

Finally, we mention some connections between these questions and other topics of independent interest in complex analysis.

3.3. *Relation to the local theory of singular points of analytic spaces*. In part (III)(j) of Theorem 1.1 we exploited the natural duality between the concepts pseudo-concave and pseudo-convex to relate global properties of a compactification $X$ of $\mathbf{C}^n$ to local properties of the isolated singular point obtained by blowing down the infinity section in the projectivized normal bundle to the hypersurface $A$ at infinity. Dual in this sense to Question 7 is

9. Let $X$ be a complex manifold, $A \subset X$ an exceptional submanifold (i.e., $A$ can be collapsed to a point). Suppose that the normal bundle to the embedding is not topologically trivial. Then is $A$ necessarily projective algebraic?

And dual to 8:

10. Let $X$ be a projective variety, nonsingular except possibly at a single point $x \in X$. Suppose that

(i) $X$ is topologically nonsingular at $x$, and

(ii) $X$ admits a resolution $\pi: \tilde{X} \to X$ of the singularity at $x$ with $\pi^{-1}(x)$ nonsingular.

Then is $X$ necessarily (analytically) nonsingular at $x$? (This question is closely related to the possible existence of nonstandard complex structures on complex projective space.)

3.4. *Relation to value distribution theory*. By definition a compactification of $\mathbf{C}^n$ is a 1-1 mapping $\phi$ of $\mathbf{C}^n$ into a compact manifold $X$ such that $\phi$ "behaves nicely at infinity". This is precisely the situation studied in the Nevanlinna theory. Especially relelvant seems the topic of order of growth of analytic objects as in **[6]** et al. But except for Kodaira's proof in **[15]** of rationality of compactifications of $\mathbf{C}^2$, only algebro-topological and differential-geometric techniques have thus far been brought to bear on the subject of this paper. Perhaps further progress could be made along the following lines.

11. Let $X \subset \mathbf{P}^N$ be a compactification of $\mathbf{C}^n$ which is a nonsingular projective variety. Suppose that the set $A = X - \mathbf{C}^n$ is the intersection of $X$ with a hyperplane in $\mathbf{P}^N$ and consists of nonsingular components meeting transversally. It is easy to show that there exists a meromorphic $n$-form $\omega$ on $X$ with no zeros or poles on $\mathbf{C}^n$. A trivial Nevanlinna type argument shows that $\omega$ cannot be holomorphic (integrate $\omega \wedge \bar{\omega}$ over large balls in $\mathbf{C}^n$ to obtain a contradiction). Can a refinement of this argument show that the order of the pole must be "large" on some component of $A$ (cf. part (f) of Theorem 1.1)?

12. Let $V$ be an affine variety (e.g. $V = \mathbf{C}^n$), $X$ a nonsingular projective variety, which is an *analytic* compactification of $V$. What is the smallest number $\lambda$ (if any) such that among all such mappings $V \to X$ the image of $V$ has order of growth $\lambda$ at infinity. This can be regarded as a GAGA type question related to the conjecture of Serre recently proved by Quillen regarding algebraic vector bundles on $\mathbf{C}^n$. In particular one can ask, are there any

other algebraic structures than the standard one on the open complex manifold $\mathbf{C}^n$?

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