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AUTOMORPHISMS OF $\mathrm{GL}_n(R)$

BY

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ABSTRACT. Let R denote a commutative ring having 2 a unit. Let $\mathrm{GL}_n(R)$ denote the general linear group of all $n \times n$ invertible matrices over R. Let Λ be an automorphism of $\mathrm{GL}_n(R)$. An automorphism Λ is "stable" if it behaves properly relative to families of commuting involutions (see §IV). We show that if R is connected, i.e., 0 and 1 are only idempotents, then all automorphisms Λ are stable. Further, if $n > 3$, R is an arbitrary commutative ring with 2 a unit, and Λ is a stable automorphism, then we obtain a description of Λ as a composition of standard automorphisms.

I. Introduction. Let R be a commutative ring and $\mathrm{GL}_n(R)$ be the general linear group of n by n invertible matrices over R. Let Λ be a group automorphism of $\mathrm{GL}_n(R)$. This paper concerns the problem of obtaining a description of Λ in terms of standard classes of automorphisms.

The standard automorphisms of $\mathrm{GL}_n(R)$ may be grouped into three classes.

(a) Let σ be a ring automorphism of R. Then σ induces an automorphism $A \to A^\sigma$ of $\mathrm{GL}_n(R)$, where if $A = [a_{ij}]$ then $A^\sigma = [\sigma(a_{ij})]$. This automorphism $A \to A^\sigma$ is usually composed with an inner automorphism described as follows: Suppose S is a fixed proper extension of R (see §II) and Q in $\mathrm{GL}_n(S)$ satisfies $Q^{-1}\mathrm{GL}_n(R)Q \subset \mathrm{GL}_n(R)$. Then $A \to Q^{-1}AQ$ for A in $\mathrm{GL}_n(R)$ is an automorphism of $\mathrm{GL}_n(R)$. The composition of these automorphisms is denoted by $\Phi_{\langle Q,\sigma\rangle}$, i.e.,

$$\Phi_{\langle Q,\sigma\rangle}(A) = Q^{-1}A^\sigma Q$$

for A in $\mathrm{GL}_n(R)$. We call $\Phi_{\langle Q,\sigma\rangle}$ a *σ-inner automorphism*.

(b) Suppose e is an idempotent of R and $1 = e + \bar{e}$. This idempotent induces a natural decomposition of $\mathrm{GL}_n(R) = \mathrm{GL}_n(R_1) \times \mathrm{GL}_n(R_2)$ where $R_1 = Re$, $R_2 = R\bar{e}$. Let $A = \langle A_1, A_2\rangle$ denote the decomposition of A in $\mathrm{GL}_n(R)$ relative to this idempotent. The idempotent gives rise to a *transpose-inverse automorphism* Ω_e satisfying

$$\Omega_e(A) = \langle A_1^*, A_2\rangle$$

where $A_1^* = (A_1^{-1})^t$.

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(c) For suitable group morphisms χ: $\mathrm{GL}_n(R) \to \mathrm{Center}(\mathrm{GL}_n(R))$, we have the class of *radial automorphisms* P_χ where

$$P_\chi(A) = \chi(A)A.$$

The "automorphism problem" may be stated as follows, "Given an automorphism Λ of $\mathrm{GL}_n(R)$, does there exist suitable σ, Q, e and χ such that

$$\Lambda = \Omega_e \circ \Phi_{\langle Q,\sigma\rangle} \circ P_\chi?"$$

The history of this problem is discussed in [6]–[9]. In this paper we show the automorphism problem has an affirmative answer if R is a connected commutative ring having 2 a unit and $n \geqslant 3$. More generally, we show that if R is any commutative ring with 2 a unit, $n \geqslant 3$ and Λ is a "stable" (see §IV) automorphism, then Λ has the above form. We show that over a connected ring all automorphisms are stable and thus deduce the above result for connected rings from the more general theorem.

We conjecture that all automorphisms are stable over any commutative ring. If this is the case, then our arguments would give an affirmative answer to the automorphism problem when $n \geqslant 3$ and 2 is a unit in R. In any earlier version of this paper, we had an incorrect proof of the above conjecture.

We now outline the content of the paper. §II discusses the extension ring S of R which is needed to split the Picard group of R and thus stabilize the form of involutions under the action of an automorphism. §III describes some common elements in $\mathrm{GL}_n(R)$ and some of their properties. In §III we also state Suslin's theorem on the normality of $E_n(R)$ in $\mathrm{GL}_n(R)$. It is Suslin's result which allows the extension of the solution of the automorphism problem from $E_n(R)$ to $\mathrm{GL}_n(R)$. A short proof of Suslin's theorem is supplied in the Appendix. §IV describes what is meant by a "stable" automorphism and shows that all automorphisms of $\mathrm{GL}_n(R)$ when R is connected are stable. §V employs the "Chinese School" approach to the automorphism problem. For a discussion of this approach see [6] and [7]. In this section it is shown that the automorphism problem has an affirmative answer if Λ is a stable automorphism, $n \geqslant 3$ and R is a commutative ring in which 2 is a unit.

II. Basic concepts and hypotheses. Throughout this paper R will denote a commutative ring having 2 a unit. All modules over R will be assumed to be finitely generated and all unadorned tensors, Hom, GL, etc. are to be interpreted as over R.

Let P be a projective R-module. Then P is said to have *rank m* if for each prime ideal q of R, the localization $R_q \otimes P = P_q$ of P at q is a free R_q-module of dimension m.

If a projective R-module P is of rank 1, then so is $P^* = \mathrm{Hom}(P, R)$ and, further, the evaluation map e: $P \otimes P^* \to R$ by $\mathrm{e}(p \otimes f) = f(p)$ is an isomorphism of R-modules. Hence, if $\mathrm{Pic}(R)$ denotes the set of isomorphism

classes $[P]$ of rank one projective R-modules P, then $\mathrm{Pic}(R)$ is a group under $[P][\bar{P}] = [P \otimes \bar{P}]$ where $[P]^{-1} = [P^*]$. Further, recall if $[P]$ is in $\mathrm{Pic}(R)$, then $\mathrm{End}(P) \simeq R$.

Let $[P]$ be in $\mathrm{Pic}(R)$. A commutative R-algebra S *splits* $[P]$ if $S \otimes P \simeq S$ as S-modules.

For the remainder of this paper we will let S denote a commutative extension of R (that is, R is injected into S and R and S have the same identity 1) such that S splits $[P]$ for all $[P]$ in $\mathrm{Pic}(R)$.

We give several examples of such extensions:

(a) Suppose T is a multiplicative subset of R containing no zero divisors. Set $S = T^{-1}R$ (the ring of fractions of R determined by T). Then S splits $[P]$ for all P in $\mathrm{Pic}(R)$ if:

(i) R is a domain and $R = R - \{0\}$. Here $T^{-1}R$ is the field of fractions of R.

(ii) R is a domain and T is the complement of a prime ideal.

(iii) R is a ring having all rank one projectives free and $T =$ units of R, e.g., R a local ring, a semilocal ring ([**1**, p. 113] shows that projective modules of constant rank over semilocal rings are free), a principal ideal domain, or a polynomial ring over a field in finite number of indeterminants. Here $T^{-1}R = R$.

(iv) T the complement of the zero divisors provided the zero divisors form an ideal. In this case, $S = T^{-1}R$ is local.

(v) R a Noetherian ring and T the complement of the zero divisors. Here $S = T^{-1}R$ may be shown to be semilocal and by the remark in (iii), S splits each $[P]$ in $\mathrm{Pic}(R)$.

In each of the above cases, S is a ring of quotients of R. It is sometimes an advantage to use a ring of quotients (as noted after V.9).

(b) An arbitrary ring R may be embedded in an extension ring S having the above splitting property. A stronger condition is to force $\mathrm{Pic}(S)$ to be trivial, i.e., all rank one projective S-modules are trivial–not only those extended from projective rank one R-modules. Consider $S = \prod_q R_q$ where the product extends over all primes q in $\mathrm{Spec}(R)$ and R_q denotes R localized at q. There is a natural injective morphism $R \to S$ by $r \to \langle r_q \rangle_{q \in \mathrm{Spec}(R)}$ where r_q is the image of r under the canonical morphism $R \to R_q$. Since R_q is local, $\mathrm{Pic}(R_q) = 1$ is trivial. It is shown in [**4**] that Pic: Commutative Rings $\to$ Abelian Groups is an ultra functor. Thus

$$\mathrm{Pic}(S) = \mathrm{Pic}\left(\prod R_q\right) \subseteq \prod \mathrm{Pic}(R_q) = 1.$$

(c) Since by (b) any ring R can be embedded in a ring S which splits each $[P]$ in $\mathrm{Pic}(R)$, it may be worthwhile to search for the most efficient ring for this purpose. In [**3**], Garfinkel described a generic splitting ring for a fixed $[P]$

in $\mathrm{Pic}(R)$. By a different approach we construct a faithfully flat "minimal" extension which splits all $[P]$ in $\mathrm{Pic}(R)$ as follows:

Let P be a rank one projective R-module. Then [**1**, II, §5.3, Theorem 1], there is a finite family $\{f_1, \ldots, f_t\}$ of R with $(f_1, \ldots, f_t) = R$ and P_{f_i} a free R_{f_i}-module for dimension one for $1 \leqslant i \leqslant n$. ($(*)_{f_i}$ denotes localization at the multiplicatively closed set $\{f_i^n | n \geqslant 0\}$.) Let $S_P = R_{f_1} \oplus \cdots \oplus R_{f_t}$. Then [**1**, II, §5.3, Proposition 3] S_P is a faithfully flat extension of R. Further

$$\begin{aligned} S_P \otimes P &\simeq (R_{f_1} \otimes P) \oplus \cdots \oplus (R_{f_t} \otimes P) \\ &\simeq R_{f_1} \oplus \cdots \oplus R_{f_t} \simeq S_P. \end{aligned}$$

Thus, the projective S_P-module $S_P \otimes P$ is free of dimension one and thus in the identity class of $\mathrm{Pic}(S_P)$.

Let $S = \bigotimes_{[P]} S_P$ denote the tensor product over all $[P]$ in $\mathrm{Pic}(R)$. By this we mean the following: The family $\{S_P | [P] \text{ in } \mathrm{Pic}(R)\}$ is a collection of R-algebras. For each finite subset Φ of $\mathrm{Pic}(R)$, let $B_\Phi = \bigotimes_{[P]} S_P$ where $[P]$ extends over Φ. If $\Phi \subseteq \Psi$ where Ψ is a finite subset of $\mathrm{Pic}(R)$, then there is a canonical R-algebra morphism $B_\Phi \to B_\Psi$. Then S is precisely the direct limit $S = \mathrm{inj}\lim B_\Phi$. In particular, S, being a direct limit of faithfully flat R-algebras, is faithfully flat. Let $[P]$ be in $\mathrm{Pic}(R)$. Then

$$S \otimes P \simeq \Big(\bigotimes_{[Q] \neq [P]} S_Q \Big) \otimes S_P \otimes P \simeq \Big(\bigotimes_{[Q] \neq [P]} S_Q \Big) \otimes S_P \simeq S.$$

Thus S splits $[P]$ for each $[P]$ in $\mathrm{Pic}(R)$ and S is a faithfully flat extension of R.

We return to the original setting. Let M be a finitely generated projective R-module. Let $\overline{M} = S \otimes M$. Since M is projective [**2**, p. 279] the canonical morphism $M \to \overline{M}$ induced by $m \to 1 \otimes m$ is injective. Thus, we consider $M \subset \overline{M}$. Concerning endomorphisms, since $\mathrm{End}_R(M)$ is projective, $\mathrm{End}_R(M) \to S \otimes \mathrm{End}_R(M)$ by $\sigma \to 1 \otimes \sigma$ is injective. Further, since M is finitely generated and projective [**2**, p. 282], $S \otimes \mathrm{End}_R(M) \simeq \mathrm{End}_S(S \otimes M) = \mathrm{End}_S(\overline{M})$ under $1 \otimes \sigma \to 1_S \otimes \sigma$. Thus, we consider $\mathrm{End}_R(M) \subset \mathrm{End}_S(\overline{M})$. The invertible S-endomorphisms of $\overline{M}$ are denoted by $\mathrm{GL}_S(\overline{M})$ (the *general linear group*) and

$$\mathrm{GL}_R(M) = \left\{ \sigma \text{ in } \mathrm{GL}_S(\overline{M}) | \sigma M = M \right\}.$$

Let $\overline{M}^* = \mathrm{Hom}_S(\overline{M}, S)$ denote the dual module of $\overline{M}$.

If $M = V$ is a free module with basis $\{b_1, \ldots, b_n\}$ then $\overline{V}$ is free with basis $\{\bar{b}_1, \ldots, \bar{b}_n\}$ where $\bar{b}_i = 1 \otimes b_i$, $1 \leqslant i \leqslant n$. We write $\mathrm{GL}_S(\overline{V})$ (resp., $\mathrm{GL}_R(V)$) as $\mathrm{GL}_n(S)$ (resp., $\mathrm{GL}_n(R)$) when viewed as a group of matrices relative to the basis $\{\bar{b}_1, \ldots, \bar{b}_n\}$ (resp., $\{b_1, \ldots, b_n\}$).

III. Elements of $GL_R(V)$ and $GL_S(\bar{V})$.

(a) *Transvections*. Let V be a free R-space of dimension $n \geqslant 3$. Let S denote the extension of R described in the previous section. Let $\bar{V} = S \otimes V$. Assume 2 is a unit in R.

Let φ: $\bar{V} \to S$ be a surjective S-morphism. Then $\bar{V}$ splits as $\bar{V} \simeq \mathrm{Ker}(\varphi) \oplus S$. If a is in $\bar{V}$ and $\varphi(a) = 0$, define $\tau_{a,\varphi}$: $\bar{V} \to \bar{V}$ by $\tau_{a,\varphi}(x) = x + \varphi(x)a$. The S-linear map $\tau_{a,\varphi}$ is called a *transvection* with *vector* a and *kernel* $H = \mathrm{Ker}(\varphi)$. A vector a in $\bar{V}$ is *unimodular* if the S-submodule Sa of $\bar{V}$ is an S-free summand of $\bar{V}$. If a is unimodular, then $\tau_{a,\varphi}$ is called a *unimodular* transvection.

III.1. Lemma. (a) $\tau_{a,\varphi} = I$ *if and only if* $a = 0$.

(b) $\sigma\tau_{a,\varphi}\sigma^{-1} = \tau_{\sigma a,\varphi\sigma^{-1}}$ *for all* σ *in* $GL_S(\bar{V})$.

(c) $\tau_{a,\varphi}\tau_{b,\varphi} = \tau_{a+b,\varphi}$.

(d) *If* $\tau_{a,\varphi}$ *is unimodular, then* $\tau_{a,\varphi} = \tau_{b,\psi}$ *if and only if there is a unit* t *in* S *with* $ta = b$ *and* $\varphi = t\psi$.

Proof. See (II.1) and (II.3) of [6].

Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis of $\bar{V}$ over S. Let $\bar{B}$ have a dual basis $\bar{B}^* = \{\bar{b}_1^*, \ldots, \bar{b}_n^*\}$ in $\bar{V}^*$ given by $\bar{b}_i^*(b_j) = \delta_{ij}$ (δ = Kronecker delta). An *elementary transvection relative to* $\bar{B}$ is a transvection of the form $\tau_{\lambda b_i, b_j^*}$ $(i \neq j)$ for some λ in R. We denote the group generated by elementary transvections relative to $\bar{B}$ by $E_{\bar{B}}(V)$. Observe, relative to $\bar{B}$, the matrix of $\tau_{\lambda b_i, b_j^*}$ is $I + \lambda E_{ij}$ where I is the identity matrix and E_{ij} is the $n \times n$ matrix having all zeroes except for 1 in the (i, j)-position. We denote the matrix $I + \lambda E_{ij}$ by $T_{ij}(\lambda)$.

If $\bar{C} = \{\bar{c}_1, \ldots, \bar{c}_n\}$ is another basis for $\bar{V}$ and σ: $\bar{V} \to \bar{V}$ is given by $\sigma\bar{b}_i = \bar{c}_i$ then

$$\tau_{\lambda\bar{c}_i, \bar{c}_j^*} = \tau_{\sigma(\lambda\bar{b}_i), \bar{b}_j^*\sigma^{-1}} = \sigma\tau_{\lambda b_i, b_j^*}\sigma^{-1}.$$

Hence $E_{\bar{C}}(\bar{V}) = \sigma E_{\bar{B}}(\bar{V})\sigma^{-1}$.

We next quote a recent startling result by Suslin (see Appendix for a proof).

III.2. Theorem (Suslin). $E_{\bar{B}}(\bar{V})$ *is normal in* $GL_S(\bar{V})$.

Hence, in the above paragraph, $E_{\bar{C}}(\bar{V}) = E_{\bar{B}}(\bar{V})$. Thus we denote $E_{\bar{B}}(\bar{V})$ by only $E(\bar{V})$. If we stress matrices, we use $E_n(S)$ for $E(\bar{V})$.

(b) *Involutions*. An element σ in $GL_S(\bar{V})$ is an *involution* if $\sigma^2 = I$. If σ is an involution, then σ determines a *positive module*

$$P(\sigma) = \{x \text{ in } \bar{V} | \sigma(x) = x\}$$

and a *negative module*

$$N(\sigma) = \{x \text{ in } \bar{V} | \sigma(x) = -x\}.$$

Since 2 is a unit, σ splits the space $\bar{V}$ by $\bar{V} = P(\sigma) \oplus N(\sigma)$ (see [**6**, p. 153]) and, relative to this splitting, σ may be realized as a 2×2 block matrix

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Let P denote a rank one projective R-module. We will say that V is *presented* by P if

$$V \simeq P \oplus \cdots \oplus P \qquad (m \text{ summands}).$$

(Since $S \otimes P \simeq S$, it is clear that the number of summands $m = n$ the dimension of V.) Thus, V is presented by P if $V = P_1 \oplus \cdots \oplus P_n$ where $P_i \simeq P$ for $1 \leqslant i \leqslant n$.

Suppose σ is an involution in $\mathrm{GL}(V)$ and $V = P(\sigma) \oplus N(\sigma)$. Suppose, further, that $P(\sigma)$ and $N(\sigma)$ are presented by a rank one projective P. Then $P(\sigma) \simeq \bigoplus \Sigma_{i=1}^{t} P$ and $N(\sigma) \simeq \bigoplus \Sigma_{j=1}^{s} P$. It is clear that $s + t = n$. We say σ is *P-presented of type* (s, t)—when the context is clear we will simply say σ has type (s, t).

Let $V = P_1 \oplus \cdots \oplus P_n$ be a presentation of V by a rank one projective R-module P. Let Σ_P denote the set of all σ in $\mathrm{GL}(V)$ satisfying $\sigma|_{P_i} = I$ or $\sigma|_{P_i} = -I$ for each P_i, $1 \leqslant i \leqslant n$. Then, the elements of Σ_P are involutions and any two commute. The cardinality $|\Sigma_P| = 2^n$. We call Σ_P the *complete set of involutions* on the P-presentation of V. If $P \simeq R$ and $V = Rb_1 \oplus \cdots \oplus Rb_n$ for $B = \{b_1, \ldots, b_n\}$ a basis, we write Σ_B rather than Σ_R.

III.3. LEMMA. *Let σ and τ be in $\mathrm{GL}_S(\bar{V})$. Suppose σ is an involution. Then $\tau\sigma\tau^{-1}$ is an involution and*

$$P(\tau\sigma\tau^{-1}) = \tau P(\sigma), \quad N(\tau\sigma\tau^{-1}) = \tau N(\sigma).$$

In particular, $\tau\sigma = \sigma\tau$ if and only if $\tau N(\sigma) = N(\sigma)$ and $\tau P(\sigma) = P(\sigma)$.

PROOF. The proof is straightforward.

Lemma III.3 and an induction argument on the cardinality of the set of involutions gives the next lemma.

III.4. LEMMA. *Let $\{\sigma_i\}_{i=1}^{r}$ be a collection of pairwise commuting involutions in $\mathrm{GL}_S(\bar{V})$. Then $V = W_1 \oplus \cdots \oplus W_s$ $(s \geqslant 1)$ where $\sigma_i|_{W_j} = \pm I$ for each i, $1 \leqslant i \leqslant r$, and each j, $1 \leqslant j \leqslant s$.*

Let $V = P_1 \oplus \cdots \oplus P_n$ be a P-presentation of V where P is a rank one projective R-module. Let Σ_P be a complete set of involutions on the P-presentation. As noted, the cardinality of Σ_P is 2^n and, further, the number of involutions of type (s, t) is $\binom{n}{s}$. Finally, employing the isomorphisms between the P_i, it is straightforward to show two involutions in Σ_P of the same type are conjugate under $\mathrm{GL}(V)$.

(c) *Skew-permutations*. Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for $\bar{V}$. Let e in R satisfy $e^2 = 1$. If $1 \leqslant i \leqslant n-1$, let $\eta^e_{i,i+1}$ denote the S-morphism given by

$$\eta^e_{i,i+1}(\bar{b}_k) = e\bar{b}_k \quad \text{for } k \neq i, i+1,$$

$$\eta^e_{i,i+1}(\bar{b}_i) = -\bar{b}_{i+1}, \quad \eta^e_{i,i+1}(\bar{b}_{i+1}) = \bar{b}_i.$$

We call $\eta^e_{i,i+1}$ a *skew-permutation of type* e. Let $H^e_{\bar{B}}$ denote the set of skew-permutations of type e on the basis $\bar{B}$.

IV. Preservation of involutions. We continue the assumptions stated in §II: R is a commutative ring having 2 a unit and S is an extension of R splitting all $[P]$ in $\mathrm{Pic}(R)$. Further, we assume V is a free R-space of dimension $n \geqslant 3$. Let $\bar{V}$ denote $S \otimes V$.

Let Λ: $\mathrm{GL}(V) \to \mathrm{GL}(V)$ be a group automorphism. We say Λ is a *stable* automorphism if, given a basis B of V, here exist rank one projective R-modules P and $\bar{P}$ satisfying $\Lambda(\Sigma_B) = \Sigma_P$ and $\Lambda^{-1}(\Sigma_B) = \Sigma_{\bar{P}}$. That is, Λ is stable if a complete set Σ_B of involutions on a basis B is carried via Λ and Λ^{-1} to complete sets Σ_P and $\Sigma_{\bar{P}}$ of involutions on P- and $\bar{P}$-presentations of V for suitable P and $\bar{P}$. It is easy to check that each of the standard automorphisms is stable. It may be that $\Lambda(\Sigma_B) = \Sigma_P$ implies $\Lambda^{-1}(\Sigma_B) = \Sigma_{\bar{P}}$, however, this was not clear to us. The condition $\Lambda^{-1}(\Sigma_B) = \Sigma_{\bar{P}}$ is necessary in the final step of V.7. We conjecture the automorphisms Λ with $\Lambda(\Sigma_B) = \Sigma_P$ form a group, thus the second condition is unnecessary (see the remark after the proof of IV.1).

Recall a ring R is *connected* if R has only trivial idempotents, namely 0 and 1.

IV.1. THEOREM. *Let R be a connected ring. Then every automorphism* Λ: $\mathrm{GL}(V) \to \mathrm{GL}(V)$ *is stable.*

PROOF. Let $B = \{b_1, \ldots, b_n\}$ be a basis for V. Let Σ_B be a complete set of involutions on B. Then $\Lambda(\Sigma_B) = \bar{\Sigma}$ is a group of 2^n commuting involutions. By III.4, $\bar{\Sigma}$ decomposes $V = P_1 \oplus \cdots \oplus P_s$ into a direct sum of projective R-modules and if $\bar{\Sigma} = \{\bar{\sigma}_i\}$ (where $\Sigma_B = \{\sigma_i\}$ and $\Lambda\sigma_i = \bar{\sigma}_i$), then $\bar{\sigma}_i|_{P_j} = \pm I$ for each i and j, $1 \leqslant i \leqslant 2^n$, $1 \leqslant j \leqslant s$.

We claim $s = n$. If $s < n$ then the decomposition $V = P_1 \oplus \cdots \oplus P_s$ will carry at most $2^s < 2^n$ involutions with restrictions to the P_j being $\pm I$. Since $\Lambda(\Sigma_B)$ has 2^n distinct involutions with restrictions $\pm I$ on the P_j, we have a contradiction to assumption $s < n$.

Suppose $s > n$, $V = P_1 \oplus \cdots \oplus P_s$. Then the P_i are projective modules over a connected ring R. Thus for each q and $\bar{q}$ in $\mathrm{Spec}(R)$, the local dimensions of P_j coincide, i.e., $\dim_{R_q}(P_j)_q = \dim_{R_{\bar{q}}}(P_j)_{\bar{q}}$—projective modules over connected rings have constant rank. Further, none of the P_i are locally 0,

for then they would be 0 globally. Hence, locally, $\dim_{R_q}(P_i)_q = \lambda_{P_i} \geqslant 1$ for each q in Spec(R) and $1 \leqslant i \leqslant s$ where λ_P is an integer dependent on P_i but not q. On the other hand, $\dim_{R_q}(V_q) = n$ for all q in Spec(R). Since free modules over commutative rings have well-defined dimension, we cannot have $s > n$ (since

$$n = \dim_{R_q}(V_q) = \sum_{i=1}^{s} \lambda_{P_i} \geqslant \sum_{i=1}^{s} 1 = s\Big).$$

Hence $s = n$ and $\bar{\Sigma}$ induces a decomposition $V = P_1 \oplus \cdots \oplus P_n$. The above paragraph also shows that $\dim_{R_q}(P_i) = 1$ for each i and every q in Spec(R). Hence each P_i is a rank one projective module.

We now claim $P_i \simeq P_j$ for every i and j. Recall, from III.3, if ρ is in GL(V) and σ is an involution, then $\rho P(\sigma) = P(\bar{\sigma})$ and $\rho N(\sigma) = N(\bar{\sigma})$ where $\bar{\sigma} = \rho\sigma\rho^{-1}$. Further, $\rho^{-1}P(\bar{\sigma}) = P(\sigma)$ and $\rho^{-1}N(\bar{\sigma}) = N(\sigma)$. Hence, $\rho|_{P(\rho)}$: $P(\sigma) \to P(\bar{\sigma})$ and $\rho|_{N(\rho)}$: $N(\sigma) \to N(\bar{\sigma})$ are isomorphisms. Thus, the conjugation classes of involutions on Σ_B induce 2^n isomorphisms of positive spaces (and negative spaces.) Since Λ preserves conjugations, Λ induces 2^n isomorphisms on the positive (or negative) spaces of $\bar{\Sigma} = \Lambda(\Sigma_B)$ on $V = P_1 \oplus \cdots \oplus P_n$. Observe if, for distinct i and j, $1 \leqslant i, j \leqslant n$, we had P_i not isomorphic to P_j, then it would not be possible to produce 2^n isomorphisms from among sums of subsets of $\{P_1, \ldots, P_n\}$. Hence, $P_i \simeq P_j$ for all i and j.

Therefore $V = P_1 \oplus \cdots \oplus P_n \simeq P \oplus \cdots \oplus P$ where $P = P_1$ and $\Lambda(\Sigma_B) = \Sigma_P$, i.e., Λ is a stable automorphism, completing the proof.

In an earlier version of this paper we had an incorrect proof that any automorphism over *any* commutative ring having 2 a unit was stable. We conjecture that this is true. If so, our characterization of stable automorphisms in the next section will describe all automorphisms when $n \geqslant 3$ and 2 is a unit.

V. Classification of stable automorphisms. We assume the hypothesis on R and S as given in §§II and IV and on V as given in §IV.

Let Λ: GL(V) $\to$ GL(V) be a stable group automorphism. Let B be a basis of V. Then there is a rank one projective R-module P with $\Lambda(\Sigma_B) = \Sigma_P$. If

$$\begin{aligned}\bar{V} = S \otimes V &= S \otimes (P_1 \oplus \cdots \oplus P_n) \qquad (P \simeq P_i,\ 1 \leqslant i \leqslant n)\\ &= (S \otimes P_1) \oplus \cdots \oplus (S \otimes P_n) = \bar{P}_1 \oplus \cdots \oplus \bar{P}_n,\end{aligned}$$

where $\bar{P}_i = S \otimes P_i$, then since S splits Pic(R), each $\bar{P}_i$ is a free S-module of dimension one. Let $\bar{P}_i = S\bar{b}_i$ and $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$. The group of involutions Σ_P when considered as elements of $\mathrm{GL}_S(\bar{V})$ becomes a complete set $\Sigma_{\bar{B}}$ of involutions on $\bar{B}$.

Let $B = \{b_1, \ldots, b_n\}$. Consider the involutions of type $(1, n-1)$ in Σ_B,

namely $\sigma_1, \ldots, \sigma_n$ where $\sigma_i(b_i) = -b_i$ and $\sigma_i(b_j) = b_j$ if $i \neq j$. A proof analogous to the proof of Theorem 3.2 of [5] shows that $\bar{\sigma}_1, \ldots, \bar{\sigma}_n$ where $\bar{\sigma}_i = \Lambda(\bar{\sigma}_i)$, $1 \leqslant i \leqslant n$, are of type $(t, n-t)$ on $\bar{B}$ where $t = 1$ or $t = n-1$.

Let ρ: $V \to V$ be a permutation matrix given by $\rho(b_i) = b_{\alpha(i)}$ where α is the cycle $(1, 2, 3, \ldots, n)$. Then $\sigma_i = \rho^{i-1}\sigma_1(\rho^{i-1})^{-1}$ for $2 \leqslant i \leqslant n$. Then the basis $\bar{B}$ and the $\bar{\sigma}_i$ may be indexed so that $\bar{\sigma}_i = \Lambda(\rho)^{i-1}\bar{\sigma}_1[\Lambda(\rho)^{i-1}]^{-1}$ for $2 \leqslant i \leqslant n$. Therefore, after reindexing $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$, define τ *in* $GL_S(\bar{V})$ by $\tau(\bar{b}_i) = b_i$ for $1 \leqslant i \leqslant n$. We have

$$\tau\Lambda(\sigma_i)\tau^{-1} = \alpha\sigma_i$$

where $\alpha = \pm I$.

To summarize,

V.1. THEOREM. *There is a* τ *in* $GL_S(\bar{V})$ *with*

$$\Lambda(\sigma_i) = \alpha\tau^{-1}\sigma_i\tau$$

for $1 \leqslant i \leqslant n$ *where* $\alpha = \pm I$.

Define involutions ψ_{ij} $(i < j)$ on B by $\psi_{ij}(b_k) = -b_k$ if $k = i, j$ and $\psi_{ij}(b_k) = b_k$ if $k \neq i, j$. Then $\{\psi_{ij} | 1 \leqslant i < j \leqslant n\}$ is a set of $\binom{n}{2}$ conjugate, pairwise commuting involutions of type $(2, n-2)$ in Σ_B. Further, $\psi_{ij}\psi_{jk} = \psi_{ik}$ and $\psi_{ij} = \sigma_i\sigma_j$.

V.2. THEOREM. *There is a* τ *in* $GL_S(\bar{V})$ *with*

$$\Lambda(\psi_{ij}) = \tau^{-1}\psi_{ij}\tau$$

for $1 \leqslant i < j \leqslant n$.

PROOF. Consider

$$\begin{aligned}\Lambda(\psi_{ij}) &= \Lambda(\sigma_i\sigma_j) = \Lambda\sigma_i\Lambda\sigma_j = (\alpha\tau^{-1}\sigma_i\tau)(\alpha\tau^{-1}\sigma_j\tau) \quad \text{by V.1} \\ &= \alpha^2\tau^{-1}\sigma_i\sigma_j\tau = \tau^{-1}\psi_{ij}\tau.\end{aligned}$$

The construction of the form of the automorphism Λ now follows the approach of the "Chinese School" (see the discussion in [6, p. 154] or [7]). This approach or variations was employed in [5], [6], [10] and [11]. In [6] we noted that from this point, the arguments could be carried through for any commutative ring having trivial idempotents (see [6, p. 155, first line]).

By localization techniques we now remove this restriction on idempotents. Commutative algebraic techniques of localization have not been extensively employed in the characterization of the automorphisms of the classical groups. However, since the images of involution have been determined, these techniques will be shown to apply.

The following lemma is well known.

V.3. LEMMA. *Let q be a prime ideal in R. If a is in R, let $(a)_q$ denote the image of a under the canonical morphism $\sigma\colon R \to R_q$. Then $(a)_q = 0$ for all q in* Spec(R) *if and only if $a = 0$. In particular, $b = a$ in R if and only if $(b)_q = (a)_q$; a is a unit if and only if $(a)_q$ is a unit (both statements for all q in* Spec(R)).

If $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is the basis of $\bar{V}$ determined in the proof of V.1 and $\tau(\bar{b}_i) = b_i$, $1 \leqslant i \leqslant n$, then set $\bar{\Lambda}(\sigma) = \tau\Lambda(\sigma)\tau^{-1}$ for σ in $\mathrm{GL}(V)$. Then $\bar{\Lambda}$: $\mathrm{GL}(V) \to \mathrm{GL}(V)$ is a group automorphism and $\bar{\Lambda}(\sigma_i) = \sigma_i$, $\bar{\Lambda}(\psi_{ij}) = \psi_{ij}$ for the involutions σ_i and ψ_{ij}, $1 \leqslant i < j \leqslant n$. Thus, without loss of generality, we may assume when necessary that the original automorphism Λ fixes the involutions σ_i and ψ_{ij}.

V.4. LEMMA. *Let a, b, c, d be elements of R. If*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = -I \quad \textit{and} \quad \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}^2 = I$$

then $a = d = 0$ and $c = -b^{-1}$.

PROOF. Prove the result for a local ring–see Lemma 3.3 of [5]–and apply V.3.

V.5. THEOREM. *There is a τ in $\mathrm{GL}_S(\bar{V})$ with*

$$\Lambda(\eta^1_{i,i+1}) = \tau^{-1}\eta^e_{i,i+1}\tau$$

for $1 \leqslant i \leqslant n-1$ where $\eta^e_{i,i+1}$ denotes the skew-permutation defined in III(c).

PROOF. The proof is based on V.3 and the proof over a local ring as given in [5, Theorem 3.4, p. 383]. We sketch the proof to illustrate the remark prior to V.3.

By the above remark (before V.4) we may assume $\Lambda\psi_{ij} = \psi_{ij}$. When $e = 1$, denote $\eta^e_{i,i+1}$ by $\eta_{i,i+1}$.

Since the $\psi_{i,i+1}$ commute with η_{12} for $i = 1$ and $3 \leqslant i \leqslant n$, we have $\psi_{i,i+1} = \Lambda\psi_{i,i+1}$ commuting with $\Lambda\eta_{12}$ for $i = 1, 3, 4, \ldots, n$. Thus, since 2 is a unit, a computation shows if $n = 3$ or $n \geqslant 5$ then

$$\Lambda\eta_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus [a_3, \ldots, a_n];$$

while, if $n = 4$, then

$$\Lambda\eta_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

The identities

$$(\Lambda\eta_{12})^2 = \Lambda\psi_{12} = \psi_{12}, \quad (\Lambda\eta_{12}\Lambda\psi_{23})^2 = I \tag{a}$$

imply from V.4 that $c = -d^{-1}$, $a = d = 0$, $a_i^2 = 1$ and, if $n = 4$, $x = y = 0$, $w^2 = z^2 = 1$.

In general, if $n \geqslant 3$, an analogous argument shows

$$\Lambda\eta_{i,i+1} = [a_1^{(i)}, \ldots, a_{i-1}^{(i)}] \oplus \begin{bmatrix} 0 & b^{(i)} \\ -b^{(i)^{-1}} & 0 \end{bmatrix} \oplus [a_{i+2}^{(i)}, \ldots, a_n^{(i)}]$$

where $a_j^{(i)^2} = 1$.

Define ρ: $\bar{V} \to \bar{V}$ by

$$\rho(\bar{b}_i) = \left[\prod_{t=i}^{n-1} b^{(t)}\right]\bar{b}_i, \qquad 1 \leqslant i \leqslant n-1,$$

and

$$\rho(\bar{b}_n) = \bar{b}_n.$$

Then $\rho^{-1}\psi_{ij}\rho = \psi_{ij}$ and $\rho^{-1}\sigma_i\rho = \sigma_i$. Further,

$$\rho^{-1}(\Lambda\eta_{i,i+1})\rho = [a_1^{(i)}, \ldots, a_{i-1}^{(i)}] \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus [a_{i+2}^{(i)}, \ldots, a_n^{(i)}].$$

To complete the proof it must be shown:

(1) $a_j^{(i)} = a_k^{(i)}$ for all j and k,

(2) $a_j^{(i)} = a_j^{(i+1)}$ for all possible choices of i.

Since $\eta_{i,i+1}$ commutes with each $\eta_{j,j+1}$ where $1 \leqslant j \leqslant i-2$, $i+2 \leqslant j \leqslant n-1$, then $\Lambda\eta_{i,i+1}$ commutes with $\Lambda\eta_{j,j+1}$ for j over the same index set. This shows $a_1^{(i)} = a_2^{(i)} = \cdots = a_{i-1}^{(i)}$ and $a_{i+1}^{(i)} = \cdots = a_n^{(i)}$. Thus

$$\Lambda\eta_{i,i+1} = a^{(i)}I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus b^{(i)}I_{n-i-1}.$$

Since $(\eta_{i-1}, \eta_{i,i+1})^3 = I$ we have $(\Lambda\eta_{i-1,i}\Lambda\eta_{i,i+1})^3 = I$. This shows

$$b^{(i-1)}a^{(i)} = 1, \quad (a^{(i)}a^{(i-1)})^3 = 1, \quad (b^{(i)}b^{(i-1)})^3 = 1.$$

Localize these equations at a prime ideal q. Then $[b^{(\lambda)}]_q$ and $[a^{(\lambda)}]_q$ are 1 or -1 for $\lambda = i$, $i-1$. The first equation shows $[b^{(i-1)}]_q = [a^{(i)}]_q$, the second shows $[a^{(i)}]_q = [a^{(i-1)}]_q$, etc. By V.3, these elements are equal to a common value e where $e^2 = 1$. Thus

$$\rho^{-1}\Lambda(\eta_{i,i+1})\rho = \eta_{i,i+1}^e,$$

completing the proof.

As in the discussion prior to V.4, Λ may be adjusted by a conjugation by ρ and we may assume without loss of generality that

$$\Lambda(\sigma_i) = \sigma_i, \quad \Lambda(\psi_{ij}) = \psi_{ij}, \quad \text{and} \quad \Lambda(\eta_{i,i+1}) = \eta_{i,i+1}^e$$

for some e in R with $e^2 = 1$.

We next compute the image of an elementary transvection τ_{b_i, b_j^*} under the action of Λ.

In the above proof it was necessary to construct a transforming matrix ρ. It is not clear that this construction, if performed locally, could be lifted to the global context. Hence, localization techniques occur only in the final step.

However, if we consider $\Lambda\tau_{b_1,b_2^*}$ and localize at a prime q in Spec(R), then the argument in the proof of (3.5) of [5], shows

$$\Lambda\tau_{b_1,b_2^*} = \begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix} \oplus [1, \ldots, 1]$$

where either $(b)_q = 1$ and then $(a)_q = 0$, or $(b)_q = 0$ and then $(a)_q = -1$ (This proof also shows that e in V.5 satisfies $(e)_q = 1$ for *all* q in Spec(R). Hence, $e = 1$.)

Further, $b^2 = b$, $a^2 = -a$ and $ab = 0$. Set $\bar{f} = b$ and $f = -a$. Then f and $\bar{f}$ are orthogonal idempotents with $(f + \bar{f})_q = 1$ for all q in Spec(R). Hence, $f + \bar{f} = 1$.

This partition of 1 by $1 = f + \bar{f}$ determines a ring decomposition of R, $R = R_1 \oplus R_2$, where $R_1 = Rf$ and $R_2 = R\bar{f}$ and, in turn, natural corresponding decompositions of $V = V_1 \oplus V_2$, $S = S_1 \oplus S_2$ and $\mathrm{GL}_n(R) = \mathrm{GL}_n(R_1) \times \mathrm{GL}_n(R_2)$ ($\mathrm{GL}(V) = \mathrm{GL}_{R_1}(V_1) \times \mathrm{GL}_{R_2}(V_2)$).

We shall be "careless" and denote the identity of R, R_1 and R_2 all by 1, i.e., $1 = \langle 1, 1\rangle$. The context will indicate in which ring the element 1 is the identity.

Thus,

$$\Lambda(\tau_{b_1,b_2^*}) = \left\langle \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \oplus [1, \ldots, 1], \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus [1, \ldots, 1] \right\rangle.$$

Let $c_i = fb_i$ for $1 \leqslant i \leqslant n$ and $d_i = \bar{f}b_i$ for $1 \leqslant i \leqslant n$. Therefore

$$\Lambda(\tau_{b_1, b_2^*}) = \langle \tau_{-c_2,c_1^*}, \tau_{d_1,d_2^*} \rangle.$$

The remainder of the proof proceeds in a fashion analogous to (3.5) of [5] where the conjugating matrices are selected in either $\mathrm{GL}_n(R_1)$ (i.e., $\mathrm{GL}_{R_1}(V_1)$) or $\mathrm{GL}_n(R_2)$ (i.e., $\mathrm{GL}_{R_2}(V_2)$).

V.6. THEOREM. *There is a* $\tau = \tau_1 \times \tau_2$ *in* $\mathrm{GL}_S(\bar{V}) = \mathrm{GL}_{S_1}(\bar{V}_1) \times \mathrm{GL}_{S_2}(\bar{V}_2)$,

$$\Lambda(\tau_{b_i,b_j^*}) = \left\langle \tau_1^{-1}\tau_{-c_j,c_i^*}\tau_1, \tau_2^{-1}\tau_{d_i,d_j^*}\tau_2 \right\rangle,$$

where $\{c_i\}$ (*resp.*, $\{d_i\}$) *is the basis of* V_1 (*resp.*, V_2) *induced by the basis* $\{b_i\}$*–and an analogous statement for* $\{c_i^*\}$, $\{d_i^*\}$ *and* $\{b_i^*\}$.

It is more convenient, due to the decomposition of R, to describe the theory and results in a matrix context. Thus, V.6 states that there is a matrix

$P = \langle P_1, P_2 \rangle$ in $\mathrm{GL}_n(S) = \mathrm{GL}_n(S_1) \times \mathrm{GL}_n(S_2)$ such that

$$\Lambda(T_{ij}(1)) = P^{-1}\langle T_{ji}(-1), T_{ij}(1)\rangle P = \left\langle P_1^{-1}T_{ji}(-1)P_1, P_2^{-1}T_{ij}(1)P_2 \right\rangle$$

where $T_{ij}(1)$ is the matrix of an elementary transvection τ_{b_i,b_j^*} (see discussion after III.1).

The above may be viewed as the composition of two automorphisms. First the application of an inner automorphism Φ_P where $\Phi_P(A) = P^{-1}AP$. Second the application of a "transpose-inverse" automorphism Ω_f "based" at an idempotent f. Namely, if $A^* = (A^{-1})^t$ and if $1 = f + \bar{f}$ is a partition of 1 by orthogonal idempotents, then

$$\Omega_f(A) = \left\langle (Af)^*, A\bar{f} \right\rangle.$$

Then, the above may be written as

$$\Lambda(T_{ij}(1)) = (\Omega_f \circ \Phi_Q)(T_{ij}(1))$$

where $P_1 = (Qf)^*$ and $P_2 = Q\bar{f}$. For convenience we will agree to apply first the inner automorphism Φ_Q and second the transpose-inverse automorphism Ω_f when both appear in compositions.

If σ: $R \to R$ is a ring automorphism, then σ induces a group automorphism $A \to A^\sigma$ on $\mathrm{GL}_n(R)$, where if $A = [a_{ij}]$ then $A^\sigma = [\sigma(a_{ij})]$. A pair $\langle Q, \sigma\rangle$, where Q is in $\mathrm{GL}_n(S)$ and $Q^{-1}\mathrm{GL}_n(R)Q \subset \mathrm{GL}_n(R)$ and σ: $R \to R$ a ring automorphism, determines a group automorphism of $\mathrm{GL}_n(R)$ by $A \to Q^{-1}A^\sigma Q$. We denote this automorphism by $\Phi_{\langle Q,\sigma\rangle}$.

We continue the above notation and conventions in the next theorem.

V.7. THEOREM. *Let* Λ: $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$ *be a stable group automorphism. Let* $\bar{\Lambda} = \Lambda|_{E_n(R)}$ *be the restriction of* Λ *to the group* $E_n(R)$ *of elementary matrices. Then there is a ring automorphism* σ: $R \to R$, *a* Q *in* $\mathrm{GL}_n(S)$ *and an idempotent* f *in* R *such that*

$$\bar{\Lambda} = \Omega_f \circ \Phi_{\langle Q,\sigma\rangle}.$$

PROOF. We sketch the argument–it is analogous to the proof of (3.6) of [5].

Initially, localize and use the argument in (3.6) [5] and previous discussion to conclude

$$P\Lambda(T_{12}(\lambda))P^{-1} = \left\langle T_{21}(\beta_1)^{-1}, T_{12}(\beta_2)\right\rangle = \Omega_f(T_{12}(\beta_1 + \beta_2)).$$

Define σ: $R \to R$ by σ: $\lambda \to \beta$ ($\beta = \beta_1 + \beta_2$). By basic commutator relations (see [5] or [6]) the other elementary transvections may be generated and

$$P\Lambda(T_{ij}(\lambda))P^{-1} = \Omega_f(T_{ij}(\sigma(\lambda))).$$

Computations analogous to the methods in the proof of (3.6) of [5] show σ is an injective ring morphism.

A straightforward induction on n will show that all matrices of the form

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

are in $E_n(R)$. Since transposition carries elementary transvections to elementary transvections, $E_n(R)$ is stable under transposition. Hence, all matrices of the form

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}$$

are in $E_n(R)$. Hence,

$$P\Lambda(T_{ij}(\lambda))P^{-1} = \Omega_f(T_{ij}(\sigma(\lambda))) = \langle T_{ji}(\beta_1)^{-1}, I\rangle \langle I, T_{ij}(\beta_2)\rangle$$

is in $E_n(R)$ where $\sigma(\lambda) = \beta_1 + \beta_2$ in $R = R_1 \times R_2$. Finally, Suslin's Theorem (III.2) implies $\Lambda(T_{ij}(\lambda))$ is in $E_n(R)$.

Applying the arguments of this section to Λ^{-1}, we have $\bar{\Lambda}^{-1}$: $E_n(R) \to E_n(R)$ and $\bar{\Lambda}^{-1}$ is an automorphism of $E_n(R)$. Further, $\bar{\Lambda}^{-1}$ induces an injective morphism $\bar{\sigma}$: $R \to \bar{R}$ with $\sigma\bar{\sigma} = \bar{\sigma}\sigma = 1$. Hence, σ is a ring automorphism of R. It is for this step that we need $\Lambda^{-1}(\Sigma_B) = \Sigma_{\bar{P}}$ in the definition of stable automorphism.

Before the final description of Λ, we introduce a third class of automorphisms of $\mathrm{GL}_n(R)$–*radial* automorphisms. An automorphism Λ: $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$ is called a *radial automorphism* if there is a group morphism χ: $\mathrm{GL}_n(R) \to \mathrm{Center}(\mathrm{GL}_n(R))$ such that $\Lambda(A) = \chi(A)A$ for all A in $\mathrm{GL}_n(R)$. The radial automorphism determines χ uniquely. Thus Λ is denoted by P_χ.

The following theorem characterizes the stable automorphisms of $\mathrm{GL}_n(R)$. Since this is the principal result of the section, we state the complete hypothesis.

V.8. THEOREM (CHARACTERIZATION OF STABLE AUTOMORPHISMS). *Let V be a free R-space of dimension $n \geqslant 3$. Let 2 be a unit in R and S be a splitting ring of* $\mathrm{Pic}(R)$. *If* Λ: $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$ *is a stable group automorphism of* $\mathrm{GL}_n(R)$, *then there exist*:

(a) *a transpose-inverse automorphism Ω_f*;

(b) *an inner automorphism with isomorphism σ, $I_{\langle Q, \sigma\rangle}$, where Q is in* $\mathrm{GL}_n(S)$ *and σ: $R \to R$ is a ring automorphism*;

(c) *a radial automorphism P_χ, such that*

$$\Lambda = \Omega_f \circ \Phi_{\langle Q, \sigma\rangle} \circ P_\chi.$$

PROOF. Let $\{b_1, \ldots, b_n\}$ be a basis of V with corresponding dual basis $\{b_1^*, \ldots, b_n^*\}$ of V. Let $\Lambda|_{E_n(R)} = \Omega_f \circ \Phi_{\langle Q, \sigma\rangle}$ by V.7. Consider the automorphism $\hat{\Lambda}$ of $GL_n(R)$ where $\hat{\Lambda} = \Phi_{\langle Q, \sigma\rangle}^{-1} \circ \Omega_f \circ \Lambda$. Then $\hat{\Lambda}(A) = A$ for any A in $E_n(R)$.

Let B be in $GL_n(R)$ and τ_{b_i, b_j^*} be an elementary transvection relative to $\{b_1, \ldots, b_n\}$. Then, by III.1(b),

$$B\tau_{b_i, b_j^*}B^{-1} = \tau_{Bb_i, b_j^* B^{-1}}.$$

By the definition of $\hat{\Lambda}$,

$$\hat{\Lambda}(\tau_{b_i, b_j^*}) = \tau_{b_i, b_j^*} \quad \text{and} \quad \hat{\Lambda}(\tau_{Bb_i, b_j^* B^{-1}}) = \tau_{Bb_i, b_j^* B^{-1}}$$

by Suslin's Theorem (III.2) and (V.7). On the other hand,

$$\hat{\Lambda}\left(B\tau_{b_i, b_j^*}B^{-1}\right) = \hat{\Lambda}(B)\hat{\Lambda}(\tau_{b_i, b_j^*})\hat{\Lambda}(B)^{-1} = \tau_{\hat{\Lambda}(B)b_i, b_j^* \hat{\Lambda}(B)^{-1}}.$$

Since Bb_i is unimodular, by III.1(d), $\hat{\Lambda}(B)b_i = \lambda_i Bb_i$ where λ_i is a unit in R.

Thus

$$\hat{\Lambda}(B)B^{-1}(b_i) = \lambda_i b_i \quad \text{for } 1 \leqslant i \leqslant n.$$

By the same argument, one can find a δ with

$$\hat{\Lambda}(B)B^{-1}(b_i + b_j) = \delta(b_i + b_j) \quad \text{for } i \neq j.$$

Hence, for $i \neq j$,

$$\begin{aligned}\delta(b_i + b_j) &= \hat{\Lambda}(B)B^{-1}(b_i + b_j) = \hat{\Lambda}(B)B^{-1}b_i + \hat{\Lambda}(B)B^{-1}b_j \\ &= \lambda_i b_i + \lambda_j b_j.\end{aligned}$$

Hence all λ_i equal a common value, say λ_B. Thus $\hat{\Lambda}(B)B^{-1} = \lambda_B I$. Let χ: $GL_n(R) \to \text{Center}(GL_n(R))$ be defined by $\chi(B) = \lambda_B I$. It is straightforward to show χ is a group morphism. Hence $\hat{\Lambda}(B) = \chi(B)B$, i.e., $\Phi_{\langle Q, \sigma\rangle}^{-1} \circ \Omega_f \circ \Lambda = P_\chi$. Thus, $\Lambda = \Omega_f \circ \Phi_{\langle Q, \sigma\rangle} \circ P_\chi$, completing the proof.

If R is a connected commutative ring, then Ω_f is either Ω_1 or Ω_0, i.e., $\Omega_f(A) = \Omega_1(A) = A^* = (A^{-1})^t$ or $\Omega_f(A) = \Omega_0(A) = A$. Since, by VI.1, all automorphisms of $GL_n(R)$ are stable when R is connected, we have the following corollary.

V.9. COROLLARY. *Let R be a connected commutative ring having 2 a unit. Let Λ: $GL_n(R) \to GL_n(R)$ be a group automorphism where $n \geqslant 3$. Then $\Lambda = \Omega_1 \circ \Phi_{\langle Q,\sigma\rangle} \circ P_\chi$ or $\Lambda = \Phi_{\langle Q,\sigma\rangle} \circ P_\chi$ where Ω_1, $\Phi_{\langle Q,\sigma\rangle}$ and P_χ satisfy conditions in* V.8.

Recall an R-submodule T of the extension ring S is invertible if there is an R-submodule T' of S with $TT' = R$. If $QV = TV$ where Q (above) is in $GL_n(S)$ and T is an invertible R-submodule of S, then the automorphism $A \to Q^{-1}AQ$ has the property that $Q^{-1}GL_n(R)Q \subseteq GL_n(R)$.

In general, the argument in [**6**, pp. 157, 151–152] will show that if

$Q^{-1}\mathrm{GL}_n(R)Q \subseteq \mathrm{GL}_n(R)$ then $QV = BV$ where $B = \{\alpha \text{ in } S \mid \alpha Q b_1 \text{ is in } V\}$ where $\{b_1, \ldots, b_n\}$ is a basis for V. If S is constructed as a ring of quotients (see example (a) of §II), it is possible to show B is an invertible R-submodule of S [**6**, p. 152].

Appendix. The purpose of this Appendix is to provide a short proof of Suslin's result that $E_n(R)$ is normal in $\mathrm{GL}_n(R)$. The original proof was communicated by letter to H. Bass by Suslin in the spring of 1976. The steps below are Suslin's with several of the arguments supplied to me by David Wright. We give only a direct proof that $E_n(R)$ is normal in $\mathrm{GL}_n(R)$ when $n \geqslant 3$ (which is needed in this paper). Suslin, by the same techniques, showed the elementary congruence subgroup $E_n(R, A)$ for A an ideal is normal in $\mathrm{GL}_n(R)$. Further, Suslin's letter provided the proof (which is more involved) that $E_n(k[X_1, \ldots, X_n]) = \mathrm{SL}_n(k[X_1, \ldots, X_n])$ when k is a field and $n \geqslant 3$, i.e., the K_1-analogue of the Serre Conjecture.

LEMMA A. *Let $\alpha = \langle a_1, \ldots, a_n\rangle^t$ and $\beta = \langle b_1, \ldots, b_n\rangle$. Suppose one of the b_i is equal to* 0 *and $\beta\alpha = \Sigma b_i a_i = 0$. Then $I + \alpha\beta$ is in $E_n(R)$.*

PROOF. Without loss of generality, we may assume $b_n = 0$. Then

$$I + \alpha\beta$$

$$= \begin{bmatrix} 1 & & & a_1 \\ & 1 & 0 & \vdots \\ & 0 & \ddots & a_{n-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & 0 & \ddots & \\ b_1 & \cdots & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -a_1 \\ & 1 & 0 & \vdots \\ & & \ddots & -a_{n-1} \\ & 0 & & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & 0 & \ddots & \\ -b_1 & \cdots & -b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ a_n b_1 & \cdots & a_n b_{n-1} & 1 \end{bmatrix}$$

and each matrix on the right is in $E_n(R)$.

LEMMA B. *Let $\alpha = \langle a_1, \ldots, a_n\rangle$ be unimodular and $\beta = \langle b_1, \ldots, b_n\rangle$. Suppose $\beta\alpha^t = 0$. Then, there exist a_{ij} in R satisfying*

$$\beta = \sum_{i<j} a_{ij}(a_j e_i - a_i e_j)$$

where e_i is the row $\langle 0, 0, \ldots, 1, 0, \ldots, 0\rangle$ where 1 *is the ith position and zeroes elsewhere.*

PROOF. Since $\alpha = \langle a_1, \ldots, a_n\rangle$ is unimodular, $(a_1, \ldots, a_n) = R$. Thus,

there exist $\{s_1, \ldots, s_n\}$ with $\Sigma s_i a_i = 1$. Then, for a fixed j,

$$b_j = \left(\sum s_i a_i\right) b_j = s_j a_j b_j + \sum_{i \neq j} s_i a_i b_i$$

$$= s_j \left(\sum_{i \neq j} (-b_i a_i) \right) + \sum_{i \neq j} s_i a_i b_i$$

since $\beta\alpha^t = \Sigma a_i b_i = 0$. Set $a_{ij} = s_j b_i - s_i b_j$. Then $\beta = \Sigma_{i<j} a_{ij}(a_j e_i - a_i e_j)$.

THEOREM (SUSLIN). *If $n \geqslant 3$, then $E_n(R)$ is normal in* $\mathrm{GL}_n(R)$.

PROOF. Suppose A is in $\mathrm{GL}_n(R)$ and $T_{ij}(\lambda)$ is an elementary transvection. Then $AT_{ij}(\lambda)A^{-1} = I + \lambda\alpha\beta$ where α is the ith column of A and β is the jth row of A^{-1}. Since $AA^{-1} = I$, we have $\beta\alpha = 0$. Suppose $\beta = \langle b_1, \ldots, b_n \rangle$ and $\alpha = \langle a_1, \ldots, a_n \rangle^t$. By Lemma B, $\beta = \Sigma_{i<j} a_{ij}(a_j e_i - a_i e_j)$. Apply Lemma A to $I + \lambda\alpha(a_{ij}(a_j e_i - a_i e_j))$ (since $n \geqslant 3$) and conclude this element is in $E_n(R)$. Then

$$I + \lambda\alpha\beta = \prod_{i<j} \left[I + \lambda\alpha\left(a_{ij}(a_j e_i - a_i e_j)\right) \right]$$

is in $E_n(R)$.

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