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GEOMETRIC CONVEXITY. III: EMBEDDING

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ABSTRACT. The straight line spaces of dimension three or higher which were considered by the first author in previous papers are shown to be isomorphic with a strongly open convex subset of a real vector space. To achieve this result we consider the classical descriptive geometry studied in various papers and textbooks by Pasch, Hilbert, Veblen, Whitehead, Coxeter, Robinson, and others, with the significant difference that the geometry considered here is not restricted to 3 dimensions. Our main theorem (which is well known in dimension 3) is that any such geometry is isomorphic to a strongly open convex subset of a real vector space whose "chords" play the role of lines.

1. Introduction. The axioms, terminology, notation, and results of *Geometric convexity*. I [7] will be used. Briefly, we consider a space $(X, \mathfrak{L})$, where X is a set of elements called *points* and $\mathfrak{L}$ is a family of subsets of X called *lines*, satisfying the three axioms:

A. Every line is totally ordered so as to be order equivalent to the reals.

B. Every two distinct points of X are contained in a unique line.

C. If a, b, c are three points of X and $x \in (a, c)$ and $y \in (x, b)$, then there exists $z \in (a, b)$ so that $y \in (c, z)$.

(As in [7], (a, b) denotes the set of all points x on the unique line $L(a, b)$ containing a and b such that $a < x < b$, where $<$ is the order on $L(a, b)$.)

A subset F of X is a *flat* if whenever x and y are two points of F the line containing A. A subset $A \subset X$ is *independent* if $\operatorname{fl}(A - a) \not\ni a$ for every $a \in A$. The *dimension* of a flat F, denoted $\dim F$, is the cardinality of a maximal independent set A such that $\operatorname{fl}(A) = F$. A *plane* is a flat of dimension 2. In general, we use the terminology of elementary geometry, e.g., two lines are coplanar if they lie in a common plane, etc.

We caution the reader that we use and refer to the total order on lines in a weak sense. For example, we reverse the order when convenient and also refer to a map as being *order-preserving* if it either preserves or reverses the order on lines. In §§4, 5, and 6 other orders will be considered. To avoid

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confusion or to provide greater emphasis, the notation $<_L$ will sometimes be used to denote the order on $L \in \mathfrak{L}$.

If $(X^*, \mathfrak{L}^*)$ is another space also satisfying Axioms A, B, and C, then $(X, \mathfrak{L})$ will be said to be *embedded in* $(X^*, \mathfrak{L}^*)$ if there exists an injective map $f\colon X \to X^*$ such that $L \in \mathfrak{L}$ if and only if $f(L)$ is a subset of a member of $\mathfrak{L}^*$ and f is (weakly) order-preserving on each line L. It will be understood that a vector space V is a space $(V, \mathfrak{L})$ such that $\mathfrak{L}$ is the family of (algebraic) 1-flats.

In §5 it will be convenient to replace the given total order on each $L \in \mathfrak{L}$ by Veblen's undefined order relation in **[22]**. We proceed as follows. For each distinct triple of points a, b, c, write abc if and only if a, b, and c lie on some line L and either $a <_L b <_L c$ or $c <_L b <_L a$. Then all of Veblen's axioms of order (Axioms 1–5) hold immediately. The Axiom of Continuity is obviously a consequence of the least upper bound property for reals, and Veblen's Axiom 6 is our Axiom C.

It is clear in turn that Veblen's axioms of order enable one to construct a total order $<_L$ on each line L for which there is no first or last element and satisfying the least upper bound property. Thus, Veblen's axioms of order are equivalent to our Axioms B, C, and the following order axiom, seemingly weaker than our Axiom A.

A′. Each line may be totally ordered so that there is no first or last element and each set bounded above has a least upper bound.

None of the results we use from **[7]** require anything stronger than Axiom A′. Some recent work of Doignon **[11]** shows that our Axioms A and A′ are equivalent for $\dim X \geqslant 2$. Our development proves independently that Axioms A and A′ are equivalent for $\dim X \geqslant 3$.

Another axiom which will be referred to is the familiar parallel postulate:

P. If L is a line and x is a point not on L, then there exists a unique line L' containing x, coplanar with L, and not intersecting L.

Our main theorems are:

THEOREM 1.1. *If* $\dim X \geqslant 3$ *and* $(X, \mathfrak{L})$ *satisfies Axioms* A′, B, *and* C *then* $(X, \mathfrak{L})$ *may be embedded in a space* $(X^*, \mathfrak{L}^*)$ *satisfying Axioms* A′, B, C, *and* P *so that* X *is a strongly open subset of* X^*.

THEOREM 1.2. *If* $\dim X \geqslant 3$ *and* $(X, \mathfrak{L})$ *satisfies Axioms* A′, B, *and* C *then* $(X, \mathfrak{L})$ *may be embedded in a vector space* V *over the reals so that* X *is a strongly open subset of* V.

Clearly any strongly open subset of a vector space over the reals satisfies Axioms A′, B, and C. Moulton **[17]** gives an example of a plane which satisfies Axioms A′, B, and C but is not Desarguesian and, therefore, cannot be embedded as a subset of a real vector space. Thus, Theorem 1.2 does not hold if merely $\dim X \geqslant 2$. Veblen [**22**], Whitehead [**23**], and Coxeter [**10**] have proven Theorem 1.2 under the assumption $\dim X = 3$.

Several authors have developments related to ours but with additional metric assumptions. For example, Busemann [**5**, p. 68, Theorem 13.1] proves Theorem 1.2 in the case $\dim X = 2$ when X is a straight G-space satisfying the Desargues Property. Also in *Geometric convexity*. II: *Topology* [**8**] the following theorem, strictly weaker than Theorem 1.2, is proved:

THEOREM 1.3. *Suppose* $(X, \mathfrak{L})$ *satisfies Axioms* A, B, *and* C *and* $\mathfrak{T}$ *is the topology on* X *with strongly open sets as base. If* $\dim X = n < \infty$ *then* $(X, \mathfrak{L})$ *is homeomorphic to Euclidean n-space.*

Nitka [**18**] and Nunnally [**19**] have results related to Theorem 1.3 but again with additional metric assumptions.

There are also different purely geometric approaches to convexity, e.g., Bryant and Webster [**3**], [**4**], and Kay and Womble [**15**]. Doignon [**11**] proves Theorem 1.2 with slightly different hypotheses, and from a substantially different point of view. Rubinšteĭn [**21**] outlines a proof of Theorem 1.2 with an approach similar to ours. Mah, Naimpally, and Whitfield [**16**] and Kay and Meyer [**14**] consider embedding questions for convexity structures which are more general than the space $(X, \mathfrak{L})$ considered here.

Some preliminary results not proven in *Geometric convexity*. I will be needed.

LEMMA 1.4. *If flats* F *and* G *meet and* $p \in F - G$ *then* $F \cap \mathrm{fl}(G, p) = \mathrm{fl}(F \cap G, p)$. *Therefore*, $\dim(F \cap \mathrm{fl}(G, p)) = 1 + \dim(F \cap G)$.

PROOF. Clearly both $F \cap G$ and p are contained in $F \cap \mathrm{fl}(G, p)$. Therefore $\mathrm{fl}(F \cap G, p) \subset F \cap \mathrm{fl}(G, p)$. To see the reverse containment fix $f \in F \cap G$ and q with $f \in (p, q)$. Define $U_p = \{x\colon [x, p) \cap G \neq \varnothing\}$ and $U_q = \{x\colon [x, q) \cap G \neq \varnothing\}$. Then by Theorem 5.8 of [**7**], $\mathrm{fl}(G, p) = U_p \cup U_q$. If $x \in F \cap \mathrm{fl}(G, p)$, without loss assume $x \in U_p$. Then $[x, p)$ meets G in some point y. Since $x, p \in F$ it follows that $y \in F$. Thus $y \in F \cap G$. Therefore, $x \in \mathrm{fl}(F \cap G, p)$ and $F \cap \mathrm{fl}(G, p) \subset \mathrm{fl}(F \cap G, p)$.

THEOREM 1.5. *If two finite dimensional flats* F *and* G *meet then*

$$\dim F + \dim G = \dim(F \cap G) + \dim \mathrm{fl}(F \cup G).$$

PROOF. The desired relation is trivial if $F \subset G$, so assume otherwise, write $G_1 = G$ and choose inductively

$$p_1 \in F - G_1, \quad G_2 = \mathrm{fl}(p_1 \cup G_1),$$
$$p_2 \in F - G_2, \quad G_3 = \mathrm{fl}(p_2 \cup G_2),$$

and, in general,

$$p_n \in F - G_n, \quad G_{n+1} = \mathrm{fl}(p_n \cup G_n).$$

We have by definition $G_1 \subset G_2 \subset \cdots \subset G_n \subset G_{n+1}$. Applying Theorem 6.2 of *Geometric convexity*. I, it follows inductively that $p_1, \ldots, p_n$ are independent in F and that if $F \subset G_{n+1}$ then $\mathrm{fl}(F \cup G) = \mathrm{fl}(F \cup G_1) = G_{n+1}$. Also, assume it has been proven that

$$\dim(F \cap G_n) + \dim G = \dim(F \cap G) + \dim G_n. \tag{1}$$

By Lemma 1.4,

$$\dim(F \cap G_{n+1}) = 1 + \dim(F \cap G_n).$$

Hence,

$$\begin{aligned} \dim(F \cap G_{n+1}) + \dim G &= 1 + \dim(F \cap G_n) + \dim G \\ &= \dim(F \cap G) + \dim G_n + 1 \end{aligned}$$

or,

$$\dim(F \cap G_{n+1}) + \dim G = \dim(F \cap G) + \dim G_{n+1}. \tag{2}$$

Note that (1) is trivial for $n = 1$, so mathematical induction implies (1) for all n. Since $\dim F < \infty$ it follows that for some n, $F \subset G_{n+1}$. Thus $F \cap G_{n+1} = F$ and $\mathrm{fl}(F \cup G) = G_{n+1}$ so that substitution into (2) yields the desired relation.

2. Adding ideal points. Throughout §2 we make the assumption that $(X, \mathfrak{L})$ is a space with $\dim X \geqslant 3$ satisfying Axioms A′, B, and C. The purpose of this section is to embed such a space in a space $(Y, \mathfrak{M})$ satisfying Axioms AP, BP, and CP to be introduced later.

LEMMA 2.1. *If lines A and D are each coplanar with lines B and C, and* $\dim \mathrm{fl}(A, B, C, D) = 4$, *then A and D are coplanar.*

PROOF. The hypotheses of Lemma 2.1 imply that no three of the lines A, B, C, D are coplanar and that the dimension of the flat spanned by any three of the lines A, B, C, D is three. For example, if A, B, C were coplanar, since C and D are coplanar, it would easily follow that $\dim \mathrm{fl}(A, B, C, D) = 3$. Thus, A, B, C are not coplanar. Since A and B are coplanar and A and C are coplanar it follows that $\dim \mathrm{fl}(A, B, C) = 3$.

Choose $b \in B$ with $b \notin \mathrm{fl}(A, C, D)$. Thus, $\dim \mathrm{fl}(A, B, D) = 3 = \dim \mathrm{fl}(A, C, D)$ and $b \notin \mathrm{fl}(A, C, D)$. Therefore $\mathrm{fl}(A, B, D) \cap \mathrm{fl}(A, C, D)$ does not contain the point b but does contain the lines A and D. By Theorem 1.5 the intersection of two three flats in four space either coincides with the two three flats, is empty, or is a plane. Thus $\mathrm{fl}(A, B, D) \cap \mathrm{fl}(A, C, D)$ is a plane containing A and D, proving A and D are coplanar.

THEOREM 2.2. *If* $\dim X \geqslant 3$ *and* A *and* B *are two lines in a plane* π, *and* C *and* D *are two lines not in* π *but each coplanar with both* A *and* B, *then* C *and* D *are coplanar.*

PROOF. If $\dim X = 3$ this is just 8.62 in Coxeter [**10**, p. 167], since in Coxeter's terms, C and D belong to the same bundle by our hypothesis. (See also Theorem 45 of Veblen [**22**, p. 372] or §20 of Whitehead [**23**, p. 19].) If $\dim X \geqslant 4$ and $\dim \mathrm{fl}(A, B, C, D) = 4$ then Theorem 2.2 is just Lemma 2.1. If $\dim \mathrm{fl}(A, B, C, D) = 3$ then $\mathrm{fl}(A, B, C, D) = \mathrm{fl}(A, B, C)$. Choose a line $E \not\subset \mathrm{fl}(A, B, C)$ coplanar with both B and C. Then, by Lemma 2.1 applied to the lines A, B, C, E it follows that A is coplanar with E. By Lemma 2.1 applied to the lines A, B, D, E it follows that D is coplanar with E. Then by Lemma 2.1 applied to the lines A, C, D, E it follows that C is coplanar with D.

COROLLARY 2.3. *Suppose* A, B, C *are distinct lines in a plane* π, D *and* E *are distinct lines not in* π *such that* D *is coplanar with each of* A *and* B, *and* E *is coplanar with each of* A, B, *and* C. *Then* C *and* D *are coplanar.*

PROOF. By Theorem 2.2 applied to A, B, D, E it follows that D and E are coplanar. Either $\mathrm{fl}(A, E)$ or $\mathrm{fl}(B, E)$ does not contain D, say $\mathrm{fl}(A, E) \not\supset D$. Hence C and D do not lie in $\mathrm{fl}(A, E)$ and Theorem 2.2 applied to A, E, C, D yields C and D coplanar.

DEFINITION 2.4. Two lines A and B lying in the same plane π define a system of lines consisting of every line of intersection of a plane through A with a plane through B as well as all lines in π that are coplanar with one of the lines of the system that do not lie in π. Such a system of lines is called a *bundle of lines* and will be denoted $\mathrm{bu}(A, B)$.

THEOREM 2.5. *Every two lines of a bundle are coplanar.*

PROOF. Suppose C and D are two lines in $\mathrm{bu}(A, B)$. Let $\pi = \mathrm{fl}(A, B)$ and assume that one of C, D is not contained by π, say $C \not\subset \pi$. If, in addition, $D \not\subset \pi$ then C and D are coplanar by Theorem 2.2. If $D \subset \pi$, then C is coplanar with each of A and B and there exists $E \in \mathrm{bu}(A, B)$ not in π and coplanar with D. Thus by Corollary 2.3, C and D are coplanar.

LEMMA 2.6. *Let* $A \neq B$, $A \neq C$, *and* L *be lines such that* $C \in \mathrm{bu}(A, B)$ *and* $L \in \mathrm{bu}(A, B)$. *Then* $L \in \mathrm{bu}(A, C)$. *Thus,* $C \in \mathrm{bu}(A, B)$ *implies* $\mathrm{bu}(A, B) \subset \mathrm{bu}(A, C)$.

PROOF. If C and L both lie in $\mathrm{fl}(A, B)$ the lemma follows from an application of Corollary 2.3. Otherwise, it suffices to prove that L is coplanar with C. First, if C and L are both not in the plane of A and B, then C and L are each coplanar with A and B, which by Theorem 2.2 implies that C and L

are coplanar. If $C \subset \mathrm{fl}(A, B)$ and $L \not\subset (A, B)$ choose $E \not\subset \mathrm{fl}(A, B)$ coplanar with each of A, B, and C. Then the result follows by Corollary 2.3. The remaining case $C \not\subset \mathrm{fl}(A, B)$ and $L \subset \mathrm{fl}(A, B)$ is similar.

The following fundamental result is a routine set-theoretic application of Lemma 2.6. The proof will be omitted.

THEOREM 2.7. *Suppose C and D are distinct lines in* $\mathrm{bu}(A, B)$. *Then* $\mathrm{bu}(A, B) = \mathrm{bu}(C, D)$.

COROLLARY 2.8. *Through any point of space passes one and only one line of a given bundle or else every line of the given bundle.*

DEFINITION 2.9. If a and b are two bundles of lines then through every point x of space there passes one line of each bundle. If these lines do not coincide they define a plane. The system of planes thus defined by the two bundles a and b is called the *sheaf of planes determined by the bundles a and b*, denoted $\mathrm{sf}(a, b)$.

By Theorem 2.7 two bundles have at most one line in common. We have:

THEOREM 2.10. *If a point x does not lie on the line common to bundles a and b then x lies in a unique plane of the sheaf.*

PROOF. By Corollary 2.8 there exists a plane π of the sheaf passing through x, with lines $A \in a$, $B \in b$ containing x and contained in π. If a second plane π^* of the sheaf contains x then π^* contains some $A^* \in a$ and $B^* \in b$. Now A^* must coincide with A or else a consists of all lines through x, and B would be common to a and b, contrary to hypothesis. Similarly, $B^* = B$ and $\pi^* = \pi$.

COROLLARY 2.11. *Every line of bundles a or b, not common to both a and b, lies in one and only one plane of the sheaf. Thus if a line L lies in two planes of the sheaf, then L will be common to both bundles a and b.*

THEOREM 2.12. *If a line L is common to bundles a and b then L lies on every plane of the sheaf* $\mathrm{sf}(a, b)$.

PROOF. Let $\pi \in \mathrm{sf}(a, b)$, and choose $x \in \pi - L$ and lines $A \in a$, $B \in b$ through x. By Theorem 2.10, $\pi = \mathrm{fl}(A, B)$. But L is coplanar with each of A, B so we have $\mathrm{fl}(L, x) \supset A \cup B$ and $\mathrm{fl}(L, x) = \mathrm{fl}(A, B) = \pi$, or $L \subset \pi$.

DEFINITION 2.13. Let Y be the set of all bundles of lines. A bundle of lines is said to be *incident* with a sheaf of planes if every line of the bundle is contained in some plane of the sheaf. We denote by $\mathfrak{M}$ the set of all families M of bundles incident with a single sheaf. Note that there is a natural identification of each $M \in \mathfrak{M}$ with the sheaf defining it. Thus we may regard $\mathfrak{M}$ as the family of *all sheafs of planes*. Letting $M(a, b) \in \mathfrak{M}$ denote the sheaf $\mathrm{sf}(a, b)$ if $a \neq b$, it is obvious that $a \in M(a, b)$ and $b \in M(a, b)$.

Note that a special class of bundles exist, which we shall call *proper*, consisting of families of concurrent lines. If a is such a bundle, then the point of intersection of all lines in a will be identified with a. In this manner $X \subset Y$. Each proper bundle of Y, then, has been identified with a point in X, to be called a *proper point*. All other points of Y will be called *ideal points*.

Now suppose $L \in \mathfrak{L}$ and consider the sheaf M of all planes containing L. If $u \in L \subset X$, then as a proper point of Y, u is the bundle of all lines through u. Since each line $U \in u$ is contained in the plane $\mathrm{fl}(L, U)$ if $L \neq U$, or in any plane through L if $L = U$, then U lies in a plane of M. Hence $u \in M$ so that $L \subset M$. Thus, each member of $\mathfrak{L}$ is a subset of a member of $\mathfrak{M}$. Accordingly, any such member of $\mathfrak{M}$ will be called a *proper line* and all other members, *ideal lines*.

Finally, we want to show that if a line $M \in \mathfrak{M}$ passes through a point of X then it is proper in the above sense. Suppose $M = M(a, b)$ and that a point $u \in X$ lies on M. Hence u, as a proper bundle, consists of all lines through u. Choose $A \in a$, $B \in b$ passing through u. If $A \neq B$ then A and B determine a plane $\pi \in M$. Since $u \in M$ then any line through u which is not in π lies in a plane π' which contains intersecting lines $A' \in a$ and $B' \in b$ passing through u. It follows that since either $A' \neq A$ or $B' \neq B$ one of the bundles a, b consists of all lines through u. Thus, either $A = B$, $a = u$, or $b = u$. In any case a and b have a line L in common and $u \in L$. Then by Theorem 2.12, M consists of all planes through L and hence $L \subset M$.

We can summarize the preceding discussion as follows: Given a space $(X, \mathfrak{L})$ with $\dim X \geqslant 3$ satisfying Axioms A′, B, and C we have constructed a space $(Y, \mathfrak{M})$ where Y is a set and $\mathfrak{M}$ is a family of subsets of Y such that:

(i) $X \subset Y$.

(ii) For each $L \in \mathfrak{L}$ there exists $M \in \mathfrak{M}$ such that $L \subset M$.

In addition, we note the property:

(iii) If $c \neq d \in M(a, b)$ then $M(a, b) = M(c, d)$.

To verify this, first let $c \in M(a, b)$ and $u \in M(a, b)$. We claim that $u \in M(a, c)$. For let $U \in u$. Then U lies in a plane π which contains intersecting lines $A \neq B$ in a, b not belonging to both a and b. Let $C \in c$ pass through $x = A \cap B$. Note that x is contained in a unique plane of the sheaf $M(a, b)$, namely π, and that C is contained in some plane of the sheaf $M(a, b)$. Therefore $C \subset \pi$ and it follows that $\pi \in M(a, c)$, or $u \in M(a, c)$. This proves

LEMMA 2.14. *Suppose a, b, c, u are bundles such $a \neq b$, $a \neq c$, $c \in M(a, b)$, and $u \in M(a, b)$. Then $u \in M(a, c)$ and $M(a, b) \subset M(a, c)$.*

It is now a routine set-theoretic argument to obtain both $M(a, b) \subset M(c, d)$ and $M(c, d) \subset M(a, b)$ and the proof of (iii).

The identification of points of X with certain points of Y and the resulting incidence relationships in X and Y can now be essentially summed up in the following statement: If $x \neq y \in X$ then $M(x, y) \cap X = L(x, y)$.

The space $(Y, \mathfrak{M})$ satisfies three axioms:

AP. The members of $\mathfrak{M}$ are projectively ordered (see Definition 4.1).

BP. Every two distinct points of Y are contained in a unique line in $\mathfrak{M}$.

CP. If a, b, c are three noncollinear points of Y and $u \in M(a, b)$, $v \in M(a, c)$ with $u \neq c$ and $v \neq b$ then $M(c, u) \cap M(b, v) \neq \varnothing$.

We defer the proof that space $(Y, \mathfrak{M})$ satisfies Axiom AP and further discussion of the concepts involved in this axiom to Theorem 4.1 and §4 below. That space $(Y, \mathfrak{M})$ satisfies Axiom BP is a consequence of property (iii) above. Theorem 2.16 contains a proof independent of Axiom AP that space $(Y, \mathfrak{M})$ satisfies Axiom CP.

LEMMA 2.15. *Suppose* $\dim X \geqslant 4$ *and* $M = M(a, b)$ *is the sheaf determined by the distinct bundles* a *and* b. *If* c *is a bundle containing the distinct lines* C *and* C^* *and* π *and* π^* *are distinct planes of the sheaf* M *with* $\pi \supset C$, $\pi^* \supset C^*$, *and neither* C *nor* C^* *common to both* π *and* π^*, *then* $c \in M$.

PROOF. For every $C' \in c$ we must find $\pi' \in M$ with $C' \subset \pi'$. Choose $o \in C - \pi^*$ and $o^* \in C^* - \pi$. Choose $A, A^* \in a$ and $B, B^* \in b$ with $o \in A \cap B$ and $o^* \in A^* \cap B^*$. Clearly $A \neq B$ and $\mathrm{fl}(A, B) \in M(a, b)$ so by Theorem 2.10, $\mathrm{fl}(A, B) = \pi$. Thus, $\dim \mathrm{fl}(o^*, A, B) = 3$.

Case 1. $C' \not\subset \mathrm{fl}(o^*, A, B)$. Choose $o' \in C' - \mathrm{fl}(o^*, A, B)$. Choose $A' \neq B'$, respectively, in bundles a, b such that $o' \in A' \cap B'$. (The case $A' = B'$ involves $M(a, b)$ containing all planes through A', hence $o' \in A' \subset \pi \subset \mathrm{fl}(o^*, A, B)$ contradicting our choice of o'.) Then $\dim \mathrm{fl}(o^*, A', B') = 3 = \dim \mathrm{fl}(o, A', B')$. Further $\mathrm{fl}(o^*, A', B')$ contains $A', B', o', A^*, B^*, C^*$, and therefore $\mathrm{fl}(o', C^*)$, but does not contain o or else $\mathrm{fl}(o^*, A, B) = \mathrm{fl}(o^*, A', B')$ and $\mathrm{fl}(o^*, A, B)$ would contain o'. Similarly, $\mathrm{fl}(o, A', B')$ contains $\mathrm{fl}(o', C)$ but does not contain o^*. Therefore

$$\mathrm{fl}(A', B') \equiv \pi' = \mathrm{fl}(o^*, A', B') \cap \mathrm{fl}(o, A', B') \supset \mathrm{fl}(o, C^*) \cap \mathrm{fl}(o', C) = C'.$$

Case 2. $C' \subset \mathrm{fl}(o^*, A, B)$. If $C' \subset \mathrm{fl}(A, B) \in M$ we are done. Otherwise, choose $o'' \notin \mathrm{fl}(o^*, A, B)$. Let C'' be the line in bundle c containing the point o''. By the argument of Case 1, there exists a plane π'' in the sheaf M containing C''. Since $o'' \notin \mathrm{fl}(o^*, A, B)$ and $C' \not\subset \mathrm{fl}(A, B)$ it follows that $\mathrm{fl}(o'', A, B) \cap \mathrm{fl}(o^*, A, B) = \mathrm{fl}(A, B)$ and $C' \not\subset \mathrm{fl}(o'', A, B)$. Therefore by the argument of Case 1, with o^*, C^*, and π^* replaced by o'', C'', and π'', there exists a plane π' in M containing C'.

THEOREM 2.16. *Space* $(Y, \mathfrak{M})$ *satisfies Axiom* CP.

PROOF. Assume that a, b, c are three noncollinear points of Y, with $u \in M(a, b)$, $v \in M(a, c)$, and $u \neq c$, $v \neq b$. If $\dim X = 3$ the theorem is proved in Veblen [22], Whitehead [23, p. 29], or Coxeter [10, §8.8]. We therefore assume $\dim X \geqslant 4$. Let $L_1, \ldots, L_{10}$ be the proper lines in common between the respective pairs of bundles a, b, c, u, v, if such exist. Since a, b, and c are noncollinear, according to Lemma 2.15, there can be at most one proper line in the bundle c contained in a plane of the sheafs $M(a, b)$, $M(b, u)$, $M(b, v)$. Hence we may choose $C \in c$ distinct from L_i, $1 \leqslant i \leqslant 10$, so that C does not lie on any of the planes of $M(a, b)$, $M(b, u)$, $M(b, v)$. There exists a proper point $o \in C - \bigcup_{i=1}^{10} L_i$. Denote by A and B the lines of the bundles a and b, respectively, containing o. Then $A \neq B$, $A \neq C$, and $B \neq C$ by our choice of o and C, and $\dim \mathrm{fl}(A, B, C) = 3$. Similar reasoning shows that $\dim \mathrm{fl}(B, C, U) = \dim \mathrm{fl}(B, C, V) = 3$ where $U \in u$ and $V \in v$ are lines through o. Since $\dim X \geqslant 4$, choose $o^* \in X - [\mathrm{fl}(A, B, C) \cup \mathrm{fl}(B, C, U) \cup \mathrm{fl}(B, C, V)]$. Let A^*, B^*, C^*, U^*, V^* be the lines of bundles a, b, c, u, v, respectively, passing through o^*. Then $\mathrm{fl}(o^*, U, C)$ contains U, U^*, C, C^*, has dimension 3, and does not contain B. Similarly $\mathrm{fl}(o^*, V, B)$ contains V, V^*, B, B^*, has dimension 3, and does not contain C. Therefore, $\mathrm{fl}(o^*, U, C) \cap \mathrm{fl}(o^*, V, B)$, being the intersection of two three flats in the 4-space $\mathrm{fl}(o^*, A, B, C)$, is a plane which contains the lines $\mathrm{fl}(U, C) \cap \mathrm{fl}(V, B) = L$ and $\mathrm{fl}(U^*, C^*) \cap \mathrm{fl}(V^*, B^*) = L^*$. Thus L and L^* are coplanar and distinct (since otherwise $L \subset \mathrm{fl}(U, C)$ would pass through o^*, contrary to the choice of o^*) and therefore determines a bundle. Also $L \subset \mathrm{fl}(U, C) \in M(u, c)$ and $L^* \subset \mathrm{fl}(U^*, C^*) \in M(u, c)$. Thus, by Lemma 2.15, the bundle determined by L and L^* is incident with $M(u, c)$. Similarly the bundle determined by L and L^* is incident with $M(v, b)$ and the theorem is proven.

3. Flats and hyperplanes in $(Y, \mathfrak{M})$. In this section we assume the space $(Y, \mathfrak{M})$ satisfies Axioms BP and CP.

DEFINITION 3.1. F is a *P-flat* if $a, b \in F$ implies $M(a, b) \subset F$. If $A \subset Y$ then $P\text{-}\mathrm{fl}(A) = \bigcap \{F\colon F \supset A, F \text{ a } P\text{-flat}\}$ = *P-flat spanned by A*. We remark that it is easy to verify by Axiom BP that any line in Y is a P-flat.

THEOREM 3.2. *If F is a P-flat and $p \notin F$, then* $P\text{-}\mathrm{fl}(p \cup F) = \bigcup \{M(p, f)\colon f \in F\}$.

PROOF. Let $W = \bigcup \{M(p, f)\colon f \in F\}$. Clearly $P\text{-}\mathrm{fl}(p \cup F) \supset W \supset p \cup F$. Thus it suffices to show that W is a P-flat. To this end let $b \neq c \in F$, $x \in M(p, b)$, and $y \in M(p, c)$ with $x \neq p \neq y$. Let $z \in M(x, y)$. Axiom CP applied to triangle pxc and points b, y yields $f \in M(x, y) \cap M(b, c)$. Either $f \neq b$ or $f \neq c$. Without loss assume $f \neq b$. Then Axiom CP applied to triangle xpf and points b, z yields $f^* \in M(b, f) \cap M(p, z)$. Thus $z \in M(p, f^*)$ with $f^* \in F$ so $z \in W$. Thus W is a P-flat.

DEFINITION 3.3. A *plane* is any P-flat of the form $P\text{-fl}(p \cup F)$ where F is a line in Y and $p \notin F$. A *3-space* is any P-flat of the form $P\text{-fl}(p \cup F)$ where F is a plane and $p \notin F$.

Any line L and point $x \notin L$ which lie in a plane determine that plane uniquely, and similarly, each 3-space is determined uniquely by a plane and a point not on it, as the next result shows. It is easy to show that in any 3-space two distinct planes which have a point in common have a line in common by application of Axiom CP.

THEOREM 3.4. *If F is a P-flat, $p \notin F$, $G = P\text{-fl}(p \cup F)$, and $q \in G - F$, then $G = P\text{-fl}(q \cup F)$.*

PROOF. By Theorem 3.2 there exists $f \in F$ such that $q \in M(p, f)$, so $p \in M(q, f)$. Thus $P\text{-fl}(q \cup F) \ni p$. Therefore $G \supset P\text{-fl}(q \cup F) \supset P\text{-fl}(p \cup F) = G$.

DEFINITION 3.5. A P-flat $H \subset Y$ is a *P-hyperplane* if $P\text{-fl}(p \cup H) = Y$ for some $p \notin H$. By Theorem 3.4 this is equivalent to $P\text{-fl}(p \cup H) = Y$ for all $p \notin H$.

THEOREM 3.6. *Given F_0 a P-flat and $p \notin F_0$ then there exists a P-hyperplane $H \supset F_0$ with $p \notin H$.*

PROOF. Let $\mathfrak{F}$ be the family of all P-flats missing p and containing F_0. Partially order $\mathfrak{F}$ by inclusion. Every chain has an upper bound so by Zorn's Lemma $\mathfrak{F}$ has a maximal element F^*. If F^* is not a P-hyperplane then $P\text{-fl}(p \cup F^*) \subsetneq Y$. Choose $x \notin P\text{-fl}(p \cup F^*)$. Let $F = P\text{-fl}(x \cup F^*)$. Clearly $F \supset F^* \supset F_0$. If $p \in F$ then by Theorem 3.4, $x \in P\text{-fl}(p \cup F^*)$. Thus $p \notin F$ and $F \in \mathfrak{F}$, contradicting the maximality of F^*. Thus F^* is a P-hyperplane.

THEOREM 3.7. *If H is a P-hyperplane, $M \in \mathfrak{M}$, and $M \not\subset H$, then $M \cap H$ is exactly one point.*

PROOF. Choose $p \in M - H$. Then $M \subset Y = P\text{-fl}(p \cup H)$. Therefore by Theorem 3.2, if $q \in M$, $q \neq p$, there exists $h \in H$ so that $q \in M(h, p)$. Therefore $h \in M(p, q) = M$. Thus, $M \cap H \neq \varnothing$. If two distinct points of M were to lie in H then we would have $M \subset H$ contrary to our assumption. Thus $M \cap H$ consists of exactly one point.

THEOREM 3.8. *If F is a P-flat and $P\text{-fl}(p \cup F)$ is a P-hyperplane for some $p \notin F$ then $P\text{-fl}(q \cup F)$ is a P-hyperplane for every $q \notin F$.*

PROOF. Assume $q \notin F$. If $q \in P\text{-fl}(p \cup F)$ then by Theorem 3.4, $P\text{-fl}(q \cup F) = P\text{-fl}(p \cup F)$ which is a P-hyperplane. If $q \notin P\text{-fl}(p \cup F)$ then also $p \notin P\text{-fl}(q \cup F)$ for otherwise $q \notin F$ and $p \in P\text{-fl}(q \cup F) - F$ so that $P\text{-fl}(p \cup F) = P\text{-fl}(q \cup F)$ by Theorem 3.4. But since $P\text{-fl}(p \cup F)$ is a P-hyperplane we have $P\text{-fl}(p \cup q \cup F) = Y$. Thus $P\text{-fl}(q \cup F)$ is a P-hyperplane.

DEFINITION 3.9. If F is a P-flat and $P\text{-fl}(p \cup F)$ is a P-hyperplane for some $p \notin F$ then F is called a *codimension*-2 *P-flat*.

It is clear from Theorem 3.8 that if F is a codimension-2 P-flat and $q \notin F$ then $P\text{-fl}(q \cup F)$ is a P-hyperplane.

THEOREM 3.10. *If $H_1 \neq H_2$ are P-hyperplanes then $H_1 \cap H_2$ is a codimension-2 P-flat.*

PROOF. Choose $p \in H_1 - H_2$ and $q \in H_2 - H_1$. Then $P\text{-fl}(p \cup H_2) = Y = P\text{-fl}(q \cup H_1)$. Let $x \notin M(p, q) \cup H_1 \cup H_2$. There exist $h_1 \in H_1$ and $h_2 \in H_2$ so that $x \in M(q, h_1) \cap M(p, h_2)$. By Axiom CP applied to triangle xpq and points h_1, h_2 there exists $h \in M(p, h_1) \cap M(q, h_2) \subset H_1 \cap H_2$. Thus $h_1 \in P\text{-fl}(p, H_1 \cap H_2)$ so $x \in P\text{-fl}(p, q, H_1 \cap H_2)$. Therefore $H_1 \cap H_2$ is a codimension-2 P-flat.

In Theorem 3.11, we assume the space $(X, \mathfrak{L})$ satisfies Axioms A′, B, and C. Also $(Y, \mathfrak{M})$ is the space constructed from $(X, \mathfrak{L})$ in §2 and proved there to satisfy Axioms BP and CP.

THEOREM 3.11. *If F is a flat in X and $f \in F$ is fixed then* $P\text{-fl}(F) = \bigcup\{M(f, x): x \in F, x \neq f\}$.

PROOF. Let $W = \bigcup\{M(f, x): x \in F, x \neq f\}$. Clearly $P\text{-fl}(F) \supset W \supset F$. Therefore it suffices to show that W is a P-flat. To this end let $x, y \in F - f$, and let $a \in M(f, x)$, $b \in M(f, y)$ with $a \neq b$. Let $c \in M(a, b)$ and $C \in c$ passing through f. We want to show that $c \in W$. If $f \in M(a, b)$, clearly $c \in W$. Therefore, we may assume without loss that $f \notin M(a, b)$. By Corollary 2.11, since $L(f, x)$ belongs to every plane of the sheaf $M(f, x) = M(f, a)$ it belongs to bundle a. Thus $A \equiv L(f, x) \in a$, and similarly $B \equiv L(f, y) \in b$. It follows that $\pi = \text{fl}(A, B)$ is contained in the sheaf $M(a, b)$ and is a subset of the flat F. Since $c \in M(a, b)$ and C is not in common with bundles a and b, it follows that C is contained in a unique plane of the sheaf $M(a, b)$. Since $C \ni f$ it follows that $C \subset \pi \subset F$. Choose $z \in C$ with $z \neq f$. Then $c \in M(f, z)$ and $c \in W$.

4. Projective order in Y. The order already assumed for lines in X will now be used to construct an order relation S on quadruples of collinear points in Y (patterned after Coxeter [**10**, pp. 13–15]). The space X will be assumed to satisfy Axioms A′, B, and C, while Y satisfies Axioms BP and CP.

Recall that two sets of collinear points $a, b, c, \ldots$ and $a', b', c' \ldots$ are *in perspective* if there exists a point p such that each of (a, a', p), (b, b', p), (c, c', p), $\ldots$ are collinear triples, and such a relationship is denoted

$$abc \ldots \overset{p}{\barwedge} a'b'c' \ldots$$

with p occasionally omitted. The desired properties of S are now stated.

01. If $S(ab, cd)$ then a, b, c, and d are distinct collinear points.

02. If a, b, c, and d are distinct collinear points then at least one of $S(ab, cd)$, $S(ac, bd)$ and $S(ad, bc)$ holds.

03. If $S(ab, cd)$ then $S(ba, cd)$.

04. If $S(ab, cd)$ and $S(ad, bx)$ then $S(ab, cx)$.

05. If a, b, and c are any 3 distinct collinear points there exists d such that $S(ab, cd)$.

06. If $S(ab, cd)$ and $abcd \barwedge a'b'c'd'$ then $S(a'b', c'd')$.

07. (Axiom of Continuity). For any 3 distinct collinear points a, b, c and any set U of elements x such that for each $x \in U$ the relation $S(ab, cx)$ holds there exists x_0 such that (i) either $x_0 = a$ or $S(ab, cx_0)$, (ii) for all $x \in U$ either $x = x_0$ or $S(ax, bx_0)$, and (iii) if $x_1 \neq x_0$ is any other element for which (i) and (ii) hold, then $S(ax_0, bx_1)$.

Definition 4.1. If a relation S exists for a space $(Y, \mathfrak{M})$ then the lines of Y are said to be *projectively ordered*.

In order to introduce such a relation S in Y first begin with a line L in X. For convenience we shall use the notation abc if either $a <_L b <_L c$ or $c <_L b <_L a$ (introduced earlier); also, let $abcd$ be used to denote the occurrence of abc, abd, acd, and bcd (that is, iff either $a <_L b <_L c <_L d$ or $d <_L c <_L b <_L a$). Now for each 4 distinct points on L if any one of the conditions $acbd$, $dacb$, $bdac$, or $cbda$ holds, write $S(ab, cd)$.

If we do this for each line in X then we immediately have properties 01–05. For 06, let $S(ab, cd)$ and $abcd \barwedge^p a'b'c'd'$. Since $p \neq a$ and $p \neq a'$ then either $aa'p$, $a'ap$, or apa', and we assume without loss that $acbd$ holds. By several applications of Axiom C in X and the corresponding Theorem 3.1 of *Geometric convexity*. I [7], we find that $aa'p$ implies either $a'c'b'd'$, $d'a'c'b'$, $b'd'a'c'$, or $c'b'd'a'$, and similarly for $a'ap$ and apa'. Hence $S(a'b', c'd')$. To extend the relation S to any line M in Y let $x \in X - M$ and write $S(ab, cd)$ for 4 distinct points a, b, c, d on M provided there exists a line $L \subset X$ which meets $M(a, x)$, $M(b, x)$, $M(c, x)$, and $M(d, x)$ at a', b', c', d', respectively, and $S(a'b', c'd')$ on L. That this relation does not depend on the particular choice of x and L chosen in X will now be established. Clearly, a different choice L^* for L leads to points a^*, b^*, c^*, and d^* as the intersections of L^* with $M(a, x)$, $M(b, x)$, $M(c, x)$, and $M(d, x)$; but since $a'b'c'd' \barwedge^x a^*b^*c^*d^*$ then by 06 for X we have $S(a^*b^*, c^*d^*)$ and thus $S(ab, cd)$ holds relative to L^*. Consider $x^* \neq x$ in X, $x^* \notin M$, and suppose again that $S(ab, cd)$ holds for points a, b, c, d on M. We appeal to the following lemma which is basic to the present development.

Lemma. *Suppose* $p \in Y$, $x \in X$ *and* 4 *distinct lines* L_i, $i = 1, 2, 3, 4$, *in* X *passing through* x *lie in a plane* π *in* X *which contains* $M(x, p) \cap X$. *There* *exists a line M through p such that $M \cap X = L$ intersects the L_i in 4 distinct points.*

PROOF. Let $L' = M(x, p) \cap X$; locate y and z on L' such that yxz, and determine any point $q \in L_1$, $q \neq x$. The theorem of Pasch in π then implies that L_2 cuts either $[y, q]$ or $[q, z]$ at a point r, say $L_2 \cap [y, q] = r$. Then L_3 also cuts $[y, q]$ or $[q, z]$ at a point s; if $L_3 \cap [q, z] = s$ then applying the theorem of Pasch to the triangle (y, z, s), L_1 and L_2 cut $[y, s]$ at q' and r', respectively. Thus it may be assumed in any case that the line $L(y, q)$ cuts L_1, L_2, L_3 at q, r, s. Similarly L_4 cuts $[y, q]$ or $[q, z]$ at a point t, and it may be assumed without loss that $L(y, q)$ cuts L_1, L_2, L_3, L_4 at q, r, s, t with the order $yqrst$, and that $p \notin [x, y]$. Locate u such that yuq, and determine the line $M = M(p, u)$. Since $L = M \cap X$ cannot cut $[x, y]$ and $L \subset \pi$, the theorems of Pasch implies directly that L cuts the open segments $(x, q) \subset L_1$, $(x, r) \subset L_2$, $(x, s) \subset L_3$, and $(x, t) \subset L_4$, completing the proof.

Now returning to $x \neq x^*$ in X and $S(ab, cd)$ on M, let $p \in M$ be distinct from a, b, c, d. If we let L_u and L_u^* denote, respectively, the lines $M(u, x) \cap X$ and $M(u, x^*) \cap X$ for $u = a, b, c, d$ then it follows easily that $\mathrm{fl}(L_a, L_b, L_c, L_d)$ is a plane containing $L_p = M(p, x) \cap X$. By the lemma there exists a line L in X collinear with p which cuts L_a, L_b, L_c, and L_d at a', b', c', d', and $S(a'b', c'd')$, so that if any of $a, b, c, d \in X$ then $aa'x$, $bb'x$, $cc'x$, and $dd'x$ hold. Locate $z \in X$ such that xx^*z. Since L_a and L_a^* belong to the bundle a these lines and z are coplanar. By the theorem of Pasch L_a^* cuts (a', z) at a point a^*; similarly, L_b^*, L_c^*, and L_d^* cut (b', z), (c', z) and (d', z) at b^*, c^* and d^*. Now apply the Theorem of Desargues to the triangles (a', b', x) and (a^*, b^*, x^*) which are in perspective with z (in Y). It follows that p, a^*, b^* are collinear in Y. Similarly, p, a^*, c^*, and d^* are collinear in Y and hence a^*, b^*, c^*, d^* lie on a line L^* in X. Since $a'b'c'd' \overline{\wedge}^z a^*b^*c^*d^*$ we have $S(a^*b^*, c^*d^*)$. Hence S is a well-defined relation for quadruples of points in Y.

It is now immediate that Properties 01–05 hold for S. To obtain 06 suppose M_1 and M_2 are any two lines in Y containing distinct points a, b, c, d and a', b', c', d' such that $S(ab, cd)$ and $abcd \overline{\wedge}^p a'b'c'd'$. Then there exists $x \in X - M_1$ and a line $L_1 \subset X$ cutting $M(x, a)$, $M(x, b)$, $M(x, c)$ and $M(x, d)$ at a_1, b_1, c_1 and d_1 such that $S(a_1b_1, c_1d_1)$. Consider the point of intersection $u_0 = M_1 \cap M_2$ and choose $y \in M(x, u_0) \cap X$ distinct from x and u_0; let L_2 be a line in X cutting $M(y, a')$, $M(y, b')$, $M(y, c')$, $M(y, d')$ at a_2, b_2, c_2, d_2. Now consider the correspondence being set up in $\mathrm{fl}(x, y, L_1)$ between the projective pencils x and y by means of the center of perspectivity p and the two axes M_1 and M_2: If $M(x, u)$ for $u \in M_1$ is any line through x then let $u' = M(p, u) \cap M_2$ and let the line $M(y, u')$ through y correspond to $M(x, u)$. Since this correspondence consists of the product of the two perspectivities $x \bar{\wedge}^{M_1} p$ and $p \bar{\wedge}^{M_2} y$ with respective axes M_1 and M_2, the correspondence thus defined is a projectivity $x \wedge y$. But this projectivity has a self-corresponding line, namely, $M(x, u_0) = M(y, u_0)$. By a well-known theorem in classical projective geometry (see, for example, the dual of 3.31 in G. de B. Robinson's *The Foundations of Geometry* [**20**, p. 28]), which uses only the Theorem of Desargues in its proof, we conclude that the correspondence $x \wedge y$ is a perspectivity with a single axis M_3. That is, corresponding lines $M(x, u)$ and $M(y, u')$ in x and y meet at some point $u'' \in M_3$. Applying this to $u = a, b, c, d$ we obtain the points a'', b'', c'', d'' on M_3. Hence, by definition of S in Y, $S(a_1b_1, c_1d_1)$ implies $S(a''b'', c''d'')$, and thus $S(a_2b_2, c_2d_2)$, or $S(a'b', c'd')$.

The remaining Property 07 can be readily established by the known properties of the order relation in X. Suppose a, b, c are three distinct points on $M \in \mathfrak{M}$ and U is any set of points such that if $x \in U$ either $x = a$ or $S(ab, cx)$. Choose $p \in X - M$ and $L' \in \mathfrak{L}$ such that L' cuts $M(p, a)$, $M(p, b)$, $M(p, c)$ at a', b', c'. By making judicious choice of L' it can be assumed that $a'b'c'$. For if $a'c'b'$, locate a'' such that $a'pa''$; then $L'' = L(a'', c')$ will meet $M(p, a)$, $M(p, b)$, and $M(p, c)$ at a'', $b'' = b'$ and c'' such that $a''b''c''$, and similarly, for $b'a'c'$. Locate a^* such that $a'pa^*$, and let segments (p, c') and (a^*, b') meet at c^*. For each $x \in U$, $M(p, x) \cap X$ is a line in X in the plane of $L(p, a')$, $L(p, b')$, $L(p, c')$ and must therefore meet segment (a', b') or $[b', a^*]$ at some point x'. If the latter, then either $x' = a^*$, $a^*x'c^*b'$, $x' = c^*$, $a^*c^*x'b'$, or $x' = b'$ and hence either $x = a$, $S(ac, bx)$, $x = c$, $S(ax, bc)$, or $x = b$, denying $S(ab, cx)$. Thus $M(p, x)$ meets (a', b') at x' for each $x \in U$, such that the order in X is $a'x'b'c'$. Let $>$ denote the unique order on $L' = L(a', b')$ such that $a' > b'$. If $U' = \{x' = M(p, x) \cap [a', b']: x \in U\}$ then let $x_0' = \sup U'$ and define $x_0 = M(p, x_0') \cap M$. It is clear that x_0 is the desired point in Property 07.

This now establishes that the space $(Y, \mathfrak{M})$ satisfies Axiom AP.

5. Proof of Theorem 1.1. In this section $(X, \mathfrak{L})$ is a space with $\dim X > 3$ satisfying Axioms A′, B, and C and $(Y, \mathfrak{M})$ is the space constructed from $(X, \mathfrak{L})$ in §2 satisfying Axioms AP, BP, and CP.

LEMMA 5.1. *If H is a hyperplane in X then $H^* = P\text{-fl}(H)$ is a P-hyperplane in Y. Conversely, if H^* is a P-hyperplane in Y which meets X then $H^* \cap X$ is a hyperplane in X.*

PROOF. Let $p \in X - H$. Choosing $q \in H$, Theorem 3.11 implies $p \notin \bigcup\{M(q, x): x \in H\} = H^*$. Also, $P\text{-fl}(p \cup H^*) \supset P\text{-fl}(p \cup H) \supset \text{fl}(p \cup H) = X$ and thus $P\text{-fl}(p \cup H^*) \supset P\text{-fl}(X) = Y$. Therefore H^* is a P-hyperplane. For the converse, suppose H^* is a P-hyperplane and consider $F = H^* \cap X \neq \varnothing$. It is clear that F is a flat in X; if F were not a hyperplane

F would be contained properly by a hyperplane H in X. Now $P\text{-fl}(F) \subset P\text{-fl}(H^*) = H^*$. Choose $f \in F$ and let $y \in H^*$; $M(f, y)$ contains $x \neq f$ in X and hence $y \in M(f, x) \subset P\text{-fl}(F)$, or $H^* \subset P\text{-fl}(F)$. It then follows by the first part that $P\text{-fl}(H) = H^*$. But if $p \in H - F$ then $p \in H^* \cap X = F$, a contradiction. Therefore $H^* \cap X$ is a hyperplane in X.

THEOREM 5.2. *There exists a P-hyperplane H in Y with $H \cap X = \varnothing$.*

PROOF. Let H_0 and H_1 be disjoint hyperplanes in X (see Theorem 9.8 of *Convexity*. I [7, p. 304]). Then $G = P\text{-fl}(H_0) \cap P\text{-fl}(H_1) \subset Y - X$ is a codimension-2 P-flat in Y. Fix $p \in H_0$, $q \in H_1$, s such that psq and let H_2 be the hyperplane $P\text{-fl}(s \cup G) \cap X$. For each $t \in M = M(p, q)$, $t \notin G$, so define the P-hyperplane $H_t = P\text{-fl}(t \cup G)$. Further, define the set $U = \{t\colon \exists t' \in H_t \cap X,\ q' \in H_1,\ s' \in H_2 \ni ps'q't'\}$. Note that $S(pq, st)$ then follows. By Property 07 there is $t_0 \in M$ such that for all $t \in U$ either $t = t_0$ or $S(pt, qt_0)$. If $H_{t_0} \cap X \neq \varnothing$ choose $t_0' \in H_{t_0} \cap X$ and let t' be such that $q't_0't'$. Since $P\text{-fl}(t' \cup G)$ is a P-hyperplane, M meets $P\text{-fl}(t' \cup G)$ at t and $t' \in P\text{-fl}(t' \cup G) = P\text{-fl}(t \cup G) = H_t$. Hence $t' \in H_t \cap X$ and $ps'q't'$, so $t \in U$ and either $t = t_0$ or $S(pt, qt_0)$. But $pq't_0't'$ implies $S(pt_0, qt)$, in contradiction. Therefore H_{t_0} is the required P-hyperplane.

The proof of Theorem 1.1 may now be completed. Define $X^* = Y - H$, where H is the hyperplane guaranteed by Theorem 5.2 and for each $M \in \mathfrak{M}$ define $M^* = M - M \cap H$, and put $\mathfrak{L}^* = \{M^* \neq \varnothing\colon M \in \mathfrak{M}\}$ (thus each line in X^* is a line in Y with one point missing). By standard arguments in the foundations of geometry Axioms B and P are immediate. We let $L^*(x, y)$ denote the unique line in X^* passing through $x \neq y$. To obtain Axiom A′, define for each three distinct points a, b, c on $M^* \in \mathfrak{L}^*$ the order relation

$$abc \text{ iff } S(ac, bm)$$

where $m = M \cap H$. Veblen's axioms of order then follow directly from 01–05 (it is necessary to use such properties of S as $S(ab, cd)$ implies $S(cd, ab)$ as proved in Coxeter [10]). One may define the total order $<$ on M^* such that $a < b < c$ in the obvious manner. It follows that Veblen's Axiom 3 yields the property that M^* has no first or last element and that our 07 yields the least upper bound property for M^*. Thus Axiom A′ has been verified.

Finally, to obtain Axiom C, let axb and xyc hold. Then the lines $L^*(a, x)$ and $L^*(b, y)$ meet H at p and q, respectively. Assuming for the moment that $p \neq q$ then $b \notin L^*(x, c) \cup L^*(a, y)$ and $y \notin L^*(a, x) \cup L^*(b, c)$. Let lines $L^*(x, y)$, $L^*(a, y)$, and $L^*(b, c)$ meet $M(p, q)$ at s, t, r, and put $M(a, y) \cap M(b, c) = z$. Note that since $M(p, q) \subset H$, r, s, and t lie on H. We observe that

$$bxap \overset{y}{\overline{\overline{\wedge}}} qstp \quad \text{and} \quad xycs \overset{b}{\overline{\overline{\wedge}}} pqrs.$$

The relations axb and xyc imply $S(ab, xp)$ and $S(xc, ys)$, so by Property 06 $S(tq, sp)$ and $S(pr, qs)$. Thus, $S(sq, rp)$ and $S(sp, qt)$, so applying 04 we find $S(sq, rt)$. Again, $S(qt, ps)$ and $S(qs, tr)$ imply $S(qt, pr)$. Hence $S(pr, qt)$ and $S(qs, tr)$, and from

$$pqrt \overset{b}{\overline{\wedge}} ayzt \quad \text{and} \quad qtsr \overset{y}{\overline{\wedge}} bzcr$$

it follows that $S(az, yt)$ and $S(bc, zr)$. That is, ayz and bzc, as desired. The proof for the case $p = q$ follows by examining the possible order relations on a single line.

The final property needed is that the two orders we now have on each line L^* in X^* and the corresponding line $L = L^* \cap X$ in X agree. Let $a, b, c \in L$ and suppose abc in X. Choose $x \in X - L$ and a' such that axa' in X; since the intersection d of $M(a, b)$ with H does not lie in X, $L' = M(x, d) \cap X$ does not meet segment $[a, b]$ of the triangle (a', a, b), so L' meets $[a', b]$ at d' and $[x, c]$ meets $[a', b]$ at c' such that $a'd'c'b$. Then by definition of S, $S(a'c', bd')$, and by 06, since $a'bc'd' \overline{\wedge}^x abcd$, we have $S(ac, bd)$. Hence abc holds in X^*. Conversely, if abc holds in X^* and $a, b, c \in X$, then a, b, c are distinct points on some line L and one of bac, abc, or acb holds in X; the only one of these compatible with the hypothesis (by the above) is the desired relation.

6. Proof of Theorem 1.2. Completing the proof of the main Theorem 1.2 amounts to essentially coordinatizing the elements of the space X^* constructed in §5 by the elements of a field (if $\dim X < \infty$). A nondimensional approach will be taken so that even though the development proceeds as in some 3-dimensional treatments (see, for example, Hartshorne [13] and Coxeter [9]), certain aspects are different.

Recall that X^* is a space satisfying Axioms A, B, C, and the parallel postulate P. Thus, X^* is what we might term an *affine space*, with all the usual properties of points, lines, planes, 3-spaces, their determination and intersectional properties, and properties of parallelism. In particular, it can be shown that parallelism is a transitive relation on the family of all lines.

Fix point $0 \in X^*$ as origin and consider the group $GL(X^*)$ of all bijective linear (line-preserving) transformations on X^*, with identity denoted by θ, and consider the normal subgroup dil X^* consisting of those transformations having the property that each line is parallel to its image. (That dil X^* is a subgroup of $GL(X^*)$ follows from the transitive law of parallelism.) Further, let tran X^* be the subgroup of translation–those elements τ of dil X^* such that either τ possess no fixed points or $\tau = \theta$–and sim X^*, the subgroup of similitudes–those elements of dil X^* which fix 0. It can be easily checked that tran X^* is a commutative normal subgroup of dil X^*. (It is well known that for projective spaces in general sim X^* is neither commutative nor normal in dil X^*.) Thus, for any $\delta \in$ dil X^* and $\tau \in$ tran X^*, $\delta\tau\delta^{-1} \in$ tran X^*.

Note that each element $\tau \neq \theta$ of tran X^* has the property that for all $x \in X^*$, $L^*(x, \tau(x))$ is parallel to a fixed line L in X^*, called the *axis* of τ. Moreover, if $\text{tran}_L X^*$ denotes the set of all translations with axis L we find that $\text{tran}_L X^*$ is a subgroup of tran X^*, a normal subgroup of dil X^*, and that $\{\theta\}$ and $\{\text{tran}_L X^* - \{\theta\}$: L is a line through 0$\}$ partitions the full group of translations. It also follows that these coaxial subgroups are isomorphic (see Lemma 6.1 below).

We adopt the notation $\tau_{a,b}$ for the unique translation which takes a to b and $\sigma_{a,b}$ with $a \neq 0$, $b \neq 0$, and $a, b, 0$ collinear, for the unique similitude which takes a to b. If we consider the translation $\sigma_{a,b}\tau\sigma_{a,b}^{-1}$ for any $\tau \in$ tran X^* and let $\tau(0) = 0'$, $\sigma_{a,b}(0') = 0''$ then it follows that

$$\sigma_{a,b}\tau\sigma_{a,b}^{-1} = \tau_{0,0''}.$$

The convention of setting $\sigma_{a,b}\tau\sigma_{a,b}^{-1} = \theta$ whenever $a = 0$ or $b = 0$ will be followed.

The next lemma is fundamental to the development and may be proved using Desargues' Theorem (see Hartshorne [13]). We omit the proof here.

LEMMA 6.1. *If* $\tau' \in \text{tran}_L X^*$ *and* $\tau'' \in \text{tran}_M X^*$ *are fixed,* $\tau' \neq \theta$ *and* $\tau'' \neq \theta$, *then an isomorphism*

$$\phi: \text{tran}_L X^* \to \text{tran}_M X^*$$

may be defined as that map which takes a member $\tau \in \text{tran}_L X^*$ *into the member* $\sigma_\tau\tau''\sigma_\tau^{-1} \in \text{tran}_M X$, *where* σ_τ *denotes the similitude which takes* $\tau'(0)$ *to* $\tau(0)$.

COROLLARY 6.2. *If* τ'' *is any member of* tran X^* *and* u, u', v, w *are collinear with* 0 *such that* $w = \tau_{0,u}\tau_{0,v}(0)$ *then*

$$\left(\sigma_{u',u}\tau''\sigma_{u',u}^{-1}\right)\left(\sigma_{u',v}\tau''\sigma_{u',v}^{-1}\right) = \sigma_{u',w}\tau''\sigma_{u',w}^{-1}.$$

PROOF. Apply Lemma 6.1 with $0, u, u', v, w$ on L and $\tau' = \tau_{0,u'}$, $\tau_1 = \tau_{0,u}$, $\tau_2 = \tau_{0,v}$. Hence $\sigma_{\tau_1} = \sigma_{u',u}$, $\sigma_{\tau_2} = \sigma_{u',v}$ and $\sigma_{\tau_1\tau_2} = \sigma_{u',w}$. The desired result follows upon substitution into the equation $\phi(\tau_1) \cdot \phi(\tau_2) = \phi(\tau_1\tau_2)$.

Next we construct a division ring R from the above stated group properties of tran X^* and sim X^*. First refine the previous notation for elements of tran X^* and sim X^* as follows:

$$\tau^x = \tau_{0,x} \quad \text{and} \quad \sigma_\lambda = \sigma_{1,\lambda}$$

where 1 is any fixed element in X^* distinct from 0 and $0, 1, \lambda$ are collinear. Put $R = L^*(0, 1)$ and define the operations $+$: $X^* \times X^* \to X^*$ and $\cdot$: $R \times X^* \to X^*$ by writing

$$\tau^{x+y} = \tau^x\tau^y, \quad \tau^{\lambda x} = \sigma_\lambda\tau^x\sigma_\lambda^{-1}.$$

In particular, if we restrict the above operations to $R \times R$ they become binary operations on R.

LEMMA 6.3. *For each* $\lambda, \mu \neq 0$ *in* R, $\sigma_{\lambda\mu} = \sigma_\lambda \sigma_\mu$.

PROOF. Observe that the translation $\sigma_{\lambda\mu}\tau^1\sigma_{\lambda\mu}^{-1}$ maps 0 to $\lambda\mu$; thus $\sigma_{\lambda\mu}\tau^1\sigma_{\lambda\mu}^{-1} = \tau^{\lambda\mu} = \sigma_\lambda\tau^\mu\sigma_\lambda^{-1}$. Now apply this equation to the element 0, proving that $\sigma_{\lambda\mu}(1) = \sigma_\lambda(\mu) = \sigma_\lambda\sigma_\mu(1)$.

At this point, all the algebraic properties which make R a ring and X^* an R-module follow routinely from the previous results. For example, to show that for all $\lambda, \mu \in R$ and $x \in X^*$, $(\lambda\mu)x = \lambda(\mu x)$ simply apply Lemma 6.3:

$$\tau^{(\lambda\mu)x} = \sigma_{\lambda\mu}\tau^x\sigma_{\lambda\mu}^{-1} = \sigma_\lambda\sigma_\mu\tau^x\sigma_\mu^{-1}\sigma_\lambda^{-1} = \sigma_\lambda\tau^{\mu x}\sigma_\lambda^{-1} = \tau^{\lambda(\mu x)}.$$

To show that $(\lambda + \mu)x = \lambda x + \mu x$ for each $\lambda, \mu \in R$, $x \in X^*$, apply Corollary 6.2 with $u' = 1$, $u = \lambda$, $v = \mu$, and $w = \tau^\lambda\tau^\mu(0) = \lambda + \mu$:

$$\left(\sigma_\lambda\tau^x\sigma_\lambda^{-1}\right)\left(\sigma_\mu\tau^x\sigma_\mu^{-1}\right) = \sigma_{\lambda+\mu}\tau^x\sigma_{\lambda+\mu}^{-1}$$

which is equivalent to

$$\tau^{\lambda x}\tau^{\mu x} = \tau^{(\lambda+\mu)x}$$

or $\tau^{\lambda x + \mu x} = \tau^{(\lambda+\mu)x}$.

The ring properties for R follow directly from the vector properties by restricting the elements of X^* to $R \subset X^*$. To show that R is a division ring, first note that 1 is a unit for R, then define for each $\lambda \neq 0$ the inverse element $\lambda^{-1} = \sigma_\lambda^{-1}(1)$. It then readily follows that $\lambda\lambda^{-1} = \lambda^{-1}\lambda = 1$. Hence R is a division ring.

The proof of Theorem 1.2 will be complete when we establish the following result.

LEMMA 6.4. *The space* $(X^*, \mathfrak{L}^*)$ *is an* R*-module such that* $\mathfrak{L}^*$ *is precisely the family of* 1*-flats defined algebraically. Moreover,* R *is an ordered division ring such that for each three collinear points* x, y, z *in the order* xyz *there exists* $0 < \lambda < 1$ *in* R *with* $y = (1 - \lambda)x + \lambda z$. *Further, the order in* R *is complete and the multiplication is commutative, making* R *the field of real numbers.*

PROOF. We have already proved that X^* is an R-module and that R is a division ring. The algebraic 1-flats are sets of the form $F(x, y) = \{(1 - \lambda)x + \lambda y: \lambda \in R\}$ for each $x \neq y$ in X^*. If $z = y - x$ then we see that

$$F(x, y) = \{x + \lambda z: \lambda \in R\} = \{\tau^x\sigma_\lambda(z): \lambda \in R\}.$$

But the latter is easily seen to be a translation of the line $L^*(0, z)$ and thus $F(x, y) = L^*(x, y) \in \mathfrak{L}^*$.

The next step is to show that R has an order compatible with the order $<_R$ as defined by the relation xyz for $x, y, z \in R$. Define the positive set P on R as simply those points λ for which either $0\lambda 1$, $\lambda = 1$, or 01λ. By geometric properties of parallelograms it follows that for $x \notin R$, 1, x, $x+1$ and $2 \equiv 1 + 1$ are the vertices of a parallelogram and $L^*(1, x+1)$ cuts $[x, 2]$ at a point y such that $xy2$. Hence, it follows that since $L^*(1, x+1)$ is parallel to $L^*(0, x)$, $L^*(1, x+1)$ meets $[0, 2]$ at 1 and 012 holds. Let $\alpha, \beta \in P$. If $0\beta 1$ then since similitudes obviously preserve order we get $\sigma_\alpha(0)\sigma_\alpha(\beta)\sigma_\alpha(1)$ or $0(\alpha\beta)\alpha$. Similarly, if 01β then $0\alpha(\alpha\beta)$, and if $\beta = 1$ then $\alpha\beta = \alpha$. Using the fact that either $0\alpha 1$, $\alpha = 1$ or 01α it follows that either $0(\alpha\beta)1$, $\alpha\beta = 1$, or $01(\alpha\beta)$. Therefore $\alpha\beta \in P$, and similarly, $\beta\alpha \in P$. By applying the similitude $\sigma_{\alpha^{-1}}$ to the cases $0\alpha 1$, $\alpha = 1$, or 01α we find either $01\alpha^{-1}$, $\alpha^{-1} = 1$, or $0\alpha^{-1}1$. Hence $\alpha^{-1} \in P$. Also, since translations preserve order, if $\gamma \in P$ then $\tau^1(\gamma) = 1 + \gamma \in P$. Thus, $\alpha^{-1}\beta \in P$ and then $1 + \alpha^{-1}\beta \in P$ so that $\alpha(1 + \alpha^{-1}\beta) = \alpha + \beta \in P$. This proves that R is an ordered division ring; we use the notation $\alpha < \beta$ iff $\beta - \alpha \in P$. It is obvious that the two orders we now have on R as an element of $\mathfrak{L}^*$ are either identical or inverse to one another.

Consider the relation xyz for three collinear points in X^*. The translation τ^{-x} maps x, y, z to 0, $y' = y - x$, $z' = z - x$ in the order $0y'z'$, and let $\sigma = \sigma_{z'y'}$; since $\sigma(1) = \lambda$ for some $\lambda \in R$ then $\sigma = \sigma_\lambda$ and σ maps 1 to λ and z' to y' so that $L^*(1, z')$ is parallel to $L^*(\lambda, y')$. Hence $0y'z'$ implies $0\lambda 1$ and $0 < \lambda < 1$ in R. But $y' = \sigma(z') = \sigma_\lambda(z') = \lambda z'$, so $y - x = \lambda(z - x)$, or $y = (1 - \lambda)x + \lambda z$.

We have shown that R is an ordered division ring. By the least upper bound property of $<_R$ and thus of $<$, R is complete, and therefore, Archimedean. The commutative law for multiplication now follows from Theorem 1 of Fuchs [**12**, p. 126].

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