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PERSISTENT MANIFOLDS ARE NORMALLY HYPERBOLIC

BY

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ABSTRACT. Let M be a smooth manifold, $f\colon M \hookleftarrow$ a C^1 diffeomorphism and $V \subset M$ a C^1 compact submanifold without boundary invariant under f (i.e. $f(V) = V$). We say that V is a persistent manifold for f if there exists a compact neighborhood U of V such that $\bigcap_{n\in\mathbf{Z}} f^n(U) = V$, and for all diffeomorphisms $g\colon M \hookleftarrow$ near to f in the C^1 topology the set $V_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ is a C^1 submanifold without boundary C^1 near to V. Several authors studied sufficient conditions for persistence of invariant manifolds. Hirsch, Pugh and Shub proved that normally hyperbolic manifolds are persistent, where normally hyperbolic means that there exist a Tf-invariant splitting $TM/V = N^sV \oplus N^uV \oplus TV$ and constants $K > 0$, $0 < \lambda < 1$ such that:

$$\|(Tf)^n/N_x^sV\| < K\lambda^n, \qquad \|(Tf)^{-n}/N_x^uV\| < K\lambda^n,$$
$$\|(Tf)^n/N_x^sV\| \cdot \|(Tf)^{-n}/T_{f^n(x)}V\| < K\lambda^n$$

for all $n > 0$, $x \in V$. In this paper we prove the converse result, namely that persistent manifolds are normally hyperbolic.

Let M be a manifold, $V \subset M$ a submanifold and $f\colon M \hookleftarrow$ a diffeomorphism satisfying $f(V) \subset V$. Several authors [1]–[9] have considered under which conditions for every diffeomorphism g nearby f there exists a submanifold $V_g \subset M$ nearby V satisfying $g(V_g) \subset V_g$. This kind of question arises frequently in ordinary, functional [18] and partial [16] differential equations as well as in the stability theory of group actions [19], bifurcations [3] and the construction of diffeomorphisms exhibiting certain persistent properties [9].

Answers to this problem are usually given in terms of exponential rates of the iterates of the derivative of f at V. In this paper we shall consider the converse problem, proving that the persistence of V (see definition below) is equivalent to these conditions.

To give the precise statement of the results let us denote by $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms of M endowed with the topology of C^1 convergence on compact subsets.

DEFINITION. V is a compact invariant manifold of $f \in \mathrm{Diff}^1(M)$ if it is a compact boundaryless C^1 submanifold of M satisfying $f(V) = V$. We say that V is persistent if there exist a neighborhood U of V in M and a

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neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that:

(a) For all $g \in \mathcal{U}$ the set $V_g = \bigcap_{n \in \mathbf{Z}} g^n(U)$ is a C^1 submanifold of M and $V_g = V$.

(b) V_g is C^1 near to V if g is C^1 near to f.

Finally let us say that V is normally hyperbolic if there exist constants $C > 0$, $0 < \lambda < 1$ and a continuous (Tf)-invariant splitting $TM/V = N^sV \oplus N^uV \oplus TV$ satisfying

$$\|(Tf)^n/N_x^sV\| \leqslant C\lambda^n, \quad \|(Tf)^{-n}/N_x^uV\| \leqslant C\lambda^n,$$
$$\|(Tf)^n/N_x^sV\| \cdot \|(Tf)^{-n}/T_{f^n(x)}V\| \leqslant C\lambda^n,$$
$$\|(Tf)^{-n}/N_x^uV\| \cdot \|(Tf)^n/T_{f^{-n}(x)}V\| \leqslant C\lambda^n$$

for all $x \in \mathbf{Z}^+$.

In [9] it is proved that normal hyperbolicity implies persistence. Here we shall prove the converse, thus obtaining

THEOREM A. *A compact invariant manifold of a diffeomorphism is persistent if and only if it is normally hyperbolic.*

This theorem follows from a stronger version given in §2 that states that a Lipschitz persistent compact invariant manifold (same definition as above requiring V_g Lipschitz near to V instead of C^1 near) is normally hyperbolic.

Recalling that a compact subset Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is said to be isolated if it has a neighborhood U such that $\bigcap_{n \in \mathbf{Z}} f^n(U) = \Lambda$, the Lipschitz version of Theorem A suggests the following conjecture:

CONJECTURE. There exists a residual subset $\mathcal{B} \subset \mathrm{Diff}^1(M)$ such that every isolated compact invariant manifold of a diffeomorphism in $\mathcal{B}$ is normally hyperbolic.

Hyperbolicity hypotheses (i.e. hypotheses involving exponential rates like normal hyperbolicity) are widely used in perturbation theory. However, the generalized feeling that they are good hypotheses in the sense that they are equivalent to the stability property they seek to grant, has, in terms of theorems, an incomplete justification. These problems when settled in $\mathrm{Diff}^r(M)$ with $r \geqslant 1$ are even harder for several reasons, among them the lack of a C^r closing lemma [14]. This is, for instance, the main stumbling block to prove a C^r version of Theorem A. Such a version would state a C^r persistent invariant compact manifold (obvious definition) is r-normally hyperbolic in the sense defined in [9].

Finally I wish to thank Jacob Palis for his advice during the preparation of my thesis [10] (of which Theorem A is part) and to C. Pugh for several useful talks on Theorem A.

1. Domination and invariant subbundles. Let K be a compact metric space. Denote by $\mathcal{L}(K)$ the set of continuous finite-dimensional vector bundles F on

K endowed with a Riemann structure, i.e. a continuous map $B\colon F \times F \to \mathbf{R}$ such that $B/(F_x \times F_x)$ is an inner product on F_x for all $x \in K$. If $F \in \mathfrak{L}(K)$ let $\mathfrak{L}(F)$ be the set of continuous vector bundle isomorphisms of F. If $\Phi \in \mathfrak{L}(F)$ we say that a subbundle $E \subset F$ is Φ-invariant if $\Phi(E) = E$. In the next proposition we give a sufficient condition for the existence of an invariant complement of E, i.e. a subbundle $G \subset F$ such that G is Φ-invariant and $\Phi(G) = G$. For the statement of the result we need the following definitions:

DEFINITION 1.1. $F_1, F_2 \in \mathfrak{L}(K)$ and $\Phi_i \in \mathfrak{L}(F_i)$, $i = 1, 2$. We say that Φ_1 dominates Φ_2, denoted $\Phi_1 > \Phi_2$, if there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|\Phi_1^n/F_{1,x}\| \cdot \|\Phi_2^{-n}/F_{2,\Phi^n(x)}\| \leqslant C\lambda^n \tag{1}$$

for all $x \in K$, $n \geqslant 0$. Moreover, if $\Phi_1 > \Phi_2$ and (1) is satisfied we say that Φ_1 (C, λ)-dominates Φ_2.

DEFINITION 1.2. Let E be a finite-dimensional Hilbert space, $E_1 \subset E$ a subspace and $E_1^{\perp}$ its orthogonal complement. If $S \subset E$ is a subspace such that $\dim S = \dim E_1$ and $S \cap E_1^{\perp} = \{0\}$, we define the angle $\alpha(E_1, S)$ by $\alpha(E_1, S) = \|L\|$, where $L\colon E_1 \to E_1^{\perp}$ is a linear map satisfying $S = \operatorname{graph}(L)$.

PROPOSITION 1.1. *Let $F \in \mathfrak{L}(K)$, $\Phi \in \mathfrak{L}(F)$, $E \subset F$ a continuous Φ-invariant subbundle, $\tilde{F} = F/E$ the quotient bundle and $\tilde{\Phi} \in \mathfrak{L}(\tilde{F})$ be the vector bundle isomorphism induced by Φ.*

If $\tilde{F}$ and E have Φ-invariant splittings $\tilde{F} = \tilde{F}_1 \oplus \cdots \oplus \tilde{F}_k$, $E = E_1 \oplus \cdots \oplus E_l$ such that for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$ either $\tilde{\Phi}/\tilde{F}_i$ (, λ)-dominates Φ/E_j or Φ/E_j (C, λ)-dominates $\tilde{\Phi}/\tilde{F}_i$, then there exists a Φ-invariant continuous subbundle $\hat{F} \subset F$ satisfying $\hat{F} \oplus E = F$ and such that for all $x \in K$,

$$\alpha(E_x^{\perp}, \hat{F}_x) \leqslant \bar{C} d^2/(1 - \lambda),$$

where

$$d = \sup\{\|\Phi/F_x\|, \|\Phi^{-1}/F_x\| \mid x \in K\}$$

and $\bar{C}$ depends only on C and

$$\delta = \sup\Big\{\alpha\big((\tilde{F}_1 \oplus \cdots \oplus \tilde{F}_j)_x^{\perp}, (\tilde{F}_{j+1} \oplus \cdots \oplus \tilde{F}_k)\big), \alpha\big((E_1 \oplus \cdots \oplus E_1)_x^{\perp}, (E_{i+1} \oplus \cdots \oplus E_l)_x\big) \mid 1 \leqslant j < k, 1 \leqslant i < l, x \in K\Big\}.$$

REMARK. It follows from this proposition that if $\pi\colon \hat{F} \to \tilde{F}$ is the canonical projection and $\hat{F}_j = \pi^{-1}\tilde{F}_j$, $j = 1, \ldots, k$, we have a Φ-invariant continuous splitting $F = E \oplus \hat{F}_1 \oplus \cdots \oplus \hat{F}_j$, and if $\tilde{\Phi}/\tilde{F}_i > \Phi/E_j$ $(\Phi/E_j > \Phi/\tilde{F}_i)$ then $\Phi/\tilde{F}_i > \Phi/E_j$ $(\Phi/E_j > \Phi/\hat{F}_i)$.

PROOF. Let G be the continuous vector bundle on K whose fiber on $x \in K$ is the space of linear transformations $L\colon E_x^{\perp} \to E_x$. Let $\Gamma^0(G)$ be the space of continuous sections of G endowed with the norm $\|\eta\| = \{\|\eta(x)\| \mid x \in K\}$. Define $\hat{\Phi}\colon \Gamma^0(G) \hookleftarrow$ by

$$\hat{\Phi}(\eta)(x) = \Phi \circ \eta\big(\Phi^{-1}(x)\big) \circ \big(\hat{\pi}\Phi/E^{\perp}_{\Phi^{-1}(x)}\big)^{-1}$$

where $\hat{\pi}\colon F \to E^{\perp}$ is the orthogonal projection. Let $\tilde{\pi}\colon F \to \tilde{F}$ be the canonical projection. Define

$$\hat{F}_j = E^{\perp} \cap \tilde{\pi}^{-1}(\tilde{F}_j), \qquad 1 \leqslant j \leqslant k.$$

Let Γ^s (resp. Γ^u) be the space of sections $\eta \in \Gamma^0(G)$ such that $\eta(x)\hat{F}_{j,x} \subset \bigoplus\{E_{i,x} | \Phi/E_i > \tilde{\Phi}/\tilde{F}_j\}$ ($\eta(x)\hat{F}_{j,x} \subset \bigoplus\{E_{i,x} | \tilde{\Phi}/\tilde{F}_j > \Phi/E_i\}$) for all $x \in K$. It is easy to see that

$$\|\hat{\Phi}^n/\Gamma^s\| \leqslant \hat{C}\lambda^n, \tag{1}$$

$$\|\hat{\Phi}^{-n}/\Gamma^u\| \leqslant \hat{C}\lambda^n \tag{2}$$

for all $n \in \mathbf{Z}^+$. If we find $\eta_0 \in \Gamma^0(G)$ satisfying

$$\eta_0(x) = \hat{\Phi}(\eta_0)(x) + \pi\Phi \circ \big(\hat{\pi}\Phi/E^{\perp}_{\Phi^{-1}(x)}\big)^{-1}, \qquad x \in K,$$

where $\pi\colon F \to E$ is the orthogonal projection, then the subbundle $\hat{F} \subset F$ with fibers $\hat{F}_x = \operatorname{graph}(\eta_0(x))$ is continuous and Φ-invariant. To solve (3) observe that (1) and (2) imply (see [9] for details) that $I - \hat{\Phi}$ has an inverse that satisfies

$$\|(I - \hat{\Phi})^{-1}\| \leqslant \frac{\bar{C}}{1 - \lambda}.$$

Let $\xi \in \Gamma^0(G)$ be defined by

$$\xi(x) = \pi\Phi \circ \big(\hat{\pi}\Phi/E^{\perp}_{\Phi^{-1}(x)}\big)^{-1}, \qquad x \in K.$$

Then $\eta_0 = (I - \hat{\Phi})^{-1}\xi$ satisfies (3) and

$$\|\eta_0\| \leqslant (C/(1 - \lambda))\|\xi\| \leqslant (C/(1 - \lambda))d^2$$

where $d = \sup\{\|\Phi/F_x\|, \|\Phi^{-1}/F_x\| \mid x \in K\}$.

PROPOSITION 1.2. *Let $F \in \mathfrak{L}(K)$, $\Phi \in \mathfrak{L}(F)$ and let $K_0 \subset K$ be a Φ-invariant compact subset. Suppose that F/K_0 has a continuous Φ-invariant splitting $F/K_0 = E_0 \oplus F_0$ such that $\Phi/E_0 > \Phi/F_0$. Then there exist a compact neighborhood U of K_0 and a continuous subbundle $E \subset F/\Lambda^s$, where $\Lambda^s = \bigcap_{n>0}\Phi^{-n}(U)$, satisfying $\Phi(E) \subset E$, $E/K_0 = E_0$.*

PROOF. Take a compact neighborhood U_0 of K_0 and let $\Lambda_0^s = \bigcap_{n \in \mathbf{Z}^+}\Phi^{-n}(U_0)$. If U_0 is small enough there exists a continuous splitting $F/\Lambda_0^s = \hat{E} \oplus \hat{F}$ such that $\hat{E}/K_0 = E_0$, $\hat{F}/K_0 = F_0$. Let $\pi_1\colon F/\Lambda_0^s \to \hat{E}$, $\pi_2\colon F/\Lambda_0^s \to \hat{F}$ be the projections associated with this splitting. Let G be the continuous vector bundle on Λ_0^s whose fiber on x is the space of linear maps

L: $\hat{E}_x \to \hat{F}_x$. If $U \subset U_0$ is a compact neighborhood of Λ_0^s, let $\Gamma^0(U)$ be the space of continuous sections η of $G/\Lambda^s(U)$, such that $\eta(x) = 0$ if $x \in K_0$, where $\Lambda^s(U) = \bigcap_{n \in \mathbf{Z}^+} \Phi^{-n}(U)$, endowed with the norm $\|\eta\| = \sup\{\|\eta(x)\| \mid x \in \Lambda^s(U)\}$. Define Φ_1: $\hat{E} \hookleftarrow$ by $\Phi_1 = \pi_1\Phi/\hat{E}$, Φ_2: $\hat{F} \hookleftarrow$ by $\Phi_2 = \pi_2\Phi/\hat{F}$ and $\hat{\Phi}$: $\Gamma^0(U) \hookleftarrow$ as $\hat{\Phi}(\eta)(x) = \Phi_2^{-1} \circ \eta(\Phi(x)) \circ \Phi_1$.

If U is small enough $\hat{\Phi}$ is well defined.

LEMMA 1.1. *There exist constants $C > 0$, $0 < \lambda < 1$ and a neighborhood $U_1 \subset U_0$ of K_0 such that for all compact neighborhoods $U \subset U_1$ of K_0 the linear map $\hat{\Phi}$: $\Gamma^0(U) \hookleftarrow$ satisfies $\|\hat{\Phi}^n\| \leqslant C\lambda^n$ for all $n \in \mathbf{Z}^+$.*

PROOF. There exists $n_0 \in \mathbf{Z}^+$ such that

$$\|\Phi^{n_0}/E_{0,x}\| \cdot \|\Phi^{-n_0}/F_{0,\Phi^{n_0}(x)}\| \leqslant \tfrac{1}{3}$$

for all $x \in K$. Then, there exists a neighborhood U_1 of K_0 such that

$$\|\Phi_1^{n_0}/\hat{E}_x\| \cdot \|\Phi_2^{-n_0}/\hat{F}_{\Phi^{n_0}(x)}\| \leqslant \tfrac{1}{2}$$

for all $x \in U_1$. Let $U \subset U_1$ be a compact neighborhood of K_0. If $x \in \Lambda^s(U)$, $\Phi^n(x) \in U$ for all $x \in \mathbf{Z}^+$. Therefore

$$\|\Phi^{kn_0}/\hat{E}_x\| \cdot \|\Phi_2^{-kn_0}/F_{\Phi^{kn_0}(x)}\| \leqslant (1/2)^k$$

for all $k \in \mathbf{Z}^+$. Then, for some constant $C > 0$, $0 < \lambda < 1$ we have

$$\|\Phi_1^n/\hat{E}_x\| \cdot \|\Phi_2^{-n}/\hat{F}_{\Phi^n(x)}\| \leqslant C\lambda^n$$

for all $x \in \Lambda^s(U)$, $n \in \mathbf{Z}^+$, and this implies

$$\|\hat{\Phi}^n\| \leqslant C\lambda^n$$

for all $n \in \mathbf{Z}^+$.

If we find $\eta_0 \in \Gamma^0(U)$ such that $\Phi(\text{graph}(\eta_0(x))) = \text{graph}(\eta_0(\Phi(x)))$ for all $x \in \Lambda^s(U)$, the lemma is proved, defining a subbundle E of $F/\Lambda^s(U)$ by $E_x = \text{graph}(\eta_0(x))$. Let $\Gamma_\varepsilon^0(U) = \{\eta \in \Gamma^0 \mid \|\eta\| \leqslant \varepsilon\}$. It is easy to see that there exist $\varepsilon_2 > \varepsilon_1 > 0$ such that for all $\delta > 0$ we can find a compact neighborhood $U_\delta \subset U$ of K_0 and a map P: $\Gamma_{\varepsilon_1}^0(U_\delta) \to \Gamma_{\varepsilon_2}^0(U_\delta)$ satisfying:

(a) $\Phi^{-1}(\text{graph}(\eta(x))) = \text{graph}(P(\eta)(\Phi^{-1}(x)))$ for all $x \in \Phi(\Lambda^s(U_\delta))$.

(b) $\|P(0)\| \leqslant \delta$.

(c) $\|(P - \hat{\Phi})(\eta_1) - (P - \hat{\Phi})(\eta_2)\| \leqslant \delta\|\eta_1 - \eta_2\|$ for all $\eta_1, \eta_2 \in \Gamma_{\varepsilon_1}^0(U_\delta)$.

Take $\delta \leqslant (1 - \lambda)/2C(1 + \varepsilon_1)$. From Lemma 1.1 it follows [4] that $I - \hat{\Phi}$ has an inverse satisfying $\|(I - \hat{\Phi})^{-1}\| \leqslant C/(1 - \lambda)$. Moreover, if $\eta \in \Gamma_{\varepsilon_1}$ and $Q = P - \hat{\Phi}$,

$$\begin{aligned}
&\|(I - \Phi)^{-1}Q(\eta)\| \\
&\qquad \leqslant \|(I - \Phi)^{-1}Q(\eta) - (I - \Phi)^{-1}Q(0)\| + \|(I - \Phi)^{-1}Q(0)\| \\
&\qquad \leqslant C\delta\varepsilon_1/(1 - \lambda) + C\delta/(1 - \lambda) \leqslant \tfrac{1}{2}\varepsilon_2;
\end{aligned}$$

hence, $(I - \Phi)^{-1}Q(\Gamma_{\varepsilon_1}^0(U_\delta)) \subset \Gamma_{\varepsilon_2}^0(U_\delta)$, and if $\eta_1, \eta_2 \in \Gamma_{\varepsilon_1}^0(U_\delta)$ we have

$$\|(I-\Phi)^{-1}Q(\eta_1)-(I-\Phi)^{-1}Q(\eta_2)\| \leqslant \frac{C\delta}{1-\lambda}\|\eta_1-\eta_2\| \leqslant \tfrac{1}{2}\|\eta_1-\eta_2\|.$$

By the fixed point theorem for contractions there exists $\eta_0 \in \Gamma^0_{\varepsilon_1}(U_\delta)$ such that $(I-\hat{\Phi})^{-1}Q(\eta_0)=\eta_0$. Therefore $P(\eta_0)=\eta_0$ and, by (a), the proposition is proved.

PROPOSITION 1.3. *Let $F \in \mathfrak{L}(K)$, $\Phi \in \mathfrak{L}(F)$ and $K \subset K_1$ be a Φ-invariant dense subset. Suppose that for all $x \in K_1$ there exists a splitting $F_x = E_x \oplus G_x$ satisfying*:

(a) $\Phi(E_x)=E_{\Phi(x)}$, $\Phi(G_x)=G_{\Phi(x)}$.

(b) *There exist $C>0$, $0<\lambda<1$ such that $\|\Phi^n/E_x\|\cdot\|\Phi^{-n}/G_{\Phi^n(x)}\| \leqslant C\lambda^n$ for all $x \in \mathbf{Z}^+$, $x \in K_1$.*

(c) $\dim E_x$ *is constant.*

Then the families of subspaces $\{E_x \mid x \in K\}$, $\{G_x \mid x \in K\}$ extend to Φ-invariant continuous subbundles E and G of F such that $F = E \oplus G$ and $\Phi/E > \Phi/G$.

PROOF. Let $\Sigma \subset K - K_1$ be a subset such that each Φ-orbit contained in $K-K_1$ intersects Σ at exactly one point. For each $x \in \Sigma$ chose a sequence $\{x_n \mid n \in \mathbf{Z}^+\} \subset K_1$ such that $x_n \to x$ when $n \to +\infty$ and the sequences $\{G_{x_n} \mid n \in \mathbf{Z}^+\}$, $\{E_{x_n} \mid n \in \mathbf{Z}^+\}$ are convergent. For $x \in \Sigma$ define

$$G_x = \lim_{n\to+\infty} G_{x_n}, \qquad E_x = \lim_{n\to+\infty} E_{x_n},$$

and for each $x \in K - K_1$, choose $x_k = \Phi^k(x) \in \Sigma$ and define

$$G_x = \Phi^{-k}(G_{x_k}), \qquad E_x = \Phi^{-k}(E_{x_k}).$$

Let P be the projective bundle associated to F and $d(\cdot,\cdot)$ a metric on P inducing its topology. If A, B are subsets of P define

$$d(A,B) = \sup\{d(a,b) \mid a \in A, b \in B\}.$$

Suppose that $\dim E_x = m$, $\dim G_x = n$ for all $x \in K$. Let $S \in \mathfrak{L}(K)$ be the vector bundle whose fiber on x is the set of subspaces (L_1, L_2) of F_x such that $\dim L_1 = m$, $\dim L_2 = n$. For $x \in K$ let H_x be the set of $(L_1, L_2) \in S_x$ such that there exists a sequence $\{x_n \mid n \in \mathbf{Z}^+\} \subset K$ satisfying $x_n \to x$, $G_{x_n} \to L_2$, $E_{x_n} \to L_1$ when $n \to +\infty$. Proving that for all $x \in K$, $H_x = \{(E_x, G_x)\}$ we are done. Observe that $\bigcup_{x\in K}\tilde{H}_x$, where

$$\tilde{H}_x = \{\theta \in P_x \mid \theta \subset L_1 \cup L_2, (L_1, L_2) \in H_x\}$$

is a closed subset of P. Moreover, if $x \in K$, and $(L_1, L_2) \in H_x$ we have

$$\|\Phi^n/L_1\|\cdot\|\Phi^{-n}/\Phi^n(L_2)\| = C\lambda^n \tag{1}$$

for all $n \in \mathbf{Z}^+$. In particular, $L_1 \cap L_2 = \{0\}$; hence $F_x = L_1 \oplus L_2$. Let $x \in K$, $(L_1, L_2) \in H_x$ with $(L_1, L_2) \neq (E_x, G_x)$; let us say $L_1 \neq E_x$. Then there exists $\theta \in P_x \cap L_1$ such that $\theta \cap E_x = \{0\}$. Since $(E_x, G_x) \in H_x$, (1) implies that $d(\Phi^n(\theta), G_{\Phi^n(x)}) \to 0$ when $n \to +\infty$. Let y be an ω-limit point of

x. Take a sequence $\{n_k \mid k \in \mathbf{Z}^+\}$ such that $\lim_{k\to+\infty} n_k = +\infty$ and the sequences $\Phi^{n_k}(L_2)$, $\Phi^{n_k}(G_x)$, $\Phi^{n_k}(E_x)$ converge to subspaces L_1', L_2', S_1', S_2' of F_x. Then $(L_1', L_2') \in H_y$, $(S_1', S_2') \in H_y$ and $L_1' \cap S_2' \neq \{0\}$. Since $L_1' \oplus L_2' = F_y$ we must have $L_2' \not\subset S_1'$. Take $\theta \in P_y$ such that $\theta \subset L_2'$ and $\theta \cap S_1' \neq \{0\}$. By (1) $d(\Phi^{-n}(\theta), \Phi^{-n}(S_1')) \to 0$ if $n \to +\infty$. Take an α-limit point z of y and a sequence $\{n_k \mid k \in \mathbf{Z}^+\} \subset \mathbf{Z}^+$ such that $\lim_{k\to+\infty} n_k = +\infty$ and $\Phi^{-n_k}(L_1')$, $\Phi^{-n_k}(L_2')$, $\Phi^{-n_k}(S_1')$, $\Phi^{-n_k}(S_2')$ converge to subspaces L_1'', L_2'', S_1'', S_2''. Then $(L_1'', L_2'') \in H_z$, $(S_1'', S_2'') \in H_z$, $L_2'' \cap S_1'' \neq \{0\}$; take $0 \neq v_1 \in L_1'' \cap S_2''$, $0 \neq v_2 \in L_2'' \cap S_1''$. Since $v_1 \in L_1''$, $v_2 \in L_2''$, (1) implies

$$\|\Phi^n(v_1)\| / \|\Phi^n(v_2)\| \leqslant K\lambda^n(\|v_1\| / \|v_2\|) \tag{2}$$

for all $n \in \mathbf{Z}^+$. But $v_1 \in S_2$, $v_2 \in S_1$. Hence (1) implies

$$\|\Phi^n(v_2)\| / \|\Phi^n(v_1)\| \leqslant K\lambda^n(\|v_2\| / \|v_1\|),$$

clearly contradicting (2).

2. Proof of Theorem A. Let M be a C^∞ boundaryless manifold and $V \subset M$ a C^1 compact boundaryless submanifold. Assume that M is a submanifold of $\mathbf{R}^n$. Let NV be a C^1 subbundle of TM/V satisfying $TV \oplus NV = TM/V$. If η is a section of NV define the Lipschitz constant of η by

$$\mathrm{Lip}(\eta) = \sup\{\|\eta(x) - \eta(y)\| / \|x - y\| \mid x, y \in V, x \neq y\}.$$

We say that η is a Lipschitz section if $\mathrm{Lip}(\eta) < +\infty$. Let $\Gamma_{\mathcal{L}}(NV)$ be the space of Lipschitz sections of NV endowed with the norm

$$\|\eta\|_{\mathcal{L}} = \sup\{\|\eta(x)\| \mid x \in V\} + \mathrm{Lip}(\eta).$$

Let $\mathrm{Diff}^1(M)$ be the space of C^1 diffeomorphisms with the topology of the C^1 convergence on compact subsets.

DEFINITION 2.1. Let $f \in \mathrm{Diff}^1(M)$. We say that V is a Lipschitz persistent invariant manifold of f if there exists a neighborhood U of V such that for all $\delta > 0$ there exists a neighborhood $\mathcal{U}_\delta$ of f such that if $g \in \mathcal{U}_\delta$ there exists $\eta \in \Gamma_{\mathcal{L}}(NV)$ with $\|\eta\|_{\mathcal{L}} \leqslant \delta$ satisfying $V_g = \mathrm{graph}(\eta)$, where $\mathrm{graph}(\eta) = \{\exp_x(\eta(x)) \mid x \in V\}$, $V_g = \bigcap_{n\in\mathbf{Z}^+} g^n(U)$.

Observe that this definition implies $V_f = V$, hence $f(V) = V$. Moreover, the Lipschitz persistence is independent of the bundle NV.

In this section we shall prove the following proposition, which clearly implies Theorem A.

PROPOSITION 2.1. *If V is a Lipschitz persistent invariant manifold of f then V is normally hyperbolic.*

To prove this proposition we shall start showing some properties of the action of Tf on the Grassmannian bundle of dim V-dimensional subspaces of the tangent space of M at points of V. More precisely, let $S_x(M)$ be the set of dim V-dimensional subspaces of T_xM. For $x \in V$, $S \in S_x(M)$ define $\alpha(S)$

$= \infty$ if $S \cap N_x V \neq \{0\}$ and $\alpha(S) = \|L\|$ if $S \cap N_x V = \{0\}$ and $L\colon T_x V \to N_x V$ is the linear map satisfying graph$(L) = S$. Let $\mathcal{O}$ be the set of diffeomorphisms of M that leave V invariant.

LEMMA 2.1. *For all $\delta > 0$ there exist $\varepsilon = \varepsilon(\delta)$ and a neighborhood $\mathcal{U} = \mathcal{U}(\delta)$ of f such that if $g \in \mathcal{U}$, $x \in V$, $S \in S_x(V)$, $m \in \mathbf{Z}^+$ and $\alpha(S) < \varepsilon$, $\alpha((Tg)^m S) \leqslant \varepsilon$, then $\alpha((Tg)^j S) \leqslant \delta$ for all $0 \leqslant j \leqslant m$.*

PROOF. Suppose the lemma false. Then there exists $\delta > 0$ such that for all neighborhoods $\mathcal{U}$ of f and $\varepsilon > 0$ we can find $y_1 \in V$, $S \in S_{y_1}(V)$, $g \in \mathcal{U} \cap \mathcal{O}$ and $0 < n < m$ such that $\alpha(S) < \varepsilon$, $\alpha((Tg)^m S) < \varepsilon$ and $\alpha((T_g)^n S) > \delta$. We can assume without loss of generality that there exists a neighborhood W_1 of y_1 such that $W_1 \cap g^j(W_1) = \varnothing$ for all $0 < j < m$. Let $W_2 = g^m(W_1)$. We can suppose that there exist C^∞ diffeomorphisms $\varphi_i\colon W_i \to \{x \in \mathbf{R}^{l_1} \times R^{l_2} \mid \|x\| \leqslant 1\}$, $i = 1, 2$, satisfying $\varphi_i(y_i) = 0$ and $\varphi_i(W_i \cap V) = \{x \in \mathbf{R}^{l_1} \mid \|x\| \leqslant 1\}$. Take $L\colon \mathbf{R}^{l_1} \to \mathbf{R}^{l_2}$, $P\colon \mathbf{R}^{l_1} \to \mathbf{R}^{l_2}$ linear maps satisfying $(T\varphi_1)S = \text{graph}(L)$, $(T\varphi_2)^m S = \text{graph}(P)$. We can assume that $T\varphi_i / T_{y_i} M$ is an isometry for $i = 1, 2$. Hence $\|P\| < \varepsilon$, $\|L\| < \varepsilon$. Let E_1 be the kernel of L and $E_2 = E_1^\perp$ its orthogonal complement. Take a C^∞ function $\psi\colon \mathbf{R} \to \mathbf{R}$ satisfying $\psi(0) = 1$, $\psi(t) = 0$ for $|t| > 1$, $|\psi'(t)| \leqslant 2$, and $|\psi(t)| \leqslant 1$ for all t.

Define: $B_i = \{x \in E_i \mid \|x\| \leqslant \frac{1}{2}\}$, $i = 1, 2$, $B_3 = \{x \in R^{l_2} \mid \|x\| \leqslant \frac{1}{2}\}$; $B = B_1 \times B_2 \times B_3$. Let $F\colon B \to \mathbf{R}^{l_2}$ defined by

$$F(x_1, x_2, x_3) = \psi(\|x_1\|^2/\lambda)\psi(\|x_2\|^2/\lambda)\psi(\|x_3\|^2/\lambda^2)Lx_2$$

where λ is a positive number. Denoting by F_1' the partial derivative of F with respect to the ith variable, it is easy to see that

$$\sup_B \|F_i'\| \to 0, \qquad i = 1, 3, \tag{1}$$

$$\sup_B \|F_2'\| \leqslant 2\varepsilon, \tag{2}$$

$$\sup_B \|F\| \to 0 \tag{3}$$

when $\lambda^{-1} \to 0$. Now define $\varphi = \varphi_2 \circ g^m \circ \varphi_1^{-1}$, $E_1' = (T\varphi)E_1$, $E_2' = (T\varphi)E_2$, $B_i' = \{x \in E_1 \mid \|x\| \leqslant 1\}$, $i = 1, 2$, $B' = B_1' \times B_2'$. Take $\tilde{H}\colon B' \to \mathbf{R}^{l_2}$ such that $\varphi(\text{graph}(F/B_1 \times B_2)) = \text{graph}(\tilde{H})$. For small values, by (1)–(3), $\tilde{H}$ is well defined and satisfies

$$\sup_{B'} \|\tilde{H}_1'\| \to 0, \tag{4}$$

$$\sup_{B'} \|\tilde{H}_2'\| \leqslant \varepsilon, \tag{5}$$

$$\sup_{B'} \|\tilde{H}\| \to 0 \tag{6}$$

when $\lambda \to 0$. Take $H\colon B' \times B_3 \to \mathbf{R}^{l_2}$ defined as $H(x_1, x_2, x_3) = \psi(4x_3)\tilde{H}(x_1, x_2)$. If F', H' denote the derivatives of F and H we can rewrite

(4)–(6) as

$$\sup_{B} \|F'\| \leqslant 2\varepsilon, \tag{7}$$

$$\sup_{B' \times B_3} \|H'\| \leqslant 2\varepsilon, \tag{8}$$

$$F'(0)/B_1 \times B_2 = L, \qquad H'(0)/B' = P.$$

Take $k \in \mathrm{Diff}^1(M)$ defined by $k(x) = x$ if $x \notin W_1 \cup W_2$, $k(x) = (\varphi_1^{-1} \circ (F + I) \circ \varphi_1)(x)$ if $x \in W_1$, $k(x) = (\varphi_2^{-1} \circ (I - H) \circ \varphi_2)(x)$ for $x \in W_2$. Follows from (7) and (8) that k is C^1 near to the identity if ε and λ are small. Hence, taking g near enough to f and ε and λ sufficiently small, it follows that $\bar{g} = k \circ g$ belongs to the neighborhood $\mathcal{U}_\delta$ given by the definition of persistence. But observe that

$$V_{\bar{g}} = \left(V - \bigcup_0^m \bar{g}^j(W_1 \cap V)\right) \cup \left(\bigcup_0^m \bar{g}^j(D)\right)$$

where $D = \varphi_1^{-1}(\mathrm{graph}(F))$. Then

$$\alpha\left(T_{\bar{g}^n(y_1)} V_{\bar{g}}\right) = \alpha\left((Tg)^n S\right) > \delta,$$

thus contradicting the definition of $\mathcal{U}_\delta$.

Now let $\tilde{N}V$ be the quotient bundle $(TM/V)/TV$ and $L(V)$ the vector bundle on V whose fibers $L_x(V)$ are the spaces of linear transformations L: $T_xV \to \tilde{N}_xV$. For $g \in \mathcal{O}$ define the vector bundle isomorphism Φ_g: $L(V) \hookleftarrow$ by $\Phi_g(L) = \tilde{N}g \circ L \circ (Tg)^{-1}$ where $\tilde{N}g$: $\tilde{N}V \hookleftarrow$ is the vector bundle isomorphism induced by Tg. Let $\Phi = \Phi_f$.

LEMMA 2.2. *Φ is a quasi-Anosov vector bundle isomorphism (i.e.* **[13]** *for all $0 \neq L \in L(V)$ the set $\{\|\Phi^n(L)\| \mid n \in \mathbf{Z}\}$ is unbounded).*

PROOF. Suppose that Φ is not quasi-Anosov. Then there exist $\bar{x} \in V$ and $0 \neq L \in L_{\bar{x}}(V)$ such that $\{\|\Phi^n(L)\| \mid n \in \mathbf{Z}\}$ is bounded, let us say by $K > 0$. Take a linear map L_0: $T_{\bar{x}}V \to N_{\bar{x}}V$ such that $\pi L_0 = L$, where π: $TM/V \to \tilde{N}V$ is the canonical projection. Let $S = \mathrm{graph}(L_0)$ and $\delta = \|L_0\| = \alpha(S)$. Let $\mathcal{U}(\delta/2)$ and $\varepsilon(\delta/2)$ be given by Lemma 2.1. As in **[20**, Lemma 1.1] we can find $1 > \varepsilon_0 > 0$ such that given any finite subset $\Sigma \subset V$ and a family of linear maps L_x: $T_xM \to T_{f(x)}M$, $x \in \Sigma$, such that $L_x(T_xV) = T_{f(x)}V$ and $\|L_x - Tf/T_xM\| \leqslant \varepsilon_0$ for all $x \in \Sigma$, then there exists $g \in \mathcal{U}(\delta/2)$ satisfying $g(V) = V$, $g(x) = f(x)$ and $Tg/T_xM = L_x$ for all $x \in \Sigma$. Choose $n \in \mathbf{Z}^+$ satisfying

$$\left(1 - \frac{\varepsilon_0}{3}\right)^n < \frac{\varepsilon_0 K^{-1} \varepsilon(\delta/2)}{12 \dim V} \tag{9}$$

and (x, L') nearby $(\bar{x}, L)$ and $g \in \mathcal{O} \cap \mathcal{U}(\delta/3)$, such that $g^j(x) \neq g^i(x)$ for all $-n \leqslant j < i \leqslant n$, $\|\Phi_g^j(L')\| \leqslant K$ for all $|j| \leqslant n$ and $\|L_0'\| > \delta/2$ where

L_0': $T_xV \to N_xV$ is a linear map satisfying $L' = \pi L_0'$. Let $S' = \text{graph}(L_0')$ and $\Sigma = \{g^j(x) | \, |j| \leqslant n\}$. By the definition of ε_0 there exists a map g: $[0, \varepsilon_0/3] \to \mathcal{O} \cap \mathcal{U}(\delta/2)$ satisfying:

(1) $g_0 = g$.

(2) $g_\lambda(g^j(x)) = g^{j+1}(x)$ for all $|j| \leqslant n$.

(3)

$$\tilde{N}g_\lambda / \tilde{N}_{g^j(x)}V = \left(1 - \frac{\varepsilon_0}{3}\right)\tilde{N}g \qquad \text{for all } 0 \leqslant j \leqslant n,$$

$$(Ng_\lambda)^{-1} / N_{g^j(x)}V = \left(1 - \frac{\varepsilon_0}{3}\right)(Ng)^{-2} \quad \text{for all } -n \leqslant j \leqslant 0.$$

(4)

$$(Tg_\lambda)^n / T_xV = (1 + \lambda)(Tg)^n, \quad (Tg_\lambda)^n / T_xV = (1 + \lambda)(Tg)^n,$$

$$(Tg_\lambda)^{-n} / T_xV = (1 + \lambda)(Tg)^{-n}.$$

We claim that for some $\lambda_0 \in [0, \varepsilon_0/3]$ we have $\alpha((Tg_{\lambda_0})^{-n}S') \leqslant \varepsilon(\delta/2)$ and $\alpha((Tg_{\lambda_0})^n S') \leqslant \varepsilon(\delta/2)$. Since $\alpha(S') > \delta/2$ this contradicts $g_{\lambda_0} \in \mathcal{U}(\delta/2) \cap \mathcal{O}$ and the lemma is proved. To find λ_0 we shall use the following property: Let L_λ^+: $T_{g^n(x)}V \to N_{g^n(x)}V$, L_λ^-: $T_{g^n(x)}V \to N_{g^{-n}(x)}V$ be linear maps such that $\text{graph}(L_\lambda^+) = (Tg_\lambda)^n S'$, $\text{graph}(L_\lambda^-) = (Tg_\lambda)^{-n}S'$ and let $\{v_1^+, \ldots, v_l^+\}$, $\{v_1^-, \ldots, v_\lambda^-\}$ be orthonormal bases of $T_{g^n(x)}V$ and $T_{g^{-n}(x)}V$. Then if $\bar{\lambda} \in [0, \varepsilon_0/3]$, $1 \leqslant j \leqslant l$ and $\|L_\lambda^\pm v_j^+\| \geqslant \varepsilon(\delta/2)$ (resp. $\|L_{\bar{\lambda}}^- v_j^-\| \geqslant \varepsilon(\delta/2)$), it follows that $\|L_\lambda^+ v_j^+\| \leqslant \varepsilon(\delta/2)$ ($\|L_\lambda^- v_j^-\| \leqslant \varepsilon(\delta/2)$) for all $\bar{\lambda} + \varepsilon_0/6 \dim V \leqslant \lambda \leqslant \varepsilon_0/3$. With this property it is easy to find the desired λ_0 as follows: Suppose that for some j we have $\|L_0^+ v_j^+\| \leqslant \varepsilon(\delta/2)$. Then $\|L_{\lambda_1}^+ v_j^+\| \leqslant \varepsilon(\delta/2)$ where $\lambda_1 = \varepsilon_0/6 \dim V$. Now suppose that $\|L_{\lambda_1}^+ v_i^+\| \geqslant \varepsilon(\delta/2)$. Then $\|L_{2\lambda}^+ v_i^+\| \leqslant \varepsilon(\delta/2)$ and $\|L_{2\lambda_1}^+ v_j^+\| \leqslant \varepsilon(\delta/2)$. At the end of this method we find $\lambda_0 = m\lambda_1$, with $0 < m < 2 \dim V$ (so $\lambda_0 \in [0, \varepsilon_0/3]$). It remains to prove the property above. For this define the linear maps A_λ: $N_xV \to N_{f^n(x)}V$, P_λ: $N_xV \to T_{f^n(x)}V$, Q_λ: $T_xV \to T_{f^n(x)}V$ such that

$$(Tg_\lambda)^n / T_xM = \begin{pmatrix} A_\lambda & P_\lambda \\ 0 & Q_\lambda \end{pmatrix}.$$

Then, if $w_j = Q_0^{-1} v_j^+$,

$$\|L_\lambda^+ v_j^+\| = \frac{\|A_\lambda L_0' w_j\|}{\|Q_0(1 + \lambda)w_j + P_\lambda L_0' w_j\|} = \frac{\|(1 - \varepsilon_0/3)^n A_0 L_0' w_j\|}{\|Q_0(1 + \lambda)w_j + P_0 L_0' w_j\|}.$$

Suppose $\|L_\lambda^\pm v_j^+\| \geqslant \varepsilon(\delta/2)$. Then, if $\lambda > \bar{\lambda} + \varepsilon_0/6 \dim V$,

$$\|Q_0(1 + \lambda)w_j + P_0 L_0' w_j\| \geqslant (\lambda - \bar{\lambda})\|Q_0 w_j\| - \|Q_0(1 + \bar{\lambda})w_j + P_0 L_0' w_j\|;$$

hence,

$$\|L_\lambda^+ v_j^+\| \leqslant \frac{1}{\dfrac{\lambda-\bar{\lambda}}{(1-\varepsilon_0/3)^n}\cdot\dfrac{\|Q_0 w_j\|}{\|A_0 L_0' w_j\|} - \dfrac{1}{\|L_\lambda^+ v_j\|}}$$

$$\leqslant \frac{1}{\dfrac{\lambda-\bar{\lambda}}{(1-\varepsilon_0/3)^n} K^{-1} - (\varepsilon(\delta/2))^{-1}}. \tag{10}$$

By (9),

$$\frac{\lambda-\bar{\lambda}}{(1-\varepsilon_0/3)^n} K^{-1} \geqslant \frac{\varepsilon_0 K^{-1}}{6 \dim V(1-\varepsilon_0/3)^n} \geqslant 2(\varepsilon(\delta/2))^{-1};$$

hence (10) implies

$$\|L_\lambda^+ v_j^+\| \leqslant \varepsilon(\delta/2).$$

In the next lemma we show that $TM/\Omega(f/V)$ (where $\Omega(f/V)$ denotes the set of nonwandering points of f/V) has a splitting satisfying the domination conditions required by the definition of normal hyperbolicity.

LEMMA 2.3. *There exists a continuous Tf-invariant splitting $TM/\Omega(f/V) = TV/\Omega(f/V) \oplus N^s \oplus N^u$ such that $Tf/N^s > Tf/TV > Tf/N^u$.*

PROOF. If $g \in \mathcal{O}$ define $E_x^s(g)$ (resp. $E_x^u(g)$) as the set of $L \in L_x(V)$ such that $\{\|\Phi_g^n(L)\| \mid n \geqslant 0\}$ ($\{\|\Phi_g^{-n}(L)\| \mid n \geqslant 0\}$) is bounded,

$$\tilde{N}_x^s(g) = \cup\{L(T_xV) \mid L \in E_x^s(g)\},$$

$$N_x^u(g) = \cup\{L(T_xV) \mid L \in E_x^u(g)\}$$

and $\mathrm{Per}(g)$ as the set of periodic points of g/V. In the proof of this lemma we shall use the following property whose proof will be given later:

LEMMA 2.4. *There exist a neighborhood $\mathcal{U}$ of f and a dense subset $\mathcal{S}$ of $\mathcal{U} \cap \mathcal{O}$ such that for all $g \in \mathcal{S}$ and $x \in \mathrm{Per}(g)$ we have $\tilde{N}_x^s(g) \cap \tilde{N}_x^u(g) = \{0\}$.*

Suppose we show that

$$\tilde{N}_x^s(f) \oplus \tilde{N}_x^u(f) = \tilde{N}_x V \tag{11}$$

for all $x \in \Omega(f/V)$ and that there exist $C > 0$, $0 < \lambda < 1$ satisfying

$$\|(\tilde{N}f)^n/\tilde{N}_x^s(f)\| \cdot \|(Tf)^{-n}/T_{f^n(x)}V\| \leqslant C\lambda^n, \tag{12}$$

$$\|(\tilde{N}f)^{-n}/\tilde{N}_x^u(f)\| \cdot \|(Tf)^n/T_{f^{-n}(x)}V\| \leqslant C\lambda^n \tag{13}$$

for all $x \in \Omega(f/V)$, $n \geqslant 0$. Then by Proposition 1.1 the maps $x \to \tilde{N}_x^s(f)$, $x \to \tilde{N}_x^u(f)$ define continuous $\tilde{N}f$-invariant subbundles $\tilde{N}^s$, $\tilde{N}^u$ of $\tilde{N}V/\Omega(f/V)$ such that $\tilde{N}f/N^s > Tf/(TV/\Omega(f/V)) > \tilde{N}f/N^u$. Hence, by Lemma 1.1, there exists a continuous Tf-invariant splitting satisfying our statement. So if we prove (11)–(13) we are done. By [**13**, Proposition 1.1 and

Lemma 1.2] there exist a neighborhood $\mathcal{U}$ of f, $C > 0$ and $0 < \lambda < 1$ such that Φ_g is quasi-Anosov and

$$\|\Phi_g^n / E_x^s(g)\| < K\lambda^n, \tag{14}$$

$$\|\Phi_g^{-n} / E_x^u(g)\| < K\lambda^n \tag{15}$$

for all $g \in \mathcal{U} \cap \mathcal{O}$, $x \in V$ and $n \geqslant 0$. Moreover, since Φ_g is quasi-Anosov, it is easy to see that $E_x^s(g) \oplus E_x^u(g) = L_x(V)$, hence $\tilde{N}_x^s(g) + \tilde{N}_x^u(g) = \tilde{N}_x V$ for all $g \in \mathcal{U} \cap \mathcal{O}$, $x \in \mathrm{Per}(g)$. Let $G_x^s(g)$, $G_x^u(g)$ be the spaces of linear maps from $T_x V$ into $\tilde{N}_x^s(g)$ and $\tilde{N}_x^u(g)$, respectively. We claim that $g \in \mathcal{U} \cap \mathcal{S}$ and $x \in \mathrm{Per}(g)$ imply

$$G_x^s(g) = E_x^s(g), \tag{16}$$

$$G_x^u(g) = E_x^u(g). \tag{17}$$

Suppose, for instance, that (16) is false. Then $\dim G_x^s(g) > \dim E_x^s(g)$. Therefore,

$$\begin{aligned}\dim T_x V \cdot \dim \tilde{N}_x V = \dim L_x(V) &= \dim E_x^s(g) + \dim E_x^u(g)\\ &< \dim G_x^s(g) + \dim G_x^u(g)\\ &= \dim T_x V(\dim \tilde{N}_x^s(g)) + \dim \tilde{N}_x^u(g)\\ &= \dim T_x V \cdot \dim \tilde{N}_x V.\end{aligned}$$

Now consider $x \in \Omega(f/V)$. There exist sequences $\{g_n | n \in \mathbf{Z}^+\} \subset \mathcal{S}$, $\{x_n | n \in \mathbf{Z}^+\} \subset V$ such that $x_n \in P(g_n)$, $g_n \to g$ and $x_n \to x$ when $n \to +\infty$ **[14]**. We can assume that there exist subspaces E^+, E^- of $L_x(V)$ such that

$$\lim E_{x_n}^s(g_n) = E^+, \tag{18}$$

$$\lim E_{x_n}^u(g_n) = E^-. \tag{19}$$

Since $E_{x_n}^s(g_n) \oplus E_{x_n}^u(g_n) = L_x(V)$ we obtain

$$\dim E^+ + \dim E^- = \dim L_x(V) \tag{20}$$

and (14), (15) imply $\|\Phi^n / E^+\| < K\lambda^n$, $\|\Phi^{-n} / E^-\| < K\lambda^n$ for all $n \geqslant 0$. Since Φ is quasi-Anosov this implies $E^+ \cap E^- = \{0\}$, which together with (20) gives $L_x(V) = E^+ \oplus E^-$, hence $E^+ = E_x^s(f)$, $E^- = E_x^u(f)$; then

$$E_x^s(f) \oplus E_x^u(f) = L_x(V). \tag{21}$$

Moreover, (16)–(19) and the elementary properties

$$G_x^s(f) = \lim G_{x_n}^s(g_n), \quad G_x^u(f) = \lim G_{x_n}^u(g_n)$$

prove that

$$G_x^s(f) = E_x^s(f), \tag{22}$$

$$G_x^u(f) = E_x^u(f). \tag{23}$$

By (21) $\tilde{N}_x^s(f) + \tilde{N}_x^u(f) = \tilde{N}_x V$ and by (21)–(23), $\tilde{N}_x^s(f) \cap \tilde{N}_x^u(f) = \{0\}$. This proves (11). (12) and (13) follow from (14), (15), (22) and (23).

PROOF OF LEMMA 2.4. Let $\mathfrak{U}$ be a neighborhood of f such that Φ_g is quasi-Anosov for all $g \in \mathcal{O} \cap \mathfrak{U}$. Let $\mathfrak{S}$ be the set of diffeomorphisms $g \in \mathfrak{U} \cap \mathcal{O}$ satisfying the following properties:

(I) $\tilde{N}_g$ is a C^1 vector bundle isomorphism.

(II) $\mathrm{Per}(g)$ is dense in $\Omega(g/V)$.

(III) Every periodic point of g/V is hyperbolic.

(IV) If $x \in \mathrm{Per}(g)$ and m is its period, all the eigenvalues of $(Tf)^m/T_xM$ are simple and two of them have equal modulus if and only if they are conjugates.

Now observe that for all $g \in \mathfrak{S}$, the set of points $x \in \mathrm{Per}(g)$ such that $\tilde{N}_x^s(g) \cap \tilde{N}_x^u(g) = \{0\}$ is closed in $\mathrm{Per}(g)$. To see this suppose that $\{x_n | n \in \mathbf{Z}^+\}$ is a sequence contained in $\mathrm{Per}(g)$, $g \in \mathfrak{S}$, and $x_n \to x \in \mathrm{Per}(g)$ when $n \to +\infty$. If $\tilde{N}_{x_n}^s(g_n) \cap \tilde{N}_{x_n}^u(g_n) = \{0\}$, we deduce, as in the proof of Lemma 2.3, that $G_{x_n}^s(g) = E_{x_n}^s(g)$, $G_{x_n}^u(g) = E_{x_n}^u(g)$ and

$$\lim E_{x_n}^s(g_n) = E_x^s(g), \quad \lim E_{x_n}^u(g_n) = E_x^u(g).$$

Since, clearly, $\lim G_{x_n}^s(g) = G_x^s(g)$ and $\lim G_{x_n}^u(g) = G_x^u(g)$, it follows that $E_x^s(g) = G_x^s(g)$, $E_x^u(g) = G_x^u(g)$. If $\tilde{N}_x^s(g) \cap \tilde{N}_x^u(g) \neq \{0\}$, these relations imply $E_x^s(g) \cap E_x^u(g) \neq \{0\}$, contradicting the fact that Φ_g is quasi-Anosov. Suppose then that for some $g_0 \in \mathfrak{S}$ there exists $a \in \mathrm{Per}(g_0)$ satisfying $\tilde{N}_a^s(g) \cap \tilde{N}_a^u(g) \neq \{0\}$. By the previous remark this implies that there exists an open subset W of $\mathrm{Per}(g_0)$ such that $\tilde{N}_x^s(g_0) \cap \tilde{N}_x^u(g_0) \neq \{0\}$ for every $x \in W$. To exhibit a contradiction between this fact and our definition of $\mathfrak{S}$ we need the following definitions and Lemmas 2.5–2.7.

DEFINITION 2.2. Let $g \in \mathfrak{S}$, $x \in \mathrm{Per}(g)$. We say that

$$\tilde{N}_x V = \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_l, \tag{24}$$

$$T_x V = T_1 \oplus \cdots \oplus T_k \tag{25}$$

are the canonical splittings at x if they satisfy the following conditions:

(I) $(Tg)^m T_i = T_i$, $(\tilde{N}g)^m \tilde{N}_j = \tilde{N}_j$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, where m is the period of x.

(II) $(Tg)^m/T_{i_1} > (Tg)^m/T_{i_2}$, $(\tilde{N}g)^m/\tilde{N}_{j_1} > (\tilde{N}g)^m/\tilde{N}_{j_2}$ for all $1 \leqslant i_1 < i_2 \leqslant k$, $1 \leqslant j_1 < j_2 \leqslant l$.

(III) For all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, $(\tilde{N}g)^m/\tilde{N}_j > (Tg)^m/T_i$ or $(Tg)^m/T_i > (\tilde{N}g)^m/\tilde{N}_j$.

(IV) For all $1 \leqslant i_1 < i_2 \leqslant k$ there exists $1 \leqslant j \leqslant l$ such that $(Tg)^m/T_{i_1} > (\tilde{N}g)^m/\tilde{N}_j > (Tg)^m/T_{i_2}$.

LEMMA 2.5. *There exists $\gamma > 0$ such that if $g \in \mathfrak{S}$, $x \in \mathrm{Per}(g)$ and* (24), (25) *are the canonical splittings at x, then*

$$\alpha\big((T_1 \oplus \cdots \oplus T_i)^\perp, (T_{i+1} \oplus \cdots \oplus T_k)\big) \leqslant \gamma,$$

$$\alpha\big((\tilde{N}_1 \oplus \cdots \oplus \tilde{N}_j)^\perp, (\tilde{N}_{j+1} \oplus \cdots \oplus \tilde{N}_l)\big) \leqslant \gamma$$

for all $1 \leqslant i < l$, $1 \leqslant j < l$.

PROOF. We shall prove only the first inequality. The second one is proved applying the same method. From (14) and (15) it follows that there exists $k > 0$ such that

$$\|L_1 - L_2\| \geqslant k \tag{26}$$

for all $g \in \mathfrak{A}$, $x \in V$, $L_1 \in E_x^s(g)$, $L_2 \in E_x^u(g)$, $\|L_1\| = \|L_2\| = 1$. Suppose that, contradicting the lemma, we can find for $\varepsilon = k/3$ a diffeomorphism $g \in \delta$ and $x \in \mathrm{Per}(g)$ with canonical splittings (24), (25) such that $\alpha(T_-^\perp, T_+) \leqslant \varepsilon$, where $T_+ = T_{i+1} \oplus \cdots \oplus T_k$, $T_- = T_1 \oplus \cdots \oplus T_i$. Suppose, for instance, that $\dim T_-^\perp \leqslant \dim T_+$. Let $T' \subset T_+$, $T'' \subset T_+$ be subspaces such that $\dim T' = \dim T_-$, $T'' = (T' \oplus T_-)^\perp$. Take an isometry G: $T_xV \hookleftarrow$ such that $\|G - I\| \leqslant 2\varepsilon$, $G(T'') = T''$, $G(T') = T_-$. Let $1 \leqslant j \leqslant l$ satisfy $(Tg)^m/T_- > (\tilde{N}g)^m\tilde{N}_j > (Tg)^m/T_+$ and L: $T_xV \to \tilde{N}_j$ be a linear map with $\|L\| = 1$, $L/T_+ = 0$, $LG/T' \neq 0$. Then $L \in E_x^u(g)$, $LG \in E_x^s(g)$ and $\|L - G\| \leqslant \|L\| \cdot \|I - G\| \leqslant 2\varepsilon = \frac{2}{3}k$, contradicting (26).

LEMMA 2.6. *There exist $C' > 0$, $0 < \lambda < 1$ such that if $g \in \mathfrak{S}$, $x \in \mathrm{Per}(g)$ and* (24), (25) *are the canonical splittings at x, then for all $1 \leqslant j \leqslant l$, $1 \leqslant i \leqslant l$ we have*

$$\|(\tilde{N}g)^n/\tilde{N}_j\| \cdot \|(Tg)^{-n}/(Tg)^nT_i\| \leqslant C'\lambda^n$$

for all $n \in \mathbf{Z}^+$ or

$$\|(\tilde{N}g)^{-n}/(\tilde{N}g)^n\tilde{N}_j\| \cdot \|(Tg)^n/T_i\| \leqslant C'\lambda^n$$

for all $n \in \mathbf{Z}^+$.

PROOF. Suppose that $(\tilde{N}g)^m/\tilde{N}_j < (Tg)^m/T_i$ for all $i \leqslant i_0$ and $(Tg)^m/T_i < (\tilde{N}g)^mN_j$ if $i' > i_0$. We claim that

$$\|(\tilde{N}g)^n/\tilde{N}_j\| \cdot \|(Tg)^{-n}/(Tg)^nT_-\| \leqslant C'\lambda^n, \tag{27'}$$

$$\|(\tilde{N}g)^{-n}/(\tilde{N}g)^nN_j\| \cdot \|(Tg)^n/T_+\| \leqslant C'\lambda^n \tag{27''}$$

for all $n \in \mathbf{Z}^+$, where $T^+ = T_1 \oplus \cdots \oplus T_{i_0}$, $T^- = T_{i_0+1} \oplus \cdots \oplus T_k$, where $C' > 0$ is a constant independent of $g \in \mathfrak{S}$, $x \in \mathrm{Per}(g)$ and $n \in \mathbf{Z}^+$ and λ satisfies (14) and (15). If we prove this claim we are done. Let us prove (27′) ((27″) follows in a similar way). Take $v \in \tilde{N}_j$, $w \in T'$ with $\|w\| = 1$. Let $\mathfrak{B}$ be an orthonormal basis of T_- containing w. Define a linear map L: $T^- \to \tilde{N}_j$ by $Lu = v$ if $u \in \mathfrak{B}$ and $Lu = 0$ if $u \in T_+$. There exists a constant $\gamma' > 0$ depending only on γ such that $\|L\| \leqslant \gamma'\|L/T^-\|$. Hence $\|L\| \leqslant \gamma'\|v\|$. Clearly $L \in E_x^s(g)$. Therefore, by (14),

$$\|(\tilde{N}g)^nv\|/\|(\tilde{N}g)^nw\| \leqslant \|\Phi_g^n(L)\| \leqslant C\lambda^n\|L\| \leqslant C\gamma'\lambda^n\|v\|. \tag{28}$$

Since

$$\|(\tilde{N}g)^n/\tilde{N}_j\|\cdot\|(Tg)^{-n}/(Tg)^n T^-\|$$
$$= \sup\left\{\frac{\|(\tilde{N}g)^n v\|}{\|v\|\cdot\|(Tg)^n w\|}\,\middle|\, v\in\tilde{N}_j,\ w\in T^-,\ \|w\|=1\right\},$$

putting $C' = C\gamma'$, (27′) follows from (28).

Finally we want to prove that if $g \in \mathcal{S}$ and x is a periodic point of g/V with period m there exists a subspace $N_x \subset T_xM$ that is a $(Tg)^m$-invariant complement of T_xV, and we shall give an upper bound for $\alpha((T_xV)^\perp, N_x)$. For this let us say that if $g \in \mathcal{S}$ and $x \in \mathrm{Per}(g)$, the multindex $\sigma = (n_1, \ldots, n_k, m_1, \ldots, m_l)$ is the signature of x if the canonical splittings at x are (24) and (25) and satisfy $\dim \tilde{N}_i = n_i$, $\dim T_j = m_j$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$. Let $\Lambda_\sigma(g)$ be the set of periodic points of g/V with signature σ and let $\overline{\Lambda}_\sigma(g)$ be its closure. By Proposition 1.2 and Lemma 2.6 there exist splittings $\tilde{N}V/\overline{\Lambda}_\sigma(g) = \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_l$, $TV/\overline{\Lambda}_\sigma(g) = T_1 \oplus \cdots \oplus T_k$ satisfying

$$\tilde{N}g/\tilde{N}_{i'} > \tilde{N}g/\tilde{N}_{i''}, \qquad Tg/T_{j'} > Tg/T_{j''}$$

for all $1 \leqslant i' \leqslant i'' \leqslant l$, $1 \leqslant j' < j'' \leqslant k$ and $\tilde{N}g/\tilde{N}_i > Tg/T_j$ or $Tg/T_j > \tilde{N}g/\tilde{N}_i$ for all $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k$. By Proposition 1.1 there exist continuous Tf-invariant subbundles $N_1, \ldots, N_l$ of $TM/\overline{\Lambda}_\sigma(g)$ such that

$$T_1 \oplus \cdots \oplus T_k \oplus N_1 \oplus \cdots \oplus N_l = TM/\overline{\Lambda}_\sigma(g).$$

By Lemma 2.6 and Proposition 1.1 there exists $\gamma_1 > 0$ depending on γ (given by Lemma 2.5), C' and λ (given by 2.6) such that

$$\alpha\big((T_xV)^\perp, (N_1 \oplus \cdots \oplus N_l)_x\big) \leqslant \gamma_1$$

for all $x \in \overline{\Lambda}_\sigma(g)$. Resuming we have proved the following property:

LEMMA 2.7. *There exists $\gamma_1 > 0$ such that for all $g \in \mathcal{S}$ and every signature σ, there exists a continuous Tf-invariant subbundle N of $TM/\overline{\Lambda}_\sigma(g)$ such that*

$$TV/\overline{\Lambda}_\sigma(g) \oplus N = TM/\overline{\Lambda}_\sigma(g), \quad \alpha\big((T_xV)^\perp, N_x\big) \leqslant \gamma_1$$

for all $x \in \overline{\Lambda}_\sigma(g)$.

Let us return now to the situation we were considering before Lemma 2.5: we have $g_0 \in \mathcal{S}$ near to f such that $\mathrm{Per}(g_0)$ contains a subset W, open in the relative topology in $\mathrm{Per}(g_0)$ such that $\tilde{N}^s_x(g_0) \cap \tilde{N}^u_x(g_0) \neq \{0\}$ for all $x \in W$. Since the set of possible signatures is finite, for some σ the set $W \cap \Lambda_\sigma(g)$ contains an open subset W_0 of $\mathrm{Per}(g_0)$. Let γ_1 be the constant given in Lemma 2.7. Take $\beta > 0$ and $\rho > 0$ satisfying the following property: If $x \in V$, $S \subset T_xM$ is a suspace satisfying $S \oplus T_xV = T_xM$, $\alpha((T_xV)^\perp, S) \leqslant 2\gamma_1$, $v \in S$, $y \in V$, $w \in N_yV$ and $\|v\| \leqslant \rho$, $\|w\| \leqslant \rho$ and $\exp(v) = \exp(w)$:

$$\|x - y\| \leqslant \beta\|v\|, \qquad \|v\| \leqslant \beta\|w\|.$$

Let $\delta = \min(\rho, \frac{2}{3}\beta)$. Since g_0 is near to f we can suppose that $g_0 \in \mathcal{U}_{\delta/2}$

(see Definition 2.1). Using g_0 we shall construct $\bar{g} \in \mathfrak{U}_{2\delta/3}$ such that $V_{\bar{g}} = \{\exp h(p)/p \in V\}$ where h is a section of NV with $\operatorname{Lip}(h) > \delta$, contradicting $\bar{g} \in \mathfrak{U}_{2\delta/3}$. For this take the subbundle N of $TM/\bar{\Lambda}_\sigma(g_0)$ given by Lemma 2.7. There exist a neighborhood B of $\bar{\Lambda}_\sigma(g)$ in M, a C^∞ subbundle N' of TM/B satisfying $N'/\bar{\Lambda}_\sigma(g_0) \oplus TV/\bar{\Lambda}_\sigma(g_0) = TM/\bar{\Lambda}_\sigma(g_0)$, $\alpha((T_xV)^\perp, N'_x) \leqslant 2\gamma_2$ for all $x \in \bar{\Lambda}_\sigma(g_0)$ and a diffeomorphism $g \in \mathfrak{U}_{3\delta/5}$ such that $g \in \mathfrak{S}$, $g/V = g_0$, $N'/\bar{\Lambda}_\sigma(g_0)$ is Tg-invariant, $\tilde{N}g = \tilde{N}g_0$ and $g(\exp v) = \exp((Tg)v)$ for all $v \in N'/\bar{\Lambda}_\sigma(g_0)$ small enough, say with $\|v\| \leqslant r$. Denote $\Lambda = \bar{\Lambda}_\sigma(g_0)$. We start the construction of $\bar{g}$ proving the following property:

LEMMA 2.8. *For all* $0 \neq v \in N'/\Lambda$ *the set* $\{\|(Tg)^n v\| \mid n \in \mathbf{Z}\}$ *is unbounded.*

PROOF. Suppose that $0 \neq v \in N'/\Lambda$ and $\{\|(Tg)^n v\| \mid n \in \mathbf{Z}\}$ is bounded, say by K. Take $w = \lambda v$ such that $K\lambda \leqslant r/2$. Then $d(g^n(\exp w), V) = d(\exp(Tg)^n w, V) \leqslant K\lambda$ for all $n \in \mathbf{Z}^+$, hence $g^n(\exp w) \in \bigcap_{n\in\mathbf{Z}} g^n(U)$ if λ is small enough (where U is given by Definition 2.1). Therefore $V_g \supset \{w\} \cup V$, contradicting Definition 2.1.

By the results in [13] or [21] this lemma implies that there exists a continuous Tf-invariant splitting $N'/\Lambda = N^s \oplus N^u$ and constants $C > 0$, $0 < \sigma < 1$ such that

$$\|(Tg)^n/N^s_x\| \leqslant C\sigma^n, \qquad \|(Tg)^{-n}/N^u_x\| \leqslant C\sigma^n$$

for all $n \in \mathbf{Z}^+$. Take $x_0 \in W_0$. The condition $\tilde{N}^s_{x_0}(g) \cap \tilde{N}^u_{x_0}(g) \neq \{0\}$ implies that there exists an eigenvalue λ of $(\tilde{N}g)^m/\tilde{N}_{x_0}V$ (m being the period of x_0) and eigenvalues λ_1, λ_2 of $(Tg)^m/T_xV$ such that $|\lambda_1| < |\lambda| < |\lambda_2|$. Otherwise $E^s_x(g)$ (resp. $E^u_x(g)$) would be the space of linear maps from T_xV in the subspace $\tilde{N}^+$ ($\tilde{N}^-$) of $\tilde{N}_xV$ spanned by the invariant subspaces associated to eigenvalues with modulus smaller (greater) than the modulus of every eigenvalue of $(Tg)^m/T_xV$; hence $\tilde{N}^s_{x_0}(g) = \tilde{N}^+$, $\tilde{N}^u_{x_0}(g) = \tilde{N}^-$. Since every period of g is hyperbolic (because $g \in \mathfrak{S}$) there exist two possible cases: $|\lambda_1| < |\lambda| < |\lambda_2|$ and $1 < |\lambda| < |\lambda_2|$. Consider the case $|\lambda_1| < |\lambda| < 1$. The other case is handled with the same method. Let $N_0 \subset N'_{x_0}$ be the subspace associated to the eigenvalue λ, and $T_0 \subset T_{x_0}V$ be the subspace associated with λ_1. There exists an embedded disc $D \subset V$ containing x_0 and satisfying $T_{x_0}D = T_0$, $g(D) \subset D$. We have two possible situations: either every neighborhood of x_0 in D contains a wandering point of g_0/V or it does not. Suppose that we have the second situation. Then $\Lambda \cap D$ is a neighborhood of x_0 in D. Take $y \in \Lambda \cap D$. By [9] there exists a continuous subbundle N'_0 of N'/D such that $(Tg)^m N'_0 \subset N'_0$ and $N'_{0,x_0} = N_0$. The properties of D imply $\lim\|(Tg)^n/N'_{0,x_0}\| = 0$ for all $x \in D$. Hence $N'_0 \subset N^s/D$. Take a C^∞ section ξ of TM such that ξ/B is a section of N', the support of ξ is contained in a neighborhood $U_0 \subset \Lambda$ of a point $y \in D$ satisfying $x_0 \notin U_0$, $\xi(y) \neq 0$, $\bigcup_1^k g^j(U_0) \cap U_0 = \varnothing$, where k satisfies $\sigma^{k+1}r \leqslant 2\|\xi(y)\|$. Define a section η of $N'V$ by

$$\eta(x) = \sum_0^\infty (Tg)^n \xi(g^{-n}(x)).$$

For all $x \in \Lambda$ we have

$$\|\eta(x)\| \leqslant \sum_0^\infty \|(Tg)^n \xi(g^{-n}(x))\| \leqslant C \sum_0^\infty \sigma^n \|((Tg)^{-n}(x))\|.$$

Suppose that $\|\xi(p)\| \leqslant r(1-\sigma)/2C$ for all $p \in V$. Then

$$\|\eta(x)\| \leqslant r/2 \tag{29}$$

for all $x \in V$. The condition $g^j(U_0) \cap U_0 = \varnothing$ for all $1 \leqslant j \leqslant k$ implies

$$\begin{aligned}\|\eta(y)\| &\geqslant \|\xi(y)\| - \sum_{k+1}^\infty \|(Tg)^n \xi(g^{-n}(y))\| \\ &\geqslant \|\xi(y)\| - C\sigma^{k+1}/(1-\sigma)\cdot r(1-\sigma)/2C > 0\end{aligned} \tag{30}$$

and it follows from the definition of η that

$$(Tg)\eta(x) + \xi(g(x)) = \eta(g(x)) \tag{31}$$

for all $x \in \Lambda$. Given $\mu \in R$ take $\bar{g}$ such that

$$\bar{g} \in \mathfrak{U}_\delta, \tag{32}$$

$$\bar{g}(\exp v) = \exp\big(\mu\xi(g(x)) + (Tg)v\big) \tag{33}$$

for all $x \in \Lambda$, $v \in N'_x$ with $\|v\| \leqslant \mu r/2$. If $|\mu|$ is small there exists $\bar{g}$ satisfying (32) and (33). By (33) and (31) we have

$$\bar{g}(\exp \mu\eta(x)) = \exp\big(\mu\xi(g(x)) + (Tg)\eta(x)\big) = \exp \mu\eta(g(x)).$$

Hence,

$$\bar{g}^n(\exp \mu\eta(x)) = \exp \mu\eta(g^n(x)) \tag{34}$$

for all $n \in \mathbf{Z}$, $x \in \Lambda$. This and (29) imply that the distance between $\bar{g}^n(\exp \mu\eta(x))$ and V is $\leqslant \mu r/2$ for all $n \in \mathbf{Z}$. Therefore, by Definition 2.1, this proves that $\exp \mu\eta(x) \in V_{\bar{g}}$ for all $x \in \Lambda$. Now take $|\lambda_1| < c_1 < c < |\lambda|$ and $K > 0$ satisfying $\|(Tg)^n v\| \geqslant Kc^n\|v\|$ for all $n \in \mathbf{Z}^+$, $v \in N'_{0,y}$ and $\|g^n(y) - g^n(x_0)\| \leqslant Kc_1^n$ for all $n \in \mathbf{Z}^+$. By the condition $g^n(y) \notin U_0$ for all $n \geqslant 1$ and (30),

$$\eta(g^n(y)) = (Tg)^n \eta(y).$$

Hence

$$\frac{\|\eta(g^n(y))\|}{\|g^n(y) - g^n(x_0)\|} = \frac{\|(Tg)^n\eta(y)\|}{\|g^n(y) - g^n(x_0)\|} \geqslant \left(\frac{c}{c_1}\right)^n \|\eta(y)\|.$$

Now let $h \in \Gamma_{\mathfrak{C}}(NV)$ such that $V_{\bar{g}} = \{\exp h(p)/p \in V\}$ and $\|h\|_{\mathfrak{C}} \leqslant \delta$. We claim that $\exp \mu\eta(y) \in V_{\bar{g}}$. This follows from (34), which gives $\bar{g}^n(\exp \mu\eta(y)) = \exp \mu\eta(g^n(y))$ for all $n \in \mathbf{Z}$. Hence if μ is small enough, $\bar{g}^n(\exp \mu\eta(y)) \in U$ (U given by Definition 2.1); thus $\exp \mu\eta(y) \in V_{\bar{g}}$. Take $z_n \in V$ such that

$\exp h(z_n) = \bar{g}^n(\exp \mu\eta(y)) = \exp \mu\eta(g^n(y))$. Observing that the g-orbit and $\bar{g}$-orbit of x_0 are the same we obtain $h(g^n(x_0)) = 0$ for all $n \in \mathbf{Z}$. Therefore,

$$\begin{aligned}\operatorname{Lip}(\eta) &\geqslant \frac{\|h(z_n) - h(g^n(x_0))\|}{\|z_n - g^n(x_0)\|} = \frac{\|h(z_n)\|}{\|z_n - g_n(x_0)\|} \\ &\geqslant \frac{1}{\|z_n - g^n(y)\|/\|h(z_n)\| + \|g^n(y) - g^n(x_0)\|/\|h(z_n)\|} \\ &\geqslant \frac{1}{\beta(1 + \|g^n(y) - g^n(x_0)\|/\|h(z_n)\|)} \geqslant \frac{1}{\beta\big(1 + (c_1/c)^n/\mu\|\eta(y)\|\big)}.\end{aligned}$$

Since by (30) $\|\eta(y)\| > 0$, taking $n \to +\infty$ we obtain $\operatorname{Lip}(\eta) > 1/2\beta = \delta$, contradicting (32). This completes the proof when x_0 is in the closure of $\operatorname{Per}(g_0) - \{x_0\}$.

It remains to consider the case when $\Omega(g/V) \cap D$ is not a neighborhood of x_0 in D. This case is considered in the next lemma.

LEMMA 2.8. *There exists a neighborhood $\mathcal{V}$ of f such that if $g \in \mathcal{V} \cap \mathcal{S}$, $x \in \operatorname{Per}(g)$ and m is the period of x, then for every eigenvalue λ_1 of $(Tg)^m/T_xV$ and every eigenvalue λ of $(\tilde{N}g)^m/N_xV$ if we have $|\lambda_1| < |\lambda| < 1$ ($|\lambda_1| > |\lambda| > 1$), then every embedded disc $D \subset V$ such that $g^m(D) \subset D$ ($g^{-m}(D) \subset D$) and T_xD is the subspace of T_xV associated to λ_1, is contained in $\Omega(g/V)$.*

PROOF. If the lemma is false we can find $g \in \mathcal{U}_\delta$, where $\delta = 1/2\beta$ (β and ρ satisfying the same properties as in the previous proof), and $x \in \operatorname{Per}(g)$ with an eigenvalue λ_1 of $(Tg)^m/T_xV$ (m the period of x) and an eigenvalue λ of $(\tilde{N}g)^m/\tilde{N}_xV$ such that, for instance, $|\lambda_1| < |\lambda| < 1$ and there exists an embedded disc $D \subset V$ satisfying $g^m(D)$, T_xD is the subspace associated to the eigenvalue λ_1 and $D - \Omega(g/V) \neq \varnothing$. Take $y \in D - \Omega(g/V)$, a neighborhood U_0 of y satisfying $g^n(U_0 \cap V) \cap U_0 \neq \varnothing$ for all $n \in \mathbf{Z}$ and a disc $D_0 \subset M$ containing y such that $\lim_{n\to+\infty}\|g^n(z) - g^n(y)\| = 0$ for all $z \in D_0$ and there exist $k > 0$, $|\lambda_1| < c < |\lambda|$ satisfying

$$\|g^n(z) - g^n(y)\| \geqslant Kc^n\|z - y\| \tag{35}$$

for all $n \in \mathbf{Z}$, $z \in D_0$. Moreover, if $|\lambda_1| < c_1 < c$ there exist $K_1 > 0$, such that

$$\|g^n(z) - g^n(x)\| \leqslant K_1c_1^n\|z - x\| \tag{36}$$

for all $z \in D$, $n \in \mathbf{Z}^+$. Choose $\bar{g} \in \mathcal{U}_\delta$ with $\bar{g}(g^{-1}(y)) = \bar{y} \in D_0 - V$, $\bar{y} \neq y$ and $g(p) = \bar{g}(p)$ for all $p \notin g^{-1}(U_0)$.

The condition $g^n(U_0) \cap U_0 = \varnothing$ for all $n \in \mathbf{Z}$ implies that $\{g^n(\bar{y}) \mid n \in \mathbf{Z}^+\} \cup \{g^{-n}(g^{-1}(y)) \mid n \in \mathbf{Z}^+\}$ is a $\bar{g}$-orbit. Hence $\bar{y} \in V_{\bar{g}}$. Let $h \in \Gamma_{\mathcal{C}}(NV)$ satisfy $V_{\bar{g}} = \{\exp h(p) \mid p \in V\}$. Let $z_n \in V$ be such that $h(z_n) = g^n(\bar{y})$. Using (35), (36) as in the previous proof, we conclude $\|h\|_{\mathcal{C}} > 1/2\beta$, contradicting $g \in \mathcal{U}_\delta$.

PROOF OF PROPOSITION 2.1. Define N_x^sV (N_x^uV) as the set of $v \in T_xM$ such

that $\lim\|(Tf)^n V\| \cdot \|(Tf)^{-n}/T_{f^n(x)}V\| = 0$ ($\lim\|(Tf)^{-n}v\| \cdot \|(Tf)^n/T_{f^{-n}(x)}V\| = 0$). Observe that when $x \in \Omega(f/V)$, $N_x^s V$, $N_x^u V$ are the fibers at x of the subbundles N^s, N^u of $TM/\Omega(f/V)$ given by Lemma 2.3. We want to prove that the maps $x \to N_x^s V$, $x \to N_x^u V$ define continuous subbundles of TM/V satisfying all the conditions required by the definition of normal hyperbolicity. Clearly $(Tf)N_x^s V = N_{f(x)}^s V$, $(Tf)N_x^u V = N_{f(x)}^u V$ for all $x \in V$. Let us prove that for all $x \in V$, $N_x^s V \cap N_x^u V = \{0\}$. Let π: $TM/V \to \tilde{N}V$ be the canonical projection. Let $v \in N_x^s V \cap N_x^u V$ and $\tilde{v} = \pi v$. If $0 \neq v$ we can take a linear map $0 \neq L$: $T_x V \to \{\lambda v \mid \lambda \in \mathbf{R}\}$. Then $L \in E_x^s(f) \cap E_x^u(f)$, contradicting Lemma 2.3. Now let us show that $T_x M = T_x V \oplus N_x^s V \oplus N_x^u V$ for all $x \in V$. We start with the following proposition:

LEMMA 2.9. *Let π_0: $TM/V \to TV$, π: $TM/V \to NV$ be the projections associated to the splitting $TM/V = TV \oplus NV$. There exists $\delta > 0$ such that if $x \in V$, $v \in T_x M$ and $\|\pi(Tf)^n v\| \leqslant \delta \|\pi_0(Tf)^n v\|$ for all $n \in \mathbf{Z}^+$ then $v \in N_x^s V \oplus T_x V$.*

PROOF. By Proposition 1.3 there exist a neighborhood U of $\Omega(f/V)$ and a continuous subbundle $\hat{N}^s$ of TV/Λ^s, where $\Lambda^s = \bigcap_{n \geqslant 0} f^{-n}(U)$, satisfying

$$(Tf)\hat{N}^s \subset \hat{N}^s, \tag{37}$$

$$\hat{N}^s/\Omega(f/V) = N^s. \tag{38}$$

From these relations it follows that $\hat{N}_x^s \subset N_x^s V$ for all $x \in V$ (in fact it is possible to prove $\hat{N}_x^s = N_x^s V$, but we shall not need this property). Moreover, we can assume that there exists a continuous subbundle $\hat{N}^u$ of TM/Λ^s such that $\hat{N}^u/\Omega(f/V) = N^u$ and $\hat{N}^u \oplus \hat{N}^s \oplus TV/\Lambda^s = TM/\Lambda^s$. Let π^s: $TM/\Lambda^s \to \hat{N}^s$, π^u: $TM/\Lambda^s \to \hat{N}^u$, π^c: $TM/\Lambda^s \to TV/\Lambda^s$ be the projections associated to this splitting. Define $\pi^{cs} = \pi^c + \pi^s$ and for $x \in \Lambda^s$, $\varepsilon > 0$ define the cones

$$S_\varepsilon(x) = \{v \in T_x M/\|\pi^{cs} v\| \leqslant \varepsilon \|\pi^u v\|\}.$$

There exist $m \in \mathbf{Z}^+$, $\varepsilon > 0$ satisfying

$$(Tf)S_\varepsilon(x) \subset S_{\varepsilon/2}(f(x)),$$

$$\|(Tf)^m v\|/\|(Tf)^m w\| \geqslant 2\|v\|/\|w\|$$

for all $x \in \Omega(f/V)$, $v \in S_\varepsilon(x)$, $0 \neq w \in N_x^s \oplus TV$. Let $U_0 \subset U$ be a neighborhood of $\Omega(f/V)$ such that if $\Lambda_0^s = \bigcap_{n>0} f^{-n}(U_0)$ then these relations remain true for all $x \in \Lambda_0^s$, $v \in S_\varepsilon(x)$, $0 \neq w \in N_x^s \oplus T_x V$. From (37) and (38) it follows that there exist $C > 0$, $0 < \sigma < 1$ satisfying

$$\|(Tf)^n v\|/\|(Tf)^n w\| \geqslant C(1/\sigma)^n \|v\|/\|w\| \tag{39}$$

for all $x \in \Lambda_0^s$, $v \in S_\varepsilon(x)$, $0 \neq w \in \hat{N}_x^s \oplus T_x V$. Take $\delta > 0$ such that $v \notin S_\varepsilon(x)$ for all $x \in V$ and $v \in T_x M$ such that $\|\pi v\| \leqslant \delta \|\pi_0 v\|$. Now suppose that $y \in V$, $v \in T_y M$ satisfy $\|\pi(Tf)^n v\| \leqslant \delta \|\pi_0(Tf)^n v\|$ for all $n \in \mathbf{Z}^+$. We

want to prove $v \in N_y^s V \oplus T_y V$. Suppose $v \notin N_y^s V \oplus T_y V$. Hence if $f^{n_1}(y) = x \in \Lambda_0^s$ we have $(Tf)^{n_1} v \notin N_x^s V \oplus T_x V$. Write $(Tf)^{n_1} v = v' + v''$ where $v' \in \hat{N}_x^u$, $v'' \in N_x^s \oplus T_x V$, $v' \neq 0$. By (39) we have

$$\|(Tf)^n v'\| / \|(Tf)^n v''\| \geqslant C(1/\sigma)^n \|v'\| / \|v''\| \tag{40}$$

for all $n \in \mathbf{Z}^+$. Then

$$\|\pi^{cs}(Tf)^{n+n_1} v\| = \|\pi^{cs}(Tf)^n (v' + v'')\| \leqslant \|\pi^{cs}(Tf)^n v'\| + \|(Tf)^n v''\|.$$

Since $v' \in \hat{N}_x^u \subset S_\varepsilon$, we obtain

$$\|\pi^{cs}(Tf)^n v'\| \leqslant \varepsilon \|\pi^u (Tf)^n v'\| = \varepsilon \|\pi^u (Tf)^n (v' + v'')\| = \varepsilon \|\pi^u (Tf)^{n+n_1} v\|;$$

hence,

$$\|\pi^{cs}(Tf)^{n+n_1} v\| \leqslant \left[\varepsilon + \frac{\|(Tf)^n v''\|}{\|\pi^u (Tf)^{n+n_1} v\|}\right] \|\pi^u (Tf)^{n+n_1} v\|.$$

This inequality, together with (40), gives

$$\|\pi^{cs}(Tf)^{n+n_1} v\| \leqslant \left(\varepsilon + \frac{\sigma^n \|v''\|}{C\|v'\|}\right) \|\pi^u (Tf)^{n+n_1} v\|.$$

Hence $(Tf)^{n+n_1} v \in S_{2\varepsilon}(x)$ for large values of n, thus contradicting the definition of δ.

Using this lemma we shall prove

$$N_x^s V + N_x^u V + T_x V = T_x M$$

for all $x \in V$. Together with the property $N_x^s V \cap N_x^u V = \{0\}$, this relation proves

$$N_x^s V \oplus N_x^u V \oplus T_x V = T_x M$$

for all $x \in V$. Suppose that for some x we have $N_x^s V + N_x^u V + T_x V \neq T_x M$. By Lemma 2.4, $x \notin \Omega(f/V)$. Let $\delta > 0$ be the number given by Lemma 2.9 and $\mathcal{U}_\delta$ the neighborhood of f given by Definition 2.1. Let W be a neighborhood of x such that $f^n(W) \cap W = \varnothing$ for all $n \in \mathbf{Z}$ and let $g \in \mathcal{U}_\delta$ be a diffeomorphism satisfying $g(y) = f(y)$ for all y in a neighborhood of $V - W$, $g(x) = f(x)$, $g(f^{-1}(x)) = x$ and

$$\dim\big((Tg)(N_{f^{-1}(x)}^u V \oplus T_{f^{-1}(x)} V) \cap (N_x^s V \oplus T_x V)\big) < \dim V; \tag{41}$$

let $h \in T_{\mathfrak{E}}(NV)$ such that $V_g = \{\exp h(p) | p \in V\}$. Since $h \in \Gamma_{\mathfrak{E}}$, given any basis $\{v_1, \ldots, v_l\}$ of $T_x V$, there exist $\{w_1, \ldots, w_l\} \subset N_x V$ such that

$$\liminf_{\lambda \to 0} \left\| \frac{h(\exp \lambda v_i) - h(x)}{\lambda} - w_i \right\| = 0$$

for $1 \leqslant i \leqslant l$, where exp: $TV \to V$ is the exponential mapping of the Riemann manifold V. The condition $\|h\|_{\mathfrak{E}} \leqslant \delta$ implies $\|w_i\| / \|v_i\| \leqslant \delta$ for all $1 \leqslant i \leqslant l$. For all $n \in \mathbf{Z}^+$, by the invariance of V_g, we have

$$\liminf_{\lambda\to 0}\left\|\frac{h(\exp\lambda\pi_0(Tg)^n(v_i+w_i))-h(g^n(x))}{\lambda}-\pi(Tf)^n(v_i+w_i)\right\|=0.$$

Since $\|h\|_{\mathcal{C}}<\delta$ we obtain

$$\|\pi(Tg)^n(v_i+w_i)\|<\delta\|\pi_0(Tg)^n(v_i+w_i)\|.$$

But g satisfies $(Tg)^n/(T_{f(x)}M)=(Tf)^n$ and $(Tg)^{-n}/T_xM=(Tf)^{-n}/T_xM$ for all $n\in\mathbf{Z}^+$. Therefore, by Lemma 2.9,

$$\{(Tg)(v_i+w_i)|\ i=1,\dots,l\}\subset N^s_{f(x)}V\oplus T_{f(x)}V,$$
$$\{v_i+w_i|\ i=1,\dots,l\}\subset N^u_xV\oplus T_xV.$$

Let S be the space spanned by $\{v_i+w_i|\ i=1,\dots,l\}$. Then

$$\dim((Tg)(N^u_xV\oplus T_xV)\cap(N^s_xV\oplus T_xV))>\dim S=\dim V,$$

contradicting (41).

Now we know that $N^s_xV\oplus N^u_xV\oplus T_xV=T_xM$ for all $x\in V$. Our next step is to show that the maps $x\to N^s_xV$, $x\to N^u_xV$ define continuous subbundles of TM/V. For this define, as before, G^s_x (G^u_x) as the set of linear maps L: $T_xV\to\pi(N^s_xV)$ (L: $T_xV\to\pi(N^u_xV)$). It is clear that

$$G^s_x\oplus G^u_x=L_x(V)$$

for all $x\in V$ and

$$G^s_x=E^s_x(f),\tag{42'}$$
$$G^u_x=E^u_x(f).\tag{42''}$$

Then Φ: $L_x(V)\hookleftarrow$ (defined as in the proof of Lemma 2.3) is a hyperbolic vector bundle isomorphism and the maps $x\to E^s_x(f)$, $x\to E^u_x(f)$ define continuous subbundles of $L_x(V)$ [13, Proposition 1.1]. From this and (42′), (42″) it follows that the maps $x\to\pi(N^s_xV)$, $x\to\pi(N^u_xV)$ define a continuous $\tilde{N}f$-invariant splitting $\tilde{N}V=\tilde{N}^sV\oplus\tilde{N}^uV$ satisfying

$$\tilde{N}f/\tilde{N}^sV>Tf/TV>\tilde{N}f/N^uV.$$

From Proposition 1.2 it follows that there exist continuous subbundles N^sV, N^uV invariant under Tf and satisfying $\pi(N^sV)=\tilde{N}^sV$, $\pi(N^uV)=\tilde{N}^uV$ and

$$Tf/N^sV>Tf/TV>Tf/N^sV.\tag{43}$$

From this it follows easily that the fibers at $x\in V$ of N^sV and N^uV are the spaces N^s_xV, N^u_xV defined before.

It remains to prove that there exist constants $C>0$, $0<\lambda<1$ satisfying

$$\|(Tf)^n/N^s_xV\|<C\lambda^n,\tag{44}$$
$$\|(Tf)^{-n}/N^u_xV\|<C\lambda^n\tag{45}$$

for all $n\in\mathbf{Z}^+$, $x\in V$. Let us prove (45). If we show that $\{\|(Tf)^nv\|\mid n\in\mathbf{Z}^+\}$ is unbounded for all $0\neq v\in N^u_xV$, $x\in\Omega(f/V)$ it follows [13, Propo-

sition 1.1] that there exist $C_1 > 0$, $0 < \lambda_1 < 1$ satisfying $\|(Tf)^{-n}/N_x^u V\| \leqslant C_1\lambda_1^n$ for all $x \in \Omega(f/V)$, $n \in \mathbf{Z}^+$. Then it is easy to see that $\{\|(Tf)^n v\| \mid n \in \mathbf{Z}^+\}$ is unbounded for all $0 \neq v \in N^u V$. Applying again [13, Proposition 1.1], there exist $C > 0$, $0 < \lambda < 1$ satisfying (45). Hence let us suppose that for some $x \in \Omega(f/V)$ there exists $0 \neq v \in N_x^u V$ such that for some $K > 0$, $\|(Tf)^n v\| \leqslant K\|v\|$ for all $n \in \mathbf{Z}^+$. It is not difficult to show that there exist $C_1 > 0$, $0 < \lambda_1 < 1$, and $r > 0$ satisfying

$$\|f^n(x_1) - f^n(x_2)\|/\|(Tf)^n w\| \leqslant C_1\lambda_1^n\|x_1 - x_2\|/\|w\|$$

for all $x_i \in V$, $i = 1, 2$, $0 \neq w \in N_{x_1}^u V$ and $n \in \mathbf{Z}^+$ such that $\|x_1 - x_2\| < \delta(n)$, where $\delta(n)$ is so small that $\|f^j(x) - f^j(x_2)\| \leqslant r$ for all $0 \leqslant j \leqslant n$, if x, x_2 satisfy $\|x_1 - x_2\| < \delta(n)$. Then if $y \in V$ and $\|y - x\| \leqslant \delta(n)$, we conclude that

$$\|f^n(y) - f^n(x)\| \leqslant C_1\lambda_1^n \frac{\|y - x\|}{\|v\|} \cdot \|(Tf)^n v\| \leqslant KC_1\lambda_1^n\|y - x\|.$$

Take $\delta(1) > r_1 > 0$ satisfying $KC_1\lambda_1^n r_1 \leqslant \delta(1)$ for all $n \in \mathbf{Z}^+$. Defining $\bar{K} = KC_1$, we obtain

$$\|f^n(y) - f^n(x)\| \leqslant \bar{K}\lambda_1^n\|y - x\|$$

for all $n \in \mathbf{Z}^+$ and $y \in V$ such that $\|y - x\| \leqslant r_1$. Since x is nonwandering this implies that x is periodic and there exists an eigenvalue λ of $(Tf)^m/N_x^u V$ (m the period of x) with $|\lambda| \leqslant 1$. Since $Tf/N^u V > Tf/TV$ it follows that $(Tf)^m/N_x^u V > (Tf)^m/T_x V$ and this proves that $|\mu| < |\lambda|$ for every eigenvalue $|\mu|$ of $(Tf)^m/T_x V$, and now it is easy to approximate f by $g \in \mathcal{S}$ such that $g(y) = f(y)$ for every y in the orbit of x and such that $(\tilde{N}g)^m/\tilde{N}_x V$ has an eigenvalue λ' with $|\lambda'| < 1$ such that $|\mu| < |\lambda'|$ for all eigenvalues of $(Tg)^m/T_x V$, thus contradicting Lemma 2.9.

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