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INTERTWINING DIFFERENTIAL OPERATORS FOR Mp(n, $\mathbf{R}$) AND $SU(n, n)$[1]

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ABSTRACT. For each of the two series of groups, three series of representations U_n, D_n, and H_n ($n \in \mathbf{Z}$) are considered. For each series of representations there is a differential operator with the property, that raised to the nth power ($n > 0$), it intertwines the representations indexed by $-n$ and n. The operators are generalizations of the d'Alembertian, the Dirac-operator and a combination of the two. Unitarity of subquotients of representations indexed by negative integers is derived from the intertwining relations.

0. Introduction. Motivated by the aspects of the conformal group as a physical symmetry group, as suggested by I. E. Segal [11], we recently studied, jointly with Michele Vergne, some representation theoretical aspects of $SU(2, 2)$ [5]. One result, that was obtained, was that powers of the d'Alembertian

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},$$

as well as powers of the Dirac operator ∇ (a 4×4 matrix for which $\nabla^2 = \square$) are intertwining between two series of representations of $SU(2, 2)$. We shall see that a similar phenomenon takes place for Mp(n, $\mathbf{R}$) and $SU(n, n)$.

Specifically, we consider an $n \times n$ matrix $\mathbf{D} = \partial/\partial x_{\alpha\beta}$ with first order differential operators as entries, corresponding to a parametrization of the space of $n \times n$ symmetric (hermitian) matrices. We prove that det $\mathbf{D}$ as well as

$$\nabla = \begin{pmatrix} 0 & \mathbf{D}' \\ c(\mathbf{D}) & 0 \end{pmatrix},$$

where $c(\mathbf{D})\mathbf{D}' = \mathbf{D}'c(\mathbf{D}) = (\det \mathbf{D})I_n$, are intertwining between two series U_l

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and D_l, $l \in \mathbf{Z}$, of representations of $\mathrm{Mp}(n, \mathbf{R})$ ($SU(n, n)$) in the sense that for $l \geq 0$

$$(\det \mathbf{D})^l U_{-l} = U_l (\det \mathbf{D})^l, \quad (\nabla)^{2l+1} D_{-1-l} = D_l (\nabla)^{2l+1}. \tag{0.1}$$

We shall refer to ∇ as the Dirac-type operator.

For $\mathbf{D}^l$ and $\det \mathbf{D}$ the equations (0.1) were expected from [**5**, V.5.1 and V.6.1]. However, we shall take a somewhat different approach, which has the equations involving $c(\mathbf{D})$ above as a straightforward consequence.

We remark that as a special case of algebraic results, B. Kostant [**7**] obtained quasi-invariance properties of the wave-operator $\square$. The operators $\det \mathbf{D}^l$, $(\mathbf{D}^l)^l$, and $c(\mathbf{D})^l$, for $l \in \mathbf{N}$, were studied by L. Gårding [**3**], and the Cauchy problem was solved. It was also noted that the principal formulas involving $(\det \mathbf{D})^l$ were invariant under the transformation $x \to axa^*$ of the space of symmetric (hermitian) matrices, for a in $\mathrm{SL}(n, \mathbf{R})$ ($\mathrm{SL}(n, \mathbf{C})$).

A basic observation in the proof of (0.1) is, that if s is the operator acting on functions f from the space of symmetric $n \times n$ matrices (space of hermitian $n \times n$ matrices) to $\mathbf{C}^n$ by $(sf)(s) = sf(s)$, then $c(\mathbf{D}) = [(\det \mathbf{D}), s]$. We use (0.1) to prove that U_{-l} and D_{-l}, for $l > 0$, act unitarily on quotient spaces of functions modulo solutions to

$$(\det \mathbf{D})^l \phi = 0 \quad \text{and} \quad \nabla^{2l-1} \phi = 0,$$

respectively. The representation D_l can be written as $D_l^+ \oplus D_l^-$ where, by (0.1), D_l^+ and D_l^- satisfy

$$\begin{aligned} \det \mathbf{D}^l c(\mathbf{D}) D^{+}_{-l-1} &= D_l^- \det \mathbf{D}^l c(\mathbf{D}), \text{ and} \\ \det \mathbf{D}^l \mathbf{D}^l D_{-l-1} &= D_l^+ \det \mathbf{D}^l \mathbf{D}^l. \end{aligned} \tag{0.2}$$

We use this to study a series H_l of representations obtained by induction from reducible, noncomplemented representations of the maximal parabolic subgroup. It is proved that these representations are related to the preceding ones by

$$\begin{bmatrix} l & 0 \\ -\mathbf{D}^l & l \end{bmatrix} H_l = \begin{bmatrix} D_{l-1}^- & 0 \\ 0 & D_l^+ \end{bmatrix} \begin{bmatrix} l & 0 \\ -\mathbf{D}^l & l \end{bmatrix}. \tag{0.3}$$

Ths study of the representations H_l was motivated in part by the work of A. Salam and G. Mack [**10**, p. 178].

The present article falls in five parts. (1) is the scalar case for $\mathrm{Mp}(n, \mathbf{R})$, corresponding to $(\det \mathbf{D})^l$, and (2) is the scalar case for $SU(n, n)$. In (3) the Dirac-type operator for $\mathrm{Mp}(n, \mathbf{R})$ is related to representations obtained by induction from reducible, complemented representations of the maximal parabolic P_- in $2n$-dimensional complex vector spaces, and (4) is the corresponding for $SU(n, n)$. Finally, in (5) representations obtained by induction from reducible, noncomplemented representations of P_- in $2n$ dimensional complex vector spaces, are proved in many cases to be unitary (and reducible). The details are carried out for $SU(n, n)$.

The first two chapters of the present paper are essentially contained in the author's Ph.D. Thesis [6], written under the direction of I. E. Segal. The author is indebted to Professor Segal for many helpful discussions. He is also thankful to Michele Vergne for friendly help and conversations.

1. The scalar case for Mp(n, **R**). We shall begin with some generalities about Sp(n, **R**) which is covered twice by Mp(n, **R**).

Sp(n, **R**) is the subgroup of Gl($2n$, **R**) consisting of those matrices g that satisfy

$$g\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} g^t = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}. \tag{1.1}$$

If we write g in terms of $n \times n$ blocks; $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then (1.1) is equivalent to

$$ad^t - bc^t = 1; \quad ab^t = ba^t; \quad cd^t = dc^t, \tag{1.2}$$

and to

$$a^td - c^tb = 1; \quad a^tc = c^ta; \quad b^td = d^tb, \tag{1.3}$$

where (1.3) is obtained by replacing g by g^{-1} in (1.1). The Lie algebra of Sp(n, **R**) is thus [4, p. 341]

$$\mathrm{sp}(n, \mathbf{R}) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1^t \end{bmatrix} \middle| \; x_1 \text{ arbitrary}; \; x_2 = x_2^t; \; x_3 = x_3^t \right\}.$$

We let $\mathfrak{D} = \{z = x + iy | x, y \text{ real } n \times n \text{ matrices}, \; x = x^t; \; y = y^t; \; y > 0\}$. $\mathfrak{D}$ is then a complex domain, and Sp(n, **R**) acts on $\mathfrak{D}$ by

$$g \cdot z = (az + b)(cz + d)^{-1}.$$

We recall from [5] that if G is a group of holomorphic transformations on $\mathfrak{D}$, V a finite dimensional complex vector space, $J(g, z)$ a continuous function $G \times \mathfrak{D} \to \mathrm{GL}(V)$ which, for each fixed g in G is holomorphic in z and satisfies

$$J(g_1 g_2, z) = J(g_1, g_2 z) J(g_2, z); \quad J(1, z) = 1, \tag{1.4}$$

then a function $K(z, w)$: $\mathfrak{D} \times \mathfrak{D} \to \mathrm{End}\, V$, holomorphic in z, antiholomorphic in w, is the reproducing kernel for the representation

$$(T_J(g)f)(z) = J(g^{-1}, z)^{-1} f(g^{-1}z) \tag{1.5}$$

on a space of holomorphic functions f: $\mathfrak{D} \to V$ if and only if

$$K(z, w) = K(w, z)^*, \tag{1.6}$$

$$K(gz, gw) = J(g, z)K(z, w)J(g, w)^*, \text{ and} \tag{1.7}$$

$$\sum_{i,j=1}^{n} \langle K(z_j, z_i)v_i, v_j\rangle \geq 0 \tag{1.8}$$

for all z_i in $\mathfrak{D}$, v_i in V, $i = 1, 2, \ldots, n$, and n in $\mathbf{N}$. We let $*$ denote the complex adjoint of an operator. Thus, $(x + iy)^* = x^t - iy^t$. If $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is in $\mathrm{Sp}(n, \mathbf{R})$ and z, w are in $\mathfrak{D}$, we then define

$$J_1(g, z) = cz + d; \qquad J_2(g, z) = (zc^t + d^t)^{-1}, \tag{1.9}$$

$$K_1(z, w) = ((z - w^*)/2i)^{-1}; \qquad K_2(z, w) = ((z - w^*)/2i). \tag{1.10}$$

Then it follows easily from (1.2) and (1.3) that J_1 and J_2 both satisfy (1.4). By the same relations, it also follows that for $i = 1, 2$

$$K_i(gz, gw) = J_i(g, z)K_i(z, w)J_i(g, w)^*. \tag{1.11}$$

We shall in this section consider the one-parameter family of actions of $\mathrm{Sp}(n, \mathbf{R})$ on the space $\mathcal{O}_1$ of holomorphic functions on $\mathfrak{D}$ defined by, for $\lambda \in \mathbf{R}$,

$$(U_\lambda(g)f)(z) = \left(\det J_1(g^{-1}, z)\right)^{-\lambda-((n+1)/2)} f(g^{-1}z). \tag{1.12}$$

It is easy to see that U_λ is holomorphically induced from the one-dimensional representation $\tau_\lambda(k) = \det(J, (k, i))^{\lambda+(n+1)/2}$ of the maximal compact subgroup K (cf. [**5**, p. 61]).

REMARK. For noninteger λ's, we only get a projective representation; by passing to the universal covering group, we get a proper representation. In the sequel we shall mostly be interested in the cases where λ is integer. In these cases, we need only pass to the double covering group of $\mathrm{Sp}(n, \mathbf{R})$; the metaplectic group $\mathrm{Mp}(n, \mathbf{R})$. We shall maintain the notation U_λ irrespective of the groups.

It follows easily from (1.2) and (1.3) that

$$g \cdot (z^*) = (g \cdot z)^*$$

for all g in $\mathrm{Sp}(n, \mathbf{R})$. ($\mathrm{Sp}(n, \mathbf{R})$ also acts on the "lower half-plane".) In particular, g leaves the Shilov boundary $\partial\mathfrak{D}$ of $\mathfrak{D}$; $\partial\mathfrak{D} = S = \{x + iy \mid y = 0\}$, invariant, and even though the action is not globally defined, we still get an action on measurable functions. We shall also maintain the notation U_λ for this action. It can be seen that there exists a subspace V_λ of C^∞-functions on S, which is invariant under this latter representation, and such that the restriction of U_λ to V_λ can be imbedded into an invariant subspace of a degenerate principal series representation (cf. [**5**, pp. 82–83], and below).

The space S of $n \times n$ symmetric real matrices is a real vector space of dimension $\frac{1}{2}n(n + 1)$. We write elements of S as $x = [x_{\alpha\beta}]_{\alpha,\beta=1}^{n}$ ($x_{\alpha\beta} = x_{\alpha\beta}$) or just $x = [x_{\alpha\beta}]$, and let dx denote Lebesgue measure on S corresponding to this parametrization. On functions from S to **C**, we define first order differential operators $a_{\alpha\beta}$ by

$$\begin{aligned}&\text{For } \alpha \neq \beta\text{: } a_{\alpha\beta}f = \frac{1}{2}\ \frac{\partial}{\partial x_{\alpha\beta}}\ f. \text{ (Thus, } a_{\alpha\beta} = a_{\beta\alpha}.)\\ &\text{For } \alpha = \beta\text{: } a_{\alpha\alpha}f = \frac{\partial}{\partial x_{\alpha\alpha}}\ f.\end{aligned} \tag{1.13}$$

We let **D** be the differential operator, whose (α, β)th entry is

$$\{\mathbf{D}\}_{\alpha,\beta} = a_{\alpha\beta}. \tag{1.14}$$

We shall in this section study the nth order differential operator det **D**. The Fourier transform is defined by

$$\hat{f}\,(k) = \gamma_1 \int_S e^{-i \operatorname{tr} xk} f(x)\, dx,$$

and the inverse Fourier transform by

$$\check{g}\,(x) = \gamma_2 \int_S e^{i \operatorname{tr} xk} g(k)\, dk.$$

For suitable pairs (γ_1, γ_1), and nice functions $f, f = \check{\hat{f}} = \hat{\check{f}}$. Since for any (real or complex) matrix z,

$$(\det \mathbf{D})e^{\operatorname{tr} xz} = \det z e^{\operatorname{tr} xz}, \tag{1.15}$$

det **D** is in particular proportional to the Fourier transform of the multiplication operator $(i)^n \det k$. This could of course also be taken to be the definition of **D**.

We want to analyze whether powers of det **D** can be intertwining operators, and if so, for which pairs of $(U_\lambda, U_{\lambda'})$'s.

We remark that if we take

$$u(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \mathrm{Sp}(n, \mathbf{R}) \quad \text{for } x \in X,$$

then $(\det \mathbf{D})^r U_\lambda(u(x)) = U_{\lambda'}(u(x))(\det \mathbf{D})^r$ for all (λ, λ') and $r \in \mathbf{N}$, since det **D** is a constant coefficient differential operator, and $\{u(x) | x \in S\}$ is the translation subgroup.

Let us return to the Lie algebra. This is generated by the subalgebras

$$\left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\middle| x \in S\right\} \quad \text{and} \quad \left\{\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\middle| y \in S\right\}.$$

In fact, we could replace the latter by the single element $\begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}$, but we shall find it convenient to study a more general $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$.

Specifically, fix p and q and let y be the matrix in S given by

$$\{y\}_{kl} = \delta_{pk}\delta_{ql} + \delta_{pl}\delta_{kq}. \tag{1.16}$$

For $x \in S$ and real t, we have

$$x(tyx+1)^{-1} = x - txyx$$

and

$$\det(tyx+1) = 1 + t \operatorname{tr} yx$$

to the first order in t. Since furthermore

$$\{xyx\}_{\alpha\beta} = x_{\alpha p} x_{q\beta} + x_{\alpha q} x_{p\beta}$$

and

$$\operatorname{tr} yx = x_{qp} + x_{pq} \quad (= 2x_{qp})$$

we see, that if we let

$$Y_0 = \sum_{\alpha,\beta=1}^{n} (x_{\alpha p} x_{q\beta} + x_{\alpha q} x_{p\beta}) a_{\alpha\beta}$$

with $a_{\alpha\beta}$ as in (1.13), and if we let $Y_m = x_{qp} + x_{pq}$, then

$$dU_\lambda\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} U_\lambda\left(\begin{pmatrix} 1 & 0 \\ +ty & 1 \end{pmatrix}\right)\Bigg|_{t=0}$$

$$= Y_0 + (\lambda + (n+1)/2) Y_m. \tag{1.17}$$

We now want to compute $[\det \mathbf{D}, x_{i_0 j_0}]$, and to do this, we recall some basic facts from linear algebra.

Let $\{A\}_{i,j}$ be an $n \times n$ matrix. Let C_{ij} be the determinant of the matrix M_{ij}, obtained from A by replacing the entries in the ith row and jth column by zeros, except for a one in the (i,j)th place. Specifically,

$$\{M_{ij}\}_{rs} = \left(a_{rs}(1-\delta_{ir})(1-\delta_{js}) + \delta_{ir}\delta_{js}\right). \tag{1.18}$$

We call M_{ij} the (i,j)-minor, and denote by C_{ij}^{rs} the determinant of the (r,s)-minor of M_{ij}. Then,

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}, \tag{1.19}$$

or, more generally,

$$\sum_{i=1}^{n} a_{ri} C_{si} = \sum_{i=1}^{n} a_{is} C_{ir} = \delta_{rs} \det A, \text{ and}$$

$$C_{ij} = \sum_{s=1}^{n} a_{rs} C_{ij}^{rs} - \sum_{s=1}^{n} a_{rs} \delta_{ir} C_{ij}^{rs} - a_{rj} C_{ij}^{rj} + a_{rj} \delta_{ir} C_{ij}^{rj} + \delta_{ir} C_{ij}^{rj}. \tag{1.20}$$

We use the above relations on the differential operator $\det \mathbf{D} = \det[a_{\alpha\beta}]$. We put $i = i_0$, $r = j_0$. Then

$$\det A = \sum_{j,s=1}^{n} a_{i_0j}a_{j_0s}C_{i_0j}^{j_0s}(1 - \delta_{i_0j_0}) + \sum_{j=1}^{n} a_{i_0j}a_{j_0j}C_{i_0j}^{j_0j}(-1 + \delta_{i_0j_0}) + \delta_{i_0j_0}\sum_{j=1}^{n} a_{i_0j}C_{i_0j}^{i_0j}.$$

From (1.13) we get

$$[a_{i_0j}, x_{i_0j_0}] = \tfrac{1}{2}(\delta_{jj_0} + \delta_{i_0j_0}\delta_{i_0j}),$$
$$[a_{j_0s}, x_{i_0j_0}] = \tfrac{1}{2}(\delta_{i_0s} + \delta_{i_0j_0}\delta_{sj_0}).$$

Finally, since by construction, $[C_{i_0j}^{j_0s}, x_{i_0j_0}] = 0$, we get

$$\begin{aligned}[\det \mathbf{D}, x_{i_0j_0}] &= \sum_{j,s=1}^{n} a_{i_0j}\tfrac{1}{2}(\delta_{i_0s} + \delta_{i_0j_0}\delta_{sj_0})C_{i_0j}^{j_0s}(1 - \delta_{i_0j_0}) \\ &\quad + \sum_{j,s=1}^{n} \tfrac{1}{2}(\delta_{jj_0} + \delta_{i_0j_0}\delta_{i_0j})a_{j_0s}C_{i_0j}^{j_0s}(1 - \delta_{i_0j_0}) \\ &\quad + \sum_{j=1}^{n} \tfrac{1}{2}(\delta_{jj_0} + \delta_{i_0j_0}\delta_{i_0j})a_{j_0j}C_{i_0j}^{j_0j}(-1 + \delta_{i_0j_0}) \\ &\quad + \sum_{j=1}^{n} \tfrac{1}{2}a_{i_0j}(\delta_{i_0j} + \delta_{i_0j_0}\delta_{jj_0})C_{i_0j}^{j_0j}(-1 + \delta_{i_0j_0}) \\ &\quad + \sum_{j=1}^{n} \tfrac{1}{2}\delta_{i_0j_0}(\delta_{jj_0} + \delta_{i_0j_0}\delta_{i_0j})C_{i_0j}^{i_0j} \\ &= \tfrac{1}{2}\sum_{j=1}^{n} a_{i_0j}C_{i_0j}^{j_0i_0}(1 - \delta_{i_0j_0}) + \tfrac{1}{2}\sum_{s=1}^{n} a_{j_0s}C_{i_0j_0}^{j_0s}(1 - \delta_{i_0j_0}) \\ &\quad + \tfrac{1}{2}a_{j_0j_0}C_{i_0j_0}^{j_0j_0}(-1 + \delta_{i_0j_0}) + \tfrac{1}{2}a_{i_0i_0}C_{i_0i_0}^{j_0i_0}(-1 + \delta_{i_0j_0}) \\ &\quad + \tfrac{1}{2}\delta_{i_0j_0}\left(C_{i_0j_0}^{i_0j_0} + C_{i_0i_0}^{i_0i_0}\right).\end{aligned}$$

Using the relations (1.20), this is readily seen to give

$$[\det \mathbf{D}, x_{i_0j_0}] = \tfrac{1}{2}(C_{i_0j_0} + C_{j_0i_0}) = C_{i_0j_0}. \tag{1.21}$$

We are now able to compute

$$
\begin{aligned}
[\det \mathbf{D}, Y_0] &= \sum_{\alpha,\beta=1}^{n} [\det \mathbf{D}, x_{\alpha p}x_{q\beta} + x_{\alpha q}x_{p\beta}]a_{\alpha\beta} \\
&= \sum_{\alpha,\beta=1}^{n} (x_{\alpha p}C_{q\beta} + C_{\alpha p}x_{q\beta})a_{\alpha\beta} + \sum_{\alpha,\beta=1}^{n} (x_{\alpha q}C_{p\beta} + C_{\alpha q}x_{p\beta})a_{\alpha\beta} \\
&= \sum_{\alpha=1}^{n} x_{\alpha p}\delta_{q\alpha} \det \mathbf{D} + \sum_{\beta=1}^{n} \delta_{p\beta} \det \mathbf{D} x_{q\beta} \\
&\quad - \tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} C_{\alpha p}(\delta_{q\alpha} + \delta_{q\beta}\delta_{\alpha\beta}) + \sum_{\alpha=1}^{n} x_{\alpha q}\delta_{p\alpha} \det \mathbf{D} \\
&\quad + \sum_{\beta=1}^{n} \delta_{q\beta} \det \mathbf{D} x_{p\beta} - \tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} C_{\alpha q}(\delta_{p\alpha} + \delta_{p\beta}\delta_{\alpha\beta}) \\
&= 2Y_m \det \mathbf{D} - (n-1)C_{pq}.
\end{aligned}
$$

By (1.21) and the above, we thus have the relations

$$
\begin{aligned}
[\det \mathbf{D}, Y_0] &= 2Y_m \det \mathbf{D} - (n-1)C_{pq}, \\
[\det \mathbf{D}, Y_m] &= 2C_{pq}, \\
[\det \mathbf{D}, C_{pq}] &= 0.
\end{aligned} \tag{1.22}
$$

It follows that

$$
\begin{aligned}
\det \mathbf{D}(Y_0 + (\lambda + (n+1)/2)Y_m) = (Y_0 + (\lambda + 2 + (n+1)/2)Y_m)&\det \mathbf{D} \\
+ (2(\lambda + (n+1)/2) - (n-1))&C_{pq},
\end{aligned}
$$

or, by induction, for $r \in \mathbf{N}$,

$$
\begin{aligned}
&(\det \mathbf{D})^r(Y_0 + (\lambda + (n+1)/2)Y_m) \\
&\quad = (Y_0 + (\lambda + (n+1)/2 + 2r)Y_m)(\det \mathbf{D})^r \\
&\qquad + (2r(\lambda + (n+1)/2) + 2r(r-1) - r(n-1))C_{pq}(\det \mathbf{D})^{r-1}.
\end{aligned} \tag{1.23}
$$

We see, that $(\det \mathbf{D})^r$ is an intertwining operator exactly when $\lambda = -r$. In this case, $\lambda' = r$. We shall from now on consider U_λ, $\lambda \in \mathbf{Z}$, as a representation of $\mathrm{Mp}(n, \mathbf{R})$, even though this only is strictly necessary for n even. The above analysis may thus be summarized as

PROPOSITION 1.1. *Let* $r \in \mathbf{N}$. *Then* $\forall X \in \mathrm{sp}(n, \mathbf{R})$:

$$
(\det \mathbf{D})^r dU_{-r}(X) = dU_r(X)(\det \mathbf{D})^r. \tag{1.24}
$$

The point now is that the class of functions on which the equation (1.24) can be integrated is sufficiently big to be of interest. Specifically, the vector space V_{-r} spanned by $\{(U_{-r}(g)\det(K_1(\cdot, w))^\alpha)(x) | g \in \mathrm{Mp}(n, \mathbf{R}),\ w \in \mathfrak{D},$

$\alpha - (n+1)/2 \in \mathbf{Z}$, and $\alpha \geqslant (n+1)/2 - r\}$ is invariant under U_{-r}, and each function f in V_{-r} is the boundary value of a holomorphic function F_f, which can be extended holomorphically across the Shilov boundary.

We remark that since the operators $(\det \mathbf{D})^r$ are boundary values of operators acting on the space $\mathcal{O}_1$ of holomorphic functions on $\mathfrak{D}$, and since these, by exactly the same arguments as before, satisfy exactly the same intertwining relations, we could just integrate the relation (1.24) on holomorphic functions on $\mathfrak{D}$. In fact, by the above properties of V_{-r}, this would be exactly the same as integrating it on V_{-r}. On holomorphic functions on $\mathfrak{D}$, however, it is rather obvious, that the relation can be integrated. We shall therefore only give a sketch.

We first observe that the Lie algebra sp(n, $\mathbf{R}$) is generated by the subalgebras

$$\left\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \middle| y \in S\right\} \quad \text{and} \quad \left\{\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \middle| \theta \in \mathbf{R}\right\}.$$

For the first algebra, the relation can easily be integrated. Thus, since Mp(n, $\mathbf{R}$) is connected, we need only consider

$$c(\theta) = \exp\left(\theta\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Clearly, the map $(\theta, z) \to c(\theta)z$ is holomorphic and hence $(\theta, z) \to f(c(\theta)z)$ is holomorphic, if f is a holomorphic function on $\mathfrak{D}$. Since the same can be said about the multipliers

$$\det(-\sin\theta z + \cos\theta)^{-((n+1)/2 - r)}$$

and

$$\det(-\sin\theta z + \cos\theta)^{-((n+1)/2 + r)},$$

it follows by power series expansions that

$$\left(\det \mathbf{D}^r U_{-r}(c(\theta))f\right)(z) = \left(U_r(c(\theta))\det \mathbf{D}^r f\right)(z). \tag{1.25}$$

PROPOSITION 1.2 [5]. *For $r \in \mathbf{N}$ and f either a holomorphic function on $\mathfrak{D}$ or an element of V_{-r} we have that*

$$(\det \mathbf{D})^r U_{-r}(g)f = U_r(g)(\det \mathbf{D})^r f, \quad \forall g \in \mathrm{Mp}(n, \mathbf{R}). \tag{1.26}$$

From here, we proceed as in the case of $SU(2, 2)$ [5]. We denote the forward light cone in S by C^+, i.e.

$$C^+ = \{x \in S | x > 0\}.$$

Then we know from [3] or [9] that there exists a constant k_1 such that for all

$\alpha > -1$ and $z \in \mathfrak{D}$

$$\int_{C^+} e^{+i\,\mathrm{tr}\, zy} \det y^\alpha \, dy$$
$$= k_1(\det z/i)^{-\alpha-((n+1)/2)} \prod_{i=1}^{n} \Gamma(\alpha + (i+1)/2).$$

It is also known [9, §§4.5 and 4.6] that if we let

$$\mathcal{O}_0 = \{0\}, \quad \mathcal{O}_1 = \{x \in S \mid x \geqslant 0, \operatorname{rank} x = 1\}, \ldots,$$
$$\mathcal{O}_{n-1} = \{x \in S \mid x \geqslant 0, \operatorname{rank} x = n-1\},$$

then there are semi-invariant measures μ_j on $\mathcal{O}_j$ and constants $d_b(j)$, such that

$$\int_{\mathcal{O}_j} e^{i\,\mathrm{tr}\, zy} \, d\mu_j(y) = d_b(j)(\det z/i)^{-j/2}. \tag{1.28}$$

The formula (1.27), together with (1.7), (1.8), (1.9), (1.10), and (1.11), immediately gives that U_α is unitary for $\alpha > -1$, and (1.28) shows that $U_{(-n-1)/2+j/2}$ for $j = 0, 1, \ldots, n-1$ has an invariant subspace, on which it acts unitarily, namely the space of Fourier transforms of holomorphic functions in $L^2(\mathcal{O}_j, \mu_j)$. From (1.14) and (1.27) it also follows that there are constants $C_{n,r}$ such that

$$(\det \mathbf{D})^r \det(x+z)^{-((n+1)/2)} = C_{n,r} \det(x+z)^{-r-((n+1)/2)}, \tag{1.29}$$

for $z \in \mathfrak{D}$ and $x \in S$. As in the case of $SU(2, 2)$ [5, pp. 91–96], this is exactly what is needed to conclude: If we define an equivalence relation on V_{-r} by

$$f \underset{r}{\sim} g \Leftrightarrow (\det \mathbf{D})^r (f - g) = 0,$$

and denote the equivalence classes by $[\cdot]_r$, then, using the unitarity of U_r for $r \geqslant 0$ and Proposition 1.2, we get

PROPOSITION 1.3. *For integers $r \geqslant 0$, there exists a subspace of equivalence classes $[\cdot]_r$, which can be given a Hilbert space structure, in which U_{-r} acts unitarily.*

We shall end this section with a look at noninteger r's. We do this by Fourier transform. We consider C^∞ functions with compact support in the interior of the forward light cone C^+. We denote the Fourier transforms of the operators $\det \mathbf{D}$, Y_0, Y_m, etc. by $\hat{D} = \widehat{\det \mathbf{D}}$, $\hat{Y}_0$, $\hat{Y}_m$, etc. In particular

$$\widehat{\det \mathbf{D}} = \det ik, \quad \hat{x}_{\alpha\beta} = ia_{\alpha\beta}, \quad \hat{a}_{\alpha\beta} = ix_{\alpha\beta}.$$

We let $\mu \in \mathbf{C}$. Then it follows from (1.20), (1.21), and (1.22) that

$$\begin{aligned}\left[\det k^{\mu}, \hat{Y}_0\right] &= 2\hat{D}\hat{C}_{pq}\det(-i)^2\mu(\mu-1)\det k^{\mu-2}\\ &\quad+\det(-i)\left[\hat{D}, \hat{Y}_0\right]\mu\det k^{\mu-1}\\ &= 2\det(-i)\hat{C}_{pq}\mu(\mu-1)\det k^{\mu-1}+2\hat{Y}_m\mu\det k^{\mu}\\ &\quad-\det(-i)(n-1)\mu\hat{C}_{pq}\det k^{\mu-1}.\end{aligned}$$

Also, $[\det k^{\mu}, \hat{Y}_m] = 2\hat{C}_{pq}\mu\det k^{\mu-1}\det(-i)$. Hence,

$$\begin{aligned}&\det k^{\mu}\left(\hat{Y}_0+(\lambda+(n+1)/2)\hat{Y}_m\right)\\ &\quad=\left(\hat{Y}_0+(\lambda+(n+1)/2+2\mu)\hat{Y}_m\right)\det k^{\mu}\\ &\qquad+\det k^{\mu-1}\hat{C}_{pq}\det(-i)\left(2(\lambda+(n+1)/2)\mu+2\mu(\mu-1)-(n-1)\mu\right).\end{aligned}$$

In particular, the last term vanishes if and only if $\lambda = -\mu$. In this case, $\lambda' = \mu$.

REMARK. We let $d\hat{U}_{-\mu}$ and $d\hat{U}_{\mu}$ be the two representations of the Lie algebra obtained by Fourier transformation. They of course extend to be representations on all C^{∞} functions in the interior of the forward light cone, and since $\det k^{\mu}$ is a bijection of $C^{\infty}(C^{+})$ onto itself, these modules are infinitesimally equivalent. However, since when we took the Fourier transforms, we completely neglected boundary behavior, if these in any manner can be integrated, the result, transformed back again by the inverse Fourier transform, will in general differ from the original U_{λ}'s by boundary terms. In this connection we observe that if we define two C^{∞} functions f and g in C^{+} to be equivalent if for each point p on the boundary $b(C^{+})$ there is an open neighborhood N_p such that $\det k^{\mu}(f-g)$ in $N_p \cap C^{+}$ is the restriction of a C^{∞} function in N_p, then $d\hat{U}_{-\mu}$ preserves equivalence classes.

We finally mention that for r real and positive, the closure of the operators $d\hat{U}_r(x)$, $x \in \mathrm{sp}(n, \mathbf{R})$, are the differentials of a unitary representation $\hat{U}_r$ in $L^2(C^{+}, \det k^{-r}dk)$. Then, since the map

$$T_r\colon L^2(C^{+}, \det k^{-r}dk) \to L^2(C^{+}, \det k^{r}\,dk)\colon (T_r f)(k) = \frac{1}{\det k^{r}}\,f(r)$$

is unitary, we can define $T_r\hat{U}_rT_r^{-1}$, which then has generators $\overline{d\hat{U}_{-r}(x)}$, $x \in \mathrm{sp}(n, \mathbf{R})$.

2. The scalar case for $SU(n, n)$. In view of the analysis of Mp(n, **R**), it is not surprising that similar generalizations hold for $SU(n, n)$. Due to the strong analogy, we shall only give a sketch.

$SU(n, n)$ is the subgroup of SL($2n$, **C**) consisting of those matrices g that satisfy

$$g\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}g^{*} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix};$$

writing g in $n \times n$ blocks, this is equivalent to

$$ad^* - bc^* = 1; \quad ab^* = ba^*; \quad cd^* = dc^*, \tag{2.1}$$

and to

$$a^*d - c^*b = 1; \quad a^*c = c^*a; \quad b^*d = d^*b, \tag{2.2}$$

where (2.2) is (2.1) for g^{-1}. The Lie algebra is given by

$$su(n, n) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1^* \end{bmatrix} \middle| \operatorname{Im} \operatorname{tr} x_1 = 0; x_2 = x_2^*; x_3 = x_3^* \right\}.$$

We let $\mathfrak{D} = \{x + iy | x = x^*; y = y^*; y > 0\}$. $SU(n, n)$ acts on $\mathfrak{D}$ by $g \cdot z = (az + b)(cz + d)^{-1}$. Similarly to the case of $\mathrm{Mp}(n, \mathbf{R})$ there are functions, defined for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z, w \in \mathfrak{D}$ by

$$J_1(g, z) = cz + d, \qquad J_2(g, z) = (zc^* + d^*)^{-1}, \tag{2.3}$$

$$K_1(z, w) = ((z - w^*)/2i)^{-1}, \quad \text{and} \quad K_2(z, w) = ((z - w^*)/2i), \tag{2.4}$$

which satisfy (1.4) and (1.11):

$$\begin{aligned} &J_i(g_1 g_2, z) = J_i(g_1, g_2 z) J_i(g_2, z) \text{ and} \\ &K_i(gz, gw) = J_i(g, z) K_i(z, w) J_i(g, w)^*, \quad i = 1, 2. \end{aligned} \tag{2.5}$$

We shall consider the representations

$$(U_\lambda(g)f)(z) = \left(\det J_1(g^{-1}, z)\right)^{-\lambda - n} f(g^{-1}z), \tag{2.6}$$

for $\lambda \in \mathbf{Z}$.

It follows from (2.1) and (2.2) that $(gz^*)^* = g \cdot z$, and we can thus restrict U_λ to act on measurable functions on the Shilov boundary $\partial \mathfrak{D} = H$ of $\mathfrak{D}$; $H = \{x | x = x^*\}$.

On functions on H, we define first order differential operators $a_{\alpha\beta}$ to be dual to the variables $x_{\alpha\beta}$ in the parametrization of elements in H. Thus,

$$a_{\alpha\beta}(x_{ab}) = \delta_{a\alpha} \delta_{b\beta}.$$

We let $\mathbf{D}$ be the differential operator whose (α, β)th entry is

$$\{\mathbf{D}\}_{\alpha,\beta} = a_{\alpha\beta}, \tag{2.7}$$

and we shall here consider $\det \mathbf{D}$. We define the Fourier transform by

$$f(k) = \gamma_3 \int e^{-i \operatorname{tr} xk} f(x)\, dx,$$

and the inverse by

$$g(x) = \gamma_4 \int e^{i \operatorname{tr} xk} g(k)\, dk.$$

Since

$$(\det \mathbf{D}) e^{\operatorname{tr} xz} = \det z e^{\operatorname{tr} xz}, \tag{2.8}$$

we observe that det **D** is proportional to the Fourier transform of $(i)^n$ det k.

As in the case of Mp(n, **R**) we shall analyze for which pairs of (λ, λ')'s powers of det **D** are intertwining operators, and exactly as in that case we are reduced to studying the action of the subalgebra

$$\left\{\begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix} \middle| y = y^* \right\}.$$

Specifically, we take $\{y\}_{kl} = \rho\delta_{pk}\delta_{ql} + \rho^*\delta_{pl}\delta_{kq}$, for a fixed $\rho \in \mathbf{C}$. For x in H and t in **R**, we then have

$$x(tyx+1)^{-1} \cong x - txyx; \quad \det(tyx+1) \cong 1 + t \operatorname{tr} xy.$$

Since $\{xyx\}_{\alpha\beta} = \rho x_{\alpha p} x_{q\beta} + \rho^* x_{\alpha q} x_{p\beta}$, and $\operatorname{tr} yx = \rho x_{qp} + \rho^* x_{pq}$, we see, that if we put

$$Y_0 = \sum_{\alpha,\beta=1}^{n} (\rho x_{\alpha p} x_{q\beta} + \rho^* x_{\alpha q} x_{p\beta}) a_{\alpha\beta},$$

and

$$Y_m = \rho x_{qp} + \rho^* x_{pq},$$

then

$$dU_\lambda\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} U_\lambda\left(\begin{bmatrix} 1 & 0 \\ ty & 1 \end{bmatrix}\right)\Bigg|_{t=0} = Y_0 + (\lambda + n) Y_m. \tag{2.9}$$

We compute $[\det \mathbf{D}, x_{i_0 j_0}]$, maintaining the notation from Mp(n, **R**):

$$\det \mathbf{D} = \sum_{j=1}^{n} a_{i_0 j} C_{i_0 j} \Rightarrow [\det \mathbf{D}, x_{i_0 j_0}] = C_{i_0 j_0}. \tag{2.10}$$

Hence,

$$\begin{aligned}
[\det \mathbf{D}, Y_0] &= \sum_{\alpha,\beta=1}^{n} [\det \mathbf{D}, \rho x_{\alpha p} x_{q\beta} + \rho^* x_{\alpha q} x_{p\beta}] a_{\alpha\beta} \\
&= \sum_{\alpha,\beta=1}^{n} \rho x_{\alpha p} C_{q\beta} a_{\alpha\beta} + \sum_{\alpha,\beta=1}^{n} \rho C_{\alpha p} x_{q\beta} a_{\alpha\beta} \\
&\quad + \sum_{\alpha,\beta=1}^{n} \rho^* x_{\alpha q} C_{p\beta} a_{\alpha\beta} + \sum_{\alpha,\beta=1}^{n} \rho^* C_{\alpha q} x_{p\beta} a_{\alpha\beta} \\
&= \sum_{\alpha=1}^{n} \rho x_{\alpha p} (\det \mathbf{D}) \delta_{q\alpha} + \sum_{\alpha,\beta=1}^{n} \rho C_{\alpha p} (a_{\alpha\beta} x_{q\beta} - \delta_{q\alpha}) \\
&\quad + \sum_{\alpha=1}^{n} \rho^* x_{\alpha q} (\det \mathbf{D}) \delta_{p\alpha} + \sum_{\alpha,\beta=1}^{n} \rho^* C_{\alpha q} (a_{\alpha\beta} x_{p\beta} - \delta_{p\alpha}) \\
&= 2 Y_m (\det \mathbf{D}) - (n-1)[\rho C_{qp} + \rho^* C_{pq}].
\end{aligned}$$

We put $C = \rho C_{qp} + \rho^* C_{pq}$. Then,

$$[\det \mathbf{D}, Y_0] = 2Y_m(\det \mathbf{D}) - (n-1)C;$$
$$[\det \mathbf{D}, Y_m] = C; \, [\det \mathbf{D}, C] = 0. \tag{2.11}$$

Hence,

$$\det \mathbf{D}(Y_0 + (\lambda + n)Y_m)$$
$$= (Y_0 + (\lambda + n + 2)Y_m)\det \mathbf{D} + (\lambda + n - (n-1))C,$$

or, more generally,

$$(\det \mathbf{D})^r(Y_0 + (\lambda + n)Y_m) = (Y_0 + (\lambda + n + 2r)Y_m)(\det \mathbf{D})^r$$
$$+ [r(\lambda + n) + (r-1)r - r(n-1)]C(\det \mathbf{D})^{r-1}. \tag{2.12}$$

We see that in order for $(\det \mathbf{D})^r$ to be intertwining, the last term must vanish, i.e. $\lambda = -r$. In this case, $\lambda' = r$. Hence we have proved

PROPOSITION 2.1. *For integers $r \geqslant 0$,*

$$(\det \mathbf{D})^r dU_{-r}(X) = dU_r(X)(\det \mathbf{D})^r \quad \forall X \in su(n, n). \tag{2.13}$$

If we let V_{-r} be the space consisting of those real analytic functions f on H, that are boundary values of holomorphic functions on $\mathfrak{D}$, and for which $U_{-r}(\left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right])f$ again is such a function, then it follows easily (cf. Chapter 1) that we have

PROPOSITION 2.2 [5]. *For r in $\mathbf{N}$, f in V_{-r}, and g in $SU(n, n)$:*

$$(\det \mathbf{D})^r U_{-r}(g)f = U_r(g)(\det \mathbf{D})^r f. \tag{2.14}$$

In this section we let $C^+ = \{x \in H | x > 0\}$. Then it is again known from [3] or [9] that there exists a constant k_2 such that for $\alpha > -1$ and z in $\mathfrak{D}$

$$\int_{C^+} e^{i \operatorname{tr} zy} \det y^\alpha \, dy = k_2(\det z/i)^{-\alpha - n} \prod_{i=1}^{n} \Gamma(i + \alpha). \tag{2.15}$$

Likewise, it is known [9] that on the orbits $\mathcal{O}_0 = \{0\}$, $\mathcal{O}_1 = \{x \in H | x \geqslant 0, \operatorname{rank} x = 1\}, \ldots, \mathcal{O}_{n-1} = \{x \in H | x \geqslant 0, \operatorname{rank} x = n-1\}$ there are semi-invariant measures μ_j such that for suitable constants $c_b(j)$

$$\int_{\mathcal{O}_j} e^{i \operatorname{tr} zy} \, d\mu_j(y) = c_b(j)(\det z)^{-j}. \tag{2.16}$$

These formulas imply that U_α is unitary for $\alpha > -1$, and that U_{-j} for $j = 1, 2, \ldots, n$ has an invariant subspace, on which it acts unitarily.

Finally, again parallel to the case of $\mathrm{Mp}(n, \mathbf{R})$ we conclude:

PROPOSITION 2.3. *On a space of equivalence classes of functions whose Fourier transforms are supported by $\overline{C^+}$; equivalence being defined by $f \underset{r}{\sim} g \Leftrightarrow$*

$(\det \mathbf{D})^r(f-g)=0$, *one can construct a pre-Hilbert space structure, which is preserved by* U_{-r}, *for all* r *in* $\mathbf{N}$.

REMARK. By taking Fourier-transforms, we can get the relation

$$(\det k)^{\mu} d\hat{U}_{-\mu} = d\hat{U}_{\mu}(\det k)^{\mu},$$

for any $\mu \in \mathbf{C}$, as an equation between Lie algebra representations on the space of C^{∞}-functions whose support is compact and contained in C^+; just as in the case of $\mathrm{Mp}(n, \mathbf{R})$.

3. The Dirac-type operator. The case of $\mathrm{Mp}(n, \mathbf{R})$. There exists a distinguished operator ∇; a $2n \times 2n$ matrix with differential operators as entries, for which $(\nabla)^2 = \det \mathbf{D}$. Recall from [5] that the Dirac operator associated to

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

can be defined as $\nabla = \begin{pmatrix} 0 & \sigma \\ c(\sigma) & 0 \end{pmatrix}$, where

$$\sigma = \begin{bmatrix} \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial x_3} \end{bmatrix},$$

and $c(\sigma)$ is the co-factor of σ;

$$c(\sigma) = \begin{bmatrix} \dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x_3} \end{bmatrix}.$$

We observe that, corresponding to (1.15), σ^t is proportional to the Fourier transform of the multiplication operator $(kf)(k) = k \cdot f(k)$ for $k \in H(2)$. Likewise, $c(\sigma)^t$ is proportional to the Fourier transform of the multiplication operator $c(k)$, where $c(k)$ is the co-factor of k; $c(k) = (\det k)k^{-1}$. These observations clearly lead to a natural candidate for ∇. Finally, we make the (key) observation, that if we promote $\square$ to be the operator $\square f = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ on functions $f\colon H(2) \to \mathbf{C}^2$, and let h and $c(h)$ be the above defined multiplication operators, then [5]

$$[\square, h] = 2c(\sigma); \qquad [\square, c(h)] = 2\sigma.$$

We shall in this chapter be concerned with two series of representations of $\mathrm{Mp}(n, \mathbf{R})$. To avoid a repetition of the technical arguments following Proposition 1.1, we shall here consider the space of holomorphic functions from $\mathfrak{D}$

to $\mathbf{C}^n$. We define, for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\lambda \in \mathbf{Z}$, and $z \in \mathfrak{D}$

$$(D_\lambda^+(g)f)(z) = \frac{(cz+d)^{-1}}{\det(cz+d)^{\lambda+(n+1)/2}} f\left(\frac{az+b}{cz+d}\right), \tag{3.1}$$

$$(D_\lambda^-(g)f)(z) = \frac{(zc^t+d^t)}{\det(zc^t+d^t)^{\lambda+(n+3)/2}} f\left(\frac{az+b}{cz+d}\right), \text{ and}$$

$$D_\lambda(g) = D_\lambda^+(g) \oplus D_\lambda^-(g).$$

We let $\mathbf{D} = \mathbf{D}^t$ be the operator from (1.13), extended in an obvious way to functions from $\mathfrak{D}$ to $\mathbf{C}^n$, corresponding to a parametrization of $\mathfrak{D}$ by matrices z for which the (i,j)th entry $z_{ij} = z_{ji} \in \mathbf{C}$. Then all the formulas from Chapter 1 remain unchanged. We let $c(\mathbf{D})$ be the $n \times n$ matrix whose (i,j)th entry is C_{ij} = the determinant of M_{ij}, as defined in (1.18). Thus, $C(\mathbf{D})^t = c(\mathbf{D})$ and $\mathbf{D} \cdot c(\mathbf{D}) = c(\mathbf{D}) \cdot \mathbf{D} = (\det \mathbf{D}) I_n$. Finally, we let $\nabla = \begin{pmatrix} 0 & \mathbf{D} \\ c(\mathbf{D}) & 0 \end{pmatrix}$. Then $(\nabla)^{2r} = (\det \mathbf{D})^r$ for $r \in \mathbf{N}$.

We want to analyze the relation between odd powers of ∇ and the representations D_λ. We begin by investigating for what (λ, λ'), for a given integer $l \geqslant 0$,

$$(\det \mathbf{D})^l \mathbf{D} D_\lambda^-(g) = (D_{\lambda'}^+(g))(\det \mathbf{D})^l \mathbf{D}. \tag{3.2}$$

As in the scalar case, we are immediately reduced to studying the elements $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ of $\mathrm{sp}(n, \mathbf{R})$. We maintain the notation of Chapter 1. Then,

$$dD_{\lambda'}^+\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} D_{\lambda'}^+\left(\begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix}\right)\bigg|_{t=0}$$

$$= yz + \left(Y_0 + (\lambda' + (n+1)/2) Y_m\right) I_n, \text{ and}$$

$$dD_\lambda^-\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} D_\lambda^-\left(\begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix}\right)\bigg|_{t=0}$$

$$= -zy + \left(Y_0 + (\lambda + (n+3)/2) Y_m\right) I_n. \tag{3.3}$$

We consider $\mathbf{D}zy$. Since $\{zy\}_{rs} = \delta_{qs} z_{rp} + \delta_{ps} z_{rq}$, we get that

$$\{\mathbf{D}zy\}_{rs} = \sum_i (\delta_{qs} a_{ri} z_{ip} + \delta_{ps} a_{ri} z_{iq})$$

$$= \sum_i (\delta_{qs} z_{ip} a_{ri} + \delta_{ps} z_{iq} a_{ri}) + \frac{n+1}{2} (\delta_{qs}\delta_{rp} + \delta_{ps}\delta_{rq}).$$

Since moreover

$$\{[\mathbf{D}, Y_0]\}_{rs} = [a_{rs}, Y_0] = \sum_\beta (\delta_{sp} z_{q\beta} a_{r\beta} + \delta_{rp} z_{q\beta} a_{s\beta}) + \sum_\alpha (z_{\alpha p}\delta_{rq} a_{\alpha s} + z_{\alpha p}\delta_{sq} a_{\alpha r}),$$

$$\{\mathbf{D} Y_m\}_{rs} = \{Y_m \mathbf{D}\}_{rs} + (\delta_{qr}\delta_{ps} + \delta_{qs}\delta_{pr}),$$

and

$$\{yz\mathbf{D}\}_{rs} = \sum_i (\delta_{pr} z_{qi} a_{is} + \delta_{rq} z_{pi} a_{is}),$$

we get, using $a_{\alpha\beta} = a_{\beta\alpha}$ and $z_{\alpha\beta} = z_{\beta\alpha}$, that

$$\Big\{(\det \mathbf{D})^l \mathbf{D}\big(-zy + \big(Y_0 + (\lambda + (n+3)/2) Y_m\big) I_n\big)\Big\}_{rs}$$
$$= \Big\{(\det \mathbf{D})^l yz\mathbf{D}\Big\}_{rs} + \Big\{(\det \mathbf{D})^l \big(Y_0 + (\lambda + (n+3)/2) Y_m\big)\mathbf{D}\Big\}_{rs} + (\det \mathbf{D})^l (\lambda + (n+3)/2 - (n+1)/2)(\delta_{qr}\delta_{ps} + \delta_{qs}\delta_{pr}).$$

To compute $\{(\det \mathbf{D})^l yz\, \mathbf{D}\}_{rs}$, observe that $\{[\det \mathbf{D}, yz]\}_{rs} = \delta_{qr} C_{sp} + \delta_{pr} C_{sq}$. It then follows easily that

$$\Big\{(\det \mathbf{D})^l yz\, \mathbf{D}\Big\}_{rs} = \Big\{yz(\det \mathbf{D})^1 \mathbf{D}\Big\}_{rs} + l(\delta_{qr}\delta_{ps} + \delta_{pr}\delta_{qs})(\det \mathbf{D})^l.$$

Using (1.23), we thus get

$$\Big\{(\det \mathbf{D})^l \mathbf{D}\big(-zy + \big(Y_0 + (\lambda + (n+3)/2) Y_m\big) I_n\big)\Big\}_{rs}$$
$$= \Big\{yz(\det \mathbf{D})^l \mathbf{D}\Big\}_{rs} + \Big\{\big(Y_0 + (\lambda + 2l + (n+3)/2) Y_m\big)(\det \mathbf{D})^l \mathbf{D}\Big\}_{rs}$$
$$+ \Big\{\big(2l(\lambda + (n+3)/2) + 2l(l-1) - l(n-1)\big) C_{pq} (\det \mathbf{D})^{l-1}\mathbf{D}\Big\}_{rs}$$
$$+ (\lambda + 1 + l)(\det \mathbf{D})^l (\delta_{qr}\delta_{ps} + \delta_{qs}\delta_{pr}). \tag{3.4}$$

In particular, to have (3.2) satisfied, we must have $\lambda = -1 - l$ and $\lambda' = l$.

To deal with the other half of the problem, observe that if $\det \mathbf{D} = (\det \mathbf{D}) I_n$, then

LEMMA 3.1.

$$C(\mathbf{D}) = [\det \mathbf{D}, z], \tag{3.5}$$

and in fact, for integers $l > 0$,

$$(l + 1)(\det \mathbf{D})^l C(\mathbf{D}) = \big[(\det \mathbf{D})^{l+1}, z\big]. \tag{3.6}$$

PROOF. (3.6) clearly follows from (3.5) which again follows from (1.21).

To complete the investigation, we need therefore only check whether, for

$g \in \mathrm{Mp}(n, \mathbf{R})$

$$\left[(\det \mathbf{D})^{l+1}, z\right] D^{+}_{-1-l}(g) = D_l^{-}(g)\left[(\det \mathbf{D})^{l+1}, z\right]. \tag{3.7}$$

This relation is trivially satisfied for $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $x \in S$, since $[(\det \mathbf{D})^{l+1}, z]$ is a constant coefficient differential operator. But these elements, together with those that project onto $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in \mathrm{Sp}(n, \mathbf{R})$ generate $\widetilde{\mathrm{Sp}(n, \mathbf{R})}$. This means that if we put $(C_0 f)(z) = f(-z^{-1})$, then we must have

$$\left[(\det \mathbf{D})^{l+1}, z\right] \frac{z^{-1}}{(\det z)^{-1-l+(n+1)/2}} C_0$$
$$= \frac{z}{(\det z)^{l+(n+3)/2}} C_0\left[(\det \mathbf{D})^{l+1}, z\right].$$

That this equation is valid follows from Proposition 1.2 since

$$\left((\det \mathbf{D})^{l+1} z - z(\det \mathbf{D})^{l+1}\right) \frac{z^{-1}}{(\det z)^{-1-l+(n+1)/2}} C_0$$
$$= (\det \mathbf{D})^{l+1} \frac{1}{(\det z)^{-1-l+(n+1)/2}} C_0$$
$$+ z(\det \mathbf{D})^{l+1} \frac{1}{(\det z)^{-1-l+(n+1)/2}} C_0 z$$
$$= \frac{1}{(\det z)^{l+(n+3)/2}} C_0 (\det \mathbf{D})^{l+1}$$
$$+ z \frac{1}{(\det z)^{l+(n+3)/2}} C_0 (\det \mathbf{D})^{l+1} z$$
$$= -\frac{z}{(\det z)^{l+(n+3)/2}} C_0 z (\det \mathbf{D})^{l+1}$$
$$+ \frac{z}{(\det z)^{l+(n+3)/2}} C_0 (\det \mathbf{D})^{l+1} z$$
$$= \frac{z}{(\det z)^{l+(n+3)/2}} C_0\left[(\det \mathbf{D})^{l+1}, z\right].$$

We have thus brought the analysis to an end. Similarly to the scalar case we can then state

PROPOSITION 3.2. *Let $l \geqslant 0$ be an integer and let f be a holomorphic function from $\mathfrak{D}$ to $\mathbf{C}^n$. Then, for all g in* $\mathrm{Mp}(n, \mathbf{R})$

$$\nabla^{2l+1} D_{-1-l}(g) f = D_l(g) \nabla^{2l+1} f.$$

To settle the question of unitarity, observe that it follows quite readily from [1], [2], and [8] that there exists a constant k_1 such that for $\alpha > -1$ and

$z \in \mathfrak{D}$:

$$\int_{C^+} e^{i \operatorname{tr} zy} c(y)(\det y)^{\alpha}\, dy$$
$$= k_1(z/i)(\det z/i)^{-\alpha-((n+3)/2)}\Gamma(\alpha+1)\prod_{i=2}^{n}\Gamma(\alpha+(i+3)/2), \quad (3.8)$$

and

$$\int_{C^+} e^{i \operatorname{tr} zy} y(\det y)^{\alpha}\, dy$$
$$= k_1(z/i)^{-1}(\det z/i)^{-\alpha-((n+1)/2)}\Gamma(\alpha+(n+3)/2)\prod_{i=1}^{n-1}\Gamma(\alpha+(i+1)/2).$$

By (1.8), (1.9), (1.10), and (1.11) it then follows that D_λ is unitary for $\lambda > -1$.

We observe that since, for $v \in \mathbf{C}^n$, $\mathbf{D}e^{i \operatorname{tr} zy}v = iye^{i \operatorname{tr} zy}v$, and $c(\mathbf{D})e^{i \operatorname{tr} zy}v = (i)^{n-1}c(y)e^{i \operatorname{tr} zy}v$, it follows from (1.27) and (3.8) by analytic continuation in α, that for all α in $\mathbf{R}$, z, w in $\mathfrak{D}$ and $v \in \mathbf{C}^n$:

$$\det \mathbf{D} \det\left(\frac{z-w^*}{i}\right)^{-\alpha((n+1)/2)} v$$
$$= (i)^n \prod_{i=1}^{n}\left(\alpha+\frac{i+1}{2}\right)\det\left(\frac{z-w^*}{i}\right)^{-\alpha-1-((n+1)/2)} v,$$

$$C(\mathbf{D})\det\left(\frac{z-w^*}{i}\right)^{-\alpha-((n+1)/2)} v$$
$$= (i)^{n-1}\prod_{i=2}^{n}\left(\alpha+\frac{i+1}{2}\right)\left(\frac{z-w^*}{i}\right)\det\left(\frac{z-w^*}{i}\right)^{-\alpha-((n+3)/2)} v, \text{ and}$$

$$\mathbf{D}\det\left(\frac{z-w^*}{i}\right)^{-\alpha-((n+1)/2)} v$$
$$= (i)\left(\alpha+\frac{n+1}{2}\right)\left(\frac{z-w^*}{i}\right)^{-1}\det\left(\frac{z-w^*}{i}\right)^{-\alpha-((n+1)/2)} v. \quad (3.9)$$

By comparison with (1.28), it then follows that D_{-1} has an invariant subspace on which it is unitary, namely the completion of the pre-Hilbert space of all finite linear combinations of the functions (see (1.10))

$$z \to K_1(z, w)\det K_1(z, w)^{-1+((n+1)/2)}v_1 \oplus K_2(z, w)\det K_2(z, w)^{-((n+1)/2)}v_2,$$

with w in $\mathfrak{D}$, and v_i in $\mathbf{C}^n$ for $i = 1, 2$, in the metric given by (1.8).

Since the co-factor of a rank r matrix for $r < n-2$ is zero, a similar result for the other orbits $\mathcal{O}_j$ will only hold true for the representations $D^{+}_{-((n-j+1)/2)}$, $j = 0, 1, \ldots, n-2$. For every integer $r \geqslant 0$, we define equivalence classes of functions f from $\mathfrak{D}$ to $\mathbf{C}^n \oplus \mathbf{C}^n$ by

$$[f]_{2r+1} = [g]_{2r+1} \Leftrightarrow \nabla^{2r+1}(f-g) = 0.$$

Then (cf. [5]) (3.9) together with Proposition 3.2 easily gives

PROPOSITION 3.3. *For each integer $r \geq 0$, there exists a subspace of the equivalence classes $[\cdot]_{2r+1}$ which can be given a Hilbert space structure in which D_{-1-r} is unitary.*

4. The Dirac-type operator. The case of $SU(n, n)$. We shall here describe the analogue of Chapter 3 for $SU(n, n)$. We consider functions defined on H; it is then straightforward to translate the results to holomorphic functions on $\mathfrak{D}$.

Let f be a measurable function from H to $\mathbf{C}^n$, $\lambda \in \mathbf{Z}$, and $x \in H$. We define

$$(D_\lambda^+(g)f)(x) = \frac{(cx+d)^{-1}}{\det(cx+d)^{\lambda+n}} f\left(\frac{ax+b}{cx+d}\right), \text{ and}$$

$$(D_\lambda^-(g)f)(x) = \frac{(xc^*+d^*)}{\det(xc^*+d^*)^{\lambda+1+n}} f\left(\frac{ax+b}{cx+d}\right), \tag{4.1}$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(n, n)$. Finally, we put

$$D_\lambda(g) = D_\lambda^+(g) \oplus D_\lambda^-(g).$$

Let $\mathbf{D}^t$ be the transpose of the matrix $\mathbf{D}$ in (2.5), and let $c(\mathbf{D})$ be the $n \times n$ matrix whose (i, j)th entry is $C_{ij} = C_{ij}(\mathbf{D})$ which is nothing more than the determinant of the matrix M_{ij} obtained from $\mathbf{D}$ as in (1.18). Then, since $C_{ij}(\mathbf{D}) = C_{ji}(\mathbf{D}^t)$,

$$\mathbf{D}^t c(\mathbf{D}) = c(\mathbf{D})\mathbf{D}^t = (\det \mathbf{D})I_n,$$

i.e. if $\nabla = \left(\begin{smallmatrix} 0 & \mathbf{D}^t \\ c(\mathbf{D}) & 0 \end{smallmatrix}\right)$, then $(\nabla)^{2r} = (\det \mathbf{D})^r$ for $r \in \mathbf{N}$. Again we investigate for what (λ, λ')

$$(\det \mathbf{D})^l \mathbf{D}^t D_\lambda^-(g) = (D_{\lambda'}^+(g))(\det \mathbf{D})^l \mathbf{D}^t \tag{4.2}$$

for a given integer $l \geq 0$, and again we are reduced to studying the elements $\left(\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right)$ of $su(n, n)$; $y \in H$. We maintain the notation of Chapter 2 and get

$$dD_{\lambda'}^+\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} D_{\lambda'}^+\left(\begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix}\right)\Bigg|_{t=0} = yx + (Y_0 + (\lambda' + n)Y_m)I_n,$$

$$dD_\lambda^-\left(\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right) = \frac{d}{dt} D_\lambda^-\left(\begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix}\right)\Bigg|_{t=0}$$

$$= -xy + (Y_0 + (\lambda + n + 1)Y_m)I_n. \tag{4.3}$$

To proceed, we observe the formulas

$$\{\mathbf{D}'xy\}_{rs} = \sum_{i=1}^{n} a_{ir}(x_{ip}\rho\delta_{qs} + x_{iq}\rho^*\delta_{ps}),$$

$$\{yx\mathbf{D}'\}_{rs} = \sum_{i=1}^{n} (\rho\delta_{pr}x_{qi} + \rho^*\delta_{rq}x_{pi})a_{si},$$

$$\{[\mathbf{D}', Y_0]\}_{rs} = [a_{sr}, Y_0] = \sum_{\beta=1}^{n} \rho\delta_{rp}x_{q\beta}a_{s\beta} + \sum_{\alpha=1}^{n} \rho x_{\alpha p}\delta_{sq}a_{\alpha r} + \sum_{\beta=1}^{n} \rho^*\delta_{rq}x_{p\beta}a_{s\beta} + \sum_{\alpha=1}^{n} \rho^* x_{\alpha q}\delta_{sp}a_{\alpha r},$$

$$\{[\mathbf{D}', Y_m]\}_{rs} = (\rho\delta_{sq}\delta_{rp} + \rho^*\delta_{sp}\delta_{rq}),$$

$$\{[\det \mathbf{D}, yx]\}_{rs} = \rho\delta_{pr}C_{qs} + \rho^*\delta_{rq}C_{ps},$$

$$\{[(\det \mathbf{D})^l, yx]\}_{rs} = l(\det \mathbf{D})^{l-1}(\rho\delta_{pr}C_{qs} + \rho^*\delta_{rq}C_{ps}). \tag{4.4}$$

From these formulas, together with (2.12), it then follows that

$$\begin{aligned}\{(\det \mathbf{D})^l\mathbf{D}'(-xy + (Y_0 + (\lambda + 1 + n)Y_m)I_n)\}_{rs} &= \{yx(\det \mathbf{D})^l\mathbf{D}'\}_{rs} + (\lambda + 1 + l)(\det \mathbf{D})^l(\rho\delta_{pr}\delta_{qs} + \rho^*\delta_{rq}\delta_{ps}) \\ &\quad + \{(Y_0 + (\lambda + 1 + n + 2l)Y_m)I_n(\det \mathbf{D})^l\mathbf{D}'\}_{rs} \\ &\quad + \{(l(\lambda + 1 + n) + l(l-1) - l(n-1))C(\det \mathbf{D})^{l-1}\mathbf{D}'\}_{rs}.\end{aligned} \tag{4.5}$$

In particular, if (4.2) is to be satisfied, then $\lambda = -l - 1$ and $\lambda' = l$.

The other half of the problem follows similarly to that of Chapter 3, once we have noted

LEMMA 4.1. *For integers* $l \geqslant 0$,

$$[(\det \mathbf{D})^{l+1}, x] = (l + 1)(\det \mathbf{D})^l c(\mathbf{D}). \tag{4.6}$$

PROOF. This follows by induction from (2.8).

We let V_{-l} be the space of those functions on H with values in $\mathbf{C}^n \oplus \mathbf{C}^n$ that are boundary values of holomorphic functions on $\mathfrak{D}$, and for every x in H can be continued across the boundary as a holomorphic function in a neighborhood N_x of x, and which maintain this property when acted upon by D_{-1-l}. We can then state, similarly to the preceding cases:

PROPOSITION 4.2. *For integers* $l \geqslant 0$, $g \in SU(n, n)$, *and* $f \in V_{-l}$:

$$\nabla^{2l+1}D_{-1-l}(g)f = D_l(g)\nabla^{2l+1}f. \tag{4.7}$$

The question of unitarity is again settled by formulas obtainable from **[1]**,

[2], and [8]. Specifically, there exists a constant k_2 such that for $\alpha > -1$ and $z \in \mathfrak{D}$:

$$\int_{C^+} e^{i\,\mathrm{tr}\,zy} c(y)(\det y)^{\alpha}\, dy$$

$$= k_2(z/i)(\det(z/i))^{-\alpha-1-n}\Gamma(1+\alpha)\prod_{i=2}^{n}\Gamma(1+i+\alpha), \tag{4.8}$$

and

$$\int_{C^+} e^{i\,\mathrm{tr}\,zy} y(\det y)^{\alpha}\, dy$$

$$= k_2(z/i)^{-1}(\det(z/i))^{-\alpha-n}\Gamma(n+1+\alpha)\prod_{i=1}^{n-1}\Gamma(i+\alpha).$$

It follows again by (1.8), (1.9), (1.10), and (1.11) that D_λ is unitary for $\lambda > -1$. It is easy to see, using $\det A^t = \det A$, that for $v \in \mathbf{C}^n$:

$$\mathbf{D}^t e^{i\,\mathrm{tr}\,xy} v = iye^{i\,\mathrm{tr}\,xy} v,$$

and

$$c(\mathbf{D})e^{i\,\mathrm{tr}\,xy} v = (i)^{n-1} c(y) e^{i\,\mathrm{tr}\,xy} v.$$

It follows from these formulas, together with (2.15) and (4.8) that for $\alpha \in \mathbf{R}$, $h \in H$, $w \in \mathfrak{D}$, and $v \in \mathbf{C}^n$:

$$\det \mathbf{D} \det((h-w^*)/i)^{-\alpha-n} v = (i)^n \prod_{i=1}^{n}(\alpha+i)\det((h-w^*)/i)^{-\alpha-1-n} v,$$

$$c(\mathbf{D})\det((h-w^*)/i)^{-\alpha-n} v$$

$$= (i)^{n-1}\prod_{i=2}^{n}(i+\alpha)((h-w^*)/i)\det((h-w^*)/i)^{-\alpha-1-n} v, \tag{4.9}$$

and

$$\mathbf{D}^t \det((h-w^*)/i)^{-\alpha-n} v$$

$$= (i)(n+\alpha)((h-w^*)/i)^{-1}\det((h-w^*)/i)^{-\alpha-n} v.$$

By comparison with (2.16), it follows that D_{-1} has an invariant subspace, on which it is unitary, namely that completion of the pre-Hilbert space of all finite linear combinations of the functions (see (2.4))

$$h \to K_1(h,w)\det K_1(h,w)^{-1+n} v_1 \oplus K_2(h,w)\det K_2(h,w)^{-n} v_2,$$

with w in $\mathfrak{D}$, and v_i in $\mathbf{C}^n$ for $i = 1, 2$, in the metric given by (1.8), whereas for the other orbits $\mathcal{O}_j$, this only holds true for $D^{+}_{-(n-j)}$; $j = 0, 1, \ldots, n-2$. For every integer $l \geqslant 0$ we define equivalence classes of functions f from H to

$\mathbf{C}^n \oplus \mathbf{C}^n$ by

$$[f]_{2l+1} = [g]_{2l+1} \Leftrightarrow \nabla^{2l+1}(f-g) = 0.$$

Again analogous to the proof for the case of $SU(2, 2)$ in [5], we get, using Proposition 4.2 and (4.9)

PROPOSITION 4.3. *For each integer $l \geq 0$, there eixsts a subspace of the equivalence classes $[\cdot]_{2l+1}$ which can be given a Hilbert space structure, in which D_{-1-l} is unitary.*

5. A combination. We conclude this article with a study of a series of representations obtained by induction from noncomplemented finite dimensional representations of the maximal parabolic subgroup P_-.

We shall give the details for the groups $SU(n, n)$, whereas the corresponding results for Mp(n, **R**), due to the large similarity, are omitted.

For $SU(n, n)$,

$$P_- = \left\{ \begin{pmatrix} a & 0 \\ c & a^{*-1} \end{pmatrix} \middle| \det a \text{ is real and } ca^* = ac^* \right\},$$

and hence there are some very natural representations of P_-, namely

$$\mu_l \begin{pmatrix} a & 0 \\ c & a^{*-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{*-1} \end{pmatrix} (\det a)^{l+n}. \tag{5.1}$$

Similarly to the preceding cases, the induced representations obtained from μ_l yield actions on measurable functions on H (see [5]) since H can be identified with the subgroup $\{(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) | x \in H\}$ of G (which is mapped onto an open dense subset of G/P_-). The actions $H_l(g)$ thus obtained are

$$(H_l(g)f)(x)$$
$$= \begin{bmatrix} (cx+d)^* & 0 \\ -c^* & (cx+d)^{-1} \end{bmatrix} \det(cx+d)^{-(n+l)} f\left(\frac{ax+b}{cx+d} \right), \tag{5.2}$$

if $g^{-1} = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, and if f is any measurable function from H to $\mathbf{C}^{2n}$.

We note that if $z \in \mathfrak{D}$ and if $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in G$, then

$$\det(cz+d) = \det(zc^* + d^*). \tag{5.3}$$

This fact is trivially true on the subgroup

$$P_+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{*-1} \end{pmatrix} \middle| \det a \text{ is real, and } ab^* = ba^* \right\}$$

and on $(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) \in G$. These elements generate the group, and hence, by (2.5), the assertion follows for all g in G.

In terms of the representations studied in Chapter 2 and Chapter 4,

$$H_l(g) = \begin{pmatrix} D_{l-1}^{-}(g) & 0 \\ -c(g^{-1})^* U_l(g) & D^{+}(g) \end{pmatrix}, \tag{5.4}$$

and from (5.3) it follows, that if we let $\mathcal{O}(2n)$ be the space of holomorphic functions from $\mathfrak{D}$ to $\mathbf{C}^{2n}$, then we can think of H_l as an action on that space, with (5.2) as boundary value. We shall do that, and omit the technicalities of passing to the boundary. In this connection, we recall that it was observed in the preceding chapters, that the differential operators $\mathbf{D}'$, $c(\mathbf{D})$, and $\det \mathbf{D}$ can be extended to operate on holomorphic functions on $\mathfrak{D}$ in such a way that the intertwining relations remain valid.

LEMMA 5.5. *Let l be an integer and let f be a holomorphic function from $\mathfrak{D}$ to $\mathbf{C}^n$. Then*

$$\mathbf{D}'\left(\frac{z}{\det z^{l+n}} f(-z^{-1}) \right)$$
$$= -l \frac{1}{\det z^{l+n}} f(-z^{-1}) + \frac{z^{-1}}{\det z^{l+n}} (\mathbf{D}'f)(-z^{-1}). \tag{5.6}$$

PROOF. Since $\mathbf{D}'$ is a matrix with first order differential operators as entries, and since $(\det z)^{-l}$ is a scalar valued function, the formula will follow from (4.7) (with $l = 0$), once we have proved that for any vector $v \in \mathbf{C}^n$, $\mathbf{D}'(\det z^{-l}v) = -lz^{-1}\det z^{-l}v$. This, however, follows from (4.9).

PROPOSITION 5.7. *Let l be an integer. Then for all f in $\mathcal{O}(2n)$ and all g in G,*

$$\begin{bmatrix} l & 0 \\ -\mathbf{D}' & l \end{bmatrix} \begin{bmatrix} D_{l-1}^{-}(g) & 0 \\ -c(g^{-1})^* U_l(g) & D_l^{+}(g) \end{bmatrix} f$$
$$= \begin{bmatrix} D_{l-1}^{-}(g) & 0 \\ 0 & D_l^{+}(g) \end{bmatrix} \begin{bmatrix} l & 0 \\ -\mathbf{D}' & l \end{bmatrix} f. \tag{5.8}$$

PROOF. We need only check the relation for $g^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For this element, the only nontrivial equation in (5.8) is (5.6).

In (5.8) the operator $\left[\begin{smallmatrix} l & 0 \\ -\mathbf{D}' & l \end{smallmatrix}\right]$ is clearly invertible on $\mathcal{O}(2n)$ when $l \neq 0$. Hence, by Chapter 4, one can put a Hilbert space structure K_l on a subspace of $\mathcal{O}(2n)$ in which H_l is unitary for $l \geqslant 1$.

When $l = 0$, in addition to (5.8), we also have

$$\begin{bmatrix} 0 & 0 \\ -\mathbf{D}' & 0 \end{bmatrix} H_0 = H_0 \begin{bmatrix} 0 & 0 \\ -\mathbf{D}' & 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} D_{-1}^{-} & 0 \\ 0 & D_0^{+} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\mathbf{D}' & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\mathbf{D}' & 0 \end{bmatrix} \begin{bmatrix} D_{-1}^{-} & 0 \\ 0 & D_0^{+} \end{bmatrix}. \tag{5.9}$$

In this case, we see that $K_0^+ = \{\binom{0}{\varphi} \in \mathcal{O}(2n)\}$ and $K_0^- = \{\binom{\varphi^A}{\varphi_B} \in \mathcal{O}(2n) | \mathbf{D}^t\varphi^A = 0\}$ are invariant subspaces. The restriction of H_0 to K_0^+ is D_0^+, which is unitary on a subspace. On K_0^-/K_0^+, H_0 is equivalent to the representation D_{-1}^- in the space of solutions to $\mathbf{D}^t\varphi = 0$. Again there is a subspace on which H_0 is unitary. Finally, there is also a subspace of $\mathcal{O}(2n)/K_0^- = (\mathcal{O}(2n)/K_0^+)/(K_0^-/K_0^+)$, where H_0 is unitary, because H_0 there is equivalent to D_0^+.

As for the case $-l < 0$, it follows from (4.7) that any operator of the form

$$\mathbf{E}_{l,\delta_1,\delta_2} = \begin{bmatrix} 0 & \delta_2 \det \mathbf{D}^{l-1}c(\mathbf{D}) \\ \delta_1 \det \mathbf{D}^l\mathbf{D}^t & 0 \end{bmatrix} \tag{5.10}$$

has the property that

$$\mathbf{E}_{l,\delta_1,\delta_2}\begin{pmatrix} D_{-l-1}^- & 0 \\ 0 & D_{-l}^+ \end{pmatrix} = \begin{pmatrix} D_{l-1}^- & 0 \\ 0 & D_l^+ \end{pmatrix}\mathbf{E}_{l,\delta_1,\delta_2}. \tag{5.11}$$

Hence, if we define, for any complex number α,

$$\begin{aligned} \mathbf{D}_{l,\alpha} &= \begin{pmatrix} \det \mathbf{D}^l & [\det \mathbf{D}^l, z] \\ \alpha \det \mathbf{D}^l\mathbf{D}^t & \det \mathbf{D}^l \end{pmatrix} \\ &= \begin{pmatrix} \det \mathbf{D}^l & l \det \mathbf{D}^{l-1}c(\mathbf{D}) \\ \alpha \det \mathbf{D}^l\mathbf{D}^t & \det \mathbf{D}^l \end{pmatrix}, \end{aligned} \tag{5.12}$$

then we get from (5.8) and (5.9):

Proposition 5.13. *For any nonnegative integer l, and for all g in G*

$$\mathbf{D}_{l,\alpha}H_{-l}(g) = H_l(g)\mathbf{D}_{l,\alpha}.$$

Remark. $\mathbf{D}_{l,0} = (\mathbf{D}_{1,0})^l$.

The trivial fact, that $\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ commutes with the representation

$$\begin{pmatrix} D_{l-1}^- & 0 \\ 0 & D_l^+ \end{pmatrix}$$

for any pair of complex numbers γ_1 and γ_2, is, by (5.8), translated into

Corollary 5.14. *Let $M_{x,l} = \begin{bmatrix} x & 0 \\ \mathbf{D}^t & x-l \end{bmatrix}$, where x is an arbitrary complex number. Then*

$$H_l(g)M_{x,l} = M_{x,l}H_l(g)$$

for any integer l and for all g in G.

We observe that, either by using (5.8) to find the Hilbert space structure K_l that makes H_l unitary for $l \geqslant 1$, or directly from (2.1) and (2.2), it follows that if we define, for z and w in $\mathfrak{D}$ and β in $\mathbf{C}$,

$$F_{l,\beta}(z, w) = \begin{bmatrix} (z - w^*)/2i & 1/2i \\ -1/2i & \beta((z - w^*)/2i)^{-1} \end{bmatrix} \det((z - w^*)/2i)^{-(l+n)}, \tag{5.15}$$

and if $J_l(g, z)$ is the automorphic factor for which

$$(H_l(g)f)(z) = J_l(g^{-1}, z)^{-1} f(g^{-1}z),$$

then

PROPOSITION 5.16. $F_{l,\beta}(gz, gw) = J_l(g, z)F_{l,\beta}(z, w)J_l(g, w)^*$.

From (4.8) and (2.15) it finally follows that there are constants $K_{l,n}$ such that, for $l \geqslant 1$

$$F_{l,\beta}(z, w) = K_{l,n}\int_{C^+} \begin{bmatrix} (l/2)\cdot k^{-1} & 1/2i \\ -1/2i & (2\cdot\beta)\cdot k/(l+n) \end{bmatrix} \det k^l e^{i\,\mathrm{tr}(z-w^*)k}\, dk \tag{5.17}$$

and hence, since

$$\begin{bmatrix} (l/2)k^{-1} & 1/2i \\ -1/2i & (2\cdot\beta)\cdot k/(l+n) \end{bmatrix}$$

is a positive operator (when $k \in C^+$) if and only if $\beta > (n + l)/4l$, we get, using (1.8),

PROPOSITION 5.18. *For $l \geqslant 1$ and $\beta > (n + l)/4l$, $F_{l,\beta}$ is the reproducing kernel for the representation H_l in K_l.*

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