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# SOME METRIC PROPERTIES OF PIECEWISE MONOTONIC MAPPINGS OF THE UNIT INTERVAL

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ABSTRACT. In this note, the result of Lasota and Yorke on the existence of invariant measures for piecewise $C^2$ functions is extended to a larger class of piecewise continuous functions. Also the result of Li and Yorke on the existence of ergodic measures for piecewise $C^2$ functions is extended for the above class of functions.

**Introduction.** In a joint paper by Lasota and Yorke [7], they proved the existence of absolutely continuous invariant measures for $\tau$ a piecewise $C^2$ mapping of the unit interval with $\inf_{x\in[0,1]}|\tau'(x)| > 1$. Later Li and Yorke [8] proved the existence of ergodic measures for the same type of mapping. In this note, it will be shown that these results can be extended to a certain class of piecewise $C^1$ mappings. Also at the end of this note, an attempt will be made to explain why there has been a sudden interest in studying piecewise monotonic mappings.

**Existence of invariant and of ergodic measures.** Denote by $(L_1, \|\cdot\|_1)$, the space of all functions $f$ defined on [0, 1] for which $|f|$ is integrable, and by $m$ Lebesgue measure on [0, 1]. Let $\tau$: $[0, 1]\mapsto[0, 1]$ be a measurable nonsingular transformation, i.e., if $A$ is measurable, $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. Given $\tau$, the Frobenius-Perron operator $P_\tau$: $L_1 \mapsto L_1$ is given by the formula:

$$P_\tau f(x) = \frac{d}{dx}\int_{\tau^{-1}[0,x]} f(s)\,ds.$$

The operator $P_\tau$ is linear, continuous, and satisfies the following conditions:

(a) $P_\tau$ is positive: $f \geqslant 0 \Rightarrow P_\tau f \geqslant 0$;

(b) $P_\tau$ preserves integrals:

$$\int_0^1 P_\tau f\,dm = \int_0^1 f\,dm, \qquad f \in L_1;$$

(c) $P_{(\tau^n)} = P_\tau^n$ where $\tau^n$ denotes the $n$th iterate of $\tau$;

(d) $P_\tau f = f$ iff the measure $d\mu = f\,dm$ is invariant under $\tau$, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for each measurable $A$.

DEFINITION 1. A transformation $\tau$: $[0, 1]\mapsto R$ will be called *piecewise*

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*nearly*-$C^1$ if there is a partition of $[0, 1]$, $\mathcal{P} = \{a_i\colon 0 = a_0 < a_1 < \cdots < a_r = 1\}$ so that for each $i = 1, \ldots, r$, $\tau_i$ the restriction of $\tau$ to the open interval $(a_{i-1}, a_i)$ is a $C^1$ function.

THEOREM 1. *Let* $\tau\colon [0, 1] \mapsto [0, 1]$ *be a piecewise nearly*-$C^1$ *function with* $\inf_{x \in [0,1]}|\tau'(x)| > 1$ *and* $\mathcal{P} = \{a_i\colon 0 = a_0 < \cdots < a_r = 1\}$ *be a partition satisfying the above definition. Suppose for each* $i$, $i = 1, \ldots, r$, $1/|\tau_i'|$ *is of bounded variation on* $[a_{i-1}, a_i]$. *Then, for any* $f \in L_1$, *the sequence* $\{n^{-1}\sum_{k=0}^{n-1} P_\tau^k f\}_{n \geq 1}$ *with* $P_\tau^0 f \equiv f$ *is convergent in* $\|\cdot\|_1$ *to a function* $f^* \in L_1$. *The limit function has the following properties*:

(1) $f \geq 0 \Rightarrow f^* \geq 0$;

(2) $\int_0^1 f^*\, dm = \int_0^1 f\, dm$;

(3) $P_\tau f^* = f^*$ *and consequently the measure* $d\mu^* = f^*\, dm$ *is invariant under* $\tau$;

(4) $f^*$ *is of bounded variation*; *moreover, there exists a constant c independent of the choice of the initial f such that the variation of the limit* $f^*$ *satisfies the inequality*:

$$V_0^1 f^* \leq c\|f\|_1$$

*where* $V_a^b f$, *or* $V_{[a,b]} f$, *denotes the variation of f over the interval* $[a, b]$.

PROOF. Write $s = \inf|\tau'| > 1$ and choose $N$ so that $s^N > 3$. If $\phi = \tau^N$, $\phi$ is piecewise nearly-$C^1$. Let $\phi_i = \phi|_{[b_{i-1}, b_i]}$ where $\mathcal{P}_\phi = \{b_i;\ 0 = b_0 < b_1 < \cdots < b_q = 1\}$ a partition of $[0, 1]$ so that $\phi$ is piecewise nearly-$C^1$. Since $s > 1$, $\phi_i'$ can be undefined only at one end point of $I_i = [b_{i-1}, b_i]$. For simplicity, one can assume that points of difficulty, or singularities, are the right end points in all cases. Because $1/|\tau_i'|$ is of bounded variation for $j = 1, \ldots, r$, $1/|\phi_i'|$ is of bounded variation over $I_i$ for $i = 1, \ldots, q$.

Notice that, if $b_i$ is a singularity, i.e., $|\phi_i'(b_i)| = \infty$, then $1/|\phi_i'(b_i)| = 0$. As $1/|\phi_i'| \leq s^{-N}$ and $1/|\phi_i'|$ is defined on $I_i$, $i = 1, \ldots, q$, $1/|\phi_i'|$ is continuous on $I_i$, in fact uniformly continuous. Now one may use the fact (see [11]): If $g$ is of bounded variation over $[a, b]$ and continuous on $[a, b]$, then $V_a^x g$ is continuous on $[a, b]$. Thus it is possible to partition each $I_i$ into a finite number of subintervals $I_{i1}, I_{i2}, \ldots, I_{ij(i)}$ so that $V_{I_{ik}} 1/|\phi_i'| \leq s^{-N}$ for $k = 1, \ldots, j(i)$. One can suppose that the partition of $\phi$ already satisfies the above condition; otherwise one can simply take the subintervals and form a new partition for $\phi$ keeping $\phi$ piecewise nearly-$C^1$ without increasing the number of singularities.

Computing the Frobenius-Perron operator, one obtains

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x) \tag{A}$$

where $\psi_i = \phi_i^{-1}$, the inverse function of $\phi_i$, $\sigma_i(x) = |\psi_i'(x)|$, and $\chi_i$ the indicator function of $J_i = \phi_i([b_{i-1}, b_i])$. Because $1/|\phi_i'(x)| \leq s^{-N}$, $x \in I_i$, $i =$

$1, \ldots, q$, one has

$$|\sigma_i(x)| \leq s^{-N}, \qquad x \in J_i,\ i = 1, \ldots, q. \tag{B}$$

By definition, $P_\phi: L_1 \mapsto L_1$, but (A) enables one to consider $P_\phi$ as a map from the space of functions defined on [0, 1] into itself. Let $f$ be a given function of bounded variation over [0, 1]. From (A) and (B),

$$V_0^1 P_\phi f \leq \sum_{i=1}^{q} V_{J_i}(f \circ \psi_i)\sigma_i + s^{-N} \sum_{i=1}^{q} \left(|f(b_{i-1})| + |f(b_i)|\right).$$

Let $y_k = \phi_i(x_k)$ where $x_k \in I_i$. "$\sup_{J_i}$" and "$\sup_{I_i}$" will indicate the suprema taken over all finite partitions of $J_i$ and of $I_i$, resp.

$$\begin{aligned} V_{J_i}(f \circ \psi_i)\sigma_i &= \sup_{J_i} \sum_{k=1}^{n} |(f \circ \psi_i)(y_k)\sigma_i(y_k) - (f \circ \psi_i)(y_{k-1})\sigma_i(y_{k-1})| \\ &\leq \sup_{J_i} \sum_{k=1}^{n} |(f \circ \psi_i)(y_k)[\sigma_i(y_k) - \sigma_i(y_{k-1})]| \\ &\quad + \sup_{J_i} \sum_{k=1}^{n} |(f \circ \psi_i)(y_k) - (f \circ \psi_i)(y_{k-1})|\, |\sigma_i(y_{k-1})| \\ &\leq \sup_{I_i} \sum_{k=1}^{n} |f(x_k)| \left| \frac{1}{|\phi_i'(x_k)|} - \frac{1}{|\phi_i'(x_{k-1})|} \right| + s^{-N} V_{I_i} f. \end{aligned}$$

If $|d_i| = \inf_{I_i} |f(x)|$, then

$$\begin{aligned} \sum_{k=1}^{n} |f(x_k)| \left| \frac{1}{|\phi_i'(x_k)|} - \frac{1}{|\phi_i'(x_{k-1})|} \right| &\leq \sum_{i=1}^{n} |f(x_k)| V_{x_{k-1}}^{x_k} \frac{1}{|\phi_i'|} \\ &\leq \sum_{k=1}^{n} \left(|f(x_k) - d_i| + |d_i|\right) V_{x_{k-1}}^{x_k} \frac{1}{|\phi_i'|} \leq \sum_{k=1}^{n} \left(V_{I_i} f + |d_i|\right) V_{x_{k-1}}^{x_k} \frac{1}{|\phi_i'|} \\ &\leq \left(V_{I_i} f + \frac{1}{h} \int_{I_i} |f|\, dm\right) V_{I_i} \frac{1}{|\phi_i|} \leq s^{-N} \left(V_{I_i} f + \frac{1}{h} \int_{I_i} |f|\, dm\right) \end{aligned}$$

where $h = \min_i (b_i - b_{i-1})$. Thus

$$V_{J_i}(f \circ \psi_i)\sigma_i \leq 2s^{-N} V_{I_i} f + \frac{s^{-N}}{h} \int_{I_i} |f|\, dm.$$

By the same reasoning as in the Lasota-Yorke paper [7],

$$\sum_{i=1}^{q} \left(|f(b_{i-1})| + |f(b_i)|\right) \leq V_0^1 f + \frac{2}{h} \|f\|_1.$$

Consequently,

$$V_0^1 P_\phi f \leq \sum_{i=1}^{q}\left(2s^{-N}V_{I_i}f + \frac{s^{-N}}{h}\int_{I_i}|f|\,dm\right) + s^{-N}\left(V_0^1 f + \frac{2}{h}\|f\|_1\right)$$

$$= 2s^{-N}V_0^1 f + \frac{s^{-N}}{h}\|f\|_1 + s^{-N}V_0^1 f + \frac{2s^{-N}}{h}\|f\|_1$$

$$= \frac{3s^{-N}}{h}\|f\|_1 + 3s^{-N}V_0^1 f$$

$$= \alpha\|f\|_1 + \beta V_0^1 f \quad \text{where } \alpha = 3s^{-N}/h \text{ and } \beta = 3s^{-N} < 1$$

with this estimate, the remainder of the argument is identical to the one used in the Lasota-Yorke note.

DEFINITION 2. A transformation $\tau$: $[0, 1] \mapsto R$ will be called *piecewise nearly-$C^2$* if there is a partition $\mathcal{P} = \{a_i: 0 = a_0 < a_1 < \cdots < a_r = 1\}$ of $[0, 1]$ so that for each $i = 1, \ldots, r$, $\tau_i = \tau_{|(a_{i-1}, a_i)}$ is a $C^2$ function.

Let $\tau$ and $\mathcal{P}$ be as in the above definition and let $S_\tau \subseteq \{0, 1, \ldots, r\}$ be those $i$'s for which $a_i \in \mathcal{P}$ and $\tau_i$ cannot be extended as a $C^2$ function to $a_i$.

COROLLARY. *Let* $\tau$: $[0, 1] \mapsto [0, 1]$ *be a piecewise nearly-$C^2$ function such that* $\inf|\tau'| > 1$. *Suppose for* $i \in S_\tau$, $1/|\tau_i'|$ *is of bounded variation over the closed interval of* $\mathcal{P}$ *containing* $a_i$. *Then the conclusions of the theorem are true.*

PROOF. One simply observes that, for $i \notin S_\tau$, $1/|\tau_i'|$ is also of bounded variation.

REMARK. Clearly, if $S_\tau = \varnothing$, then one has the result of Lasota and Yorke.

With the existence of invariant measures shown, one can now turn his attention to the matter of the existence of ergodic measures as shown for piecewise $C^2$ mappings by Li and Yorke. Let $\tau$: $[0, 1] \mapsto [0, 1]$ be a piecewise nearly-$C^1$ function satisfying the theorem. Denote by $\mathcal{C}$ the set $\{x \in [0, 1]: \tau'(x) \text{ exists}\}$ and let $\{x_1, \ldots, x_k\} = [0, 1]/\mathcal{C}$, i.e., the points of discontinuity for $\tau$ and $\tau'$.

DEFINITION 3 [**8**]. $f \in L_1$ is of *bounded variation* in $L_1$ if $f$ equals a.e.-$m$ some function of bounded variation.

It has been shown that the invariant functions of $P_\tau$ exist, and each is of bounded variation in $L_1$. Let $F = \{f \in L_1: f \text{ is invariant under } \tau\}$. Each $f \in F$ represents a class of functions which are equal a.e.-$m$ to a function of bounded variation.

THEOREM 2. *There exist a finite collection of sets* $I_1, \ldots, I_n$ *and a set of functions* $\{f_1, \ldots, f_n\} \subset F$ *for which*

(1) *each* $I_i$, $i = 1, \ldots, n$, *is a finite union of closed intervals*;

(2) $I_i \cap I_j$ *contains at most a finite number of points when* $i \neq j$.

(3) *each* $I_i$ *contains at least one point of discontinuity* $x_j$, $j = 1, \ldots, k$, *in its interior*; *hence* $n \leq k$.

(4) *$f_i(x) = 0$ for $x \notin I_i$, $i = 1, \ldots, n$, and $f_i(x) > 0$ for almost all $x \in I_i$;*
(5) *$\int_{I_i} f\, dm = 1$ for all $i$;*
(6) *if $g \in F$ satisfies* (4) *and* (5) *for some $i$, then $g = f$; a.e.-$m$;*
(7) *every $f \in F$ can be written as $f = \sum_{i=1}^{n} a_i f_i$ with suitably chosen $\{a_i\}$.*

PROOF. The proof is exactly the same as the one in the Li-Yorke paper for the piecewise-$C^2$ case because the essential facts needed in that case were
(1) $\inf_{x \in \mathcal{Q}} |\tau'(x)| > 1$,
(2) $\tau$ is piecewise continuous,
(3) the invariant functions of $\tau$ are equal a.e.-$m$ to a function of bounded variation.

DEFINITION 4. For $x \in [0, 1]$, let $\Lambda(x) = \bigcap_{N \geq 1} \overline{\{\tau^n(x)\}}_{n \geq N}$, i.e., the set of limit points of $\tau^n(x)$.

THEOREM 3. *For almost every $x \in [0, 1]$, $\Lambda(x) = I_i$ for some $i = 1, \ldots, n$ where the $I_i$'s are the sets in Theorem* 2.

PROOF. The proof is identical to the one in the Li-Yorke paper.

The above two theorems imply the existence of ergodic measures for the piecewise nearly-$C^1$ functions satisfying Theorem 1 as found in the Li-Yorke paper.

REMARKS 1. Recently Rufus Bowen **[1]** has shown sufficient conditions for which piecewise $C^2$ functions give rise to a Bernoulli dynamical system. If $\tau$ happens to be piecewise[1] $C^{1+\delta}$ where $\delta \in [0, 1]$ satisfying Theorem 1 and the condition in Bowen's paper, then it is straightforward to prove $\tau$ gives rise to a Bernoulli dynamical system.

2. Other proofs of the existences of invariant and of ergodic measures for piecewise $C^2$ mappings are found in **[2]** and in **[4]**. The proofs for $\delta = 1$ are found in **[5]**.

**Piecewise monotonic transformations and the Lorenz attractor.** The interest in transformations of the unit interval into itself, particularly piecewise monotonic transformations, has been stimulated by a relationship between what appears to be a piecewise monotonic transformation of the interval into itself and an object first observed by Edward Lorenz **[9]**. The object has come to be known as the "Lorenz attractor". What makes this attractor of interest is that it comes from the study of a physically-existing dynamical system, namely atmospheric convection. Moreover the Lorenz attractor is not an "Axiom-A" attractor (see e.g. **[12]** for definition of Axiom-A) which has been extensively studied in the past in dynamical systems.

An outline of how a transformation of the unit interval can be derived

[1]By a $C^{1+\delta}$ function for $\delta \in [0, 1]$, one means a function whose derivative is Hölder with exponent $\delta$. When $\delta = 1$, the derivative is then called Lipschitz.

from the attractor will be given now. (For full details of the derivation see [6].)

Consider the square $R = \{(x, y, z): |x| < 6\sqrt{2}, |y| < 6\sqrt{2}, z = 27\}$ which contains two of the three stationary solutions $q_-$, $q_+$ of the Lorenz equations (see Figure 1). Since $R$ is a cross-section for the attractor, $C_1$ and $C_2$ represent the intersection of the attractor with $R$. Consider now a very narrow neighborhood of $C_1 \cup C_2$ in $R$ (see Figure 2).

$$\dot{x} = -10x + 10y$$
$$\dot{y} = -xz + 28x - y$$
$$\dot{z} = xy - \frac{8}{3} z$$

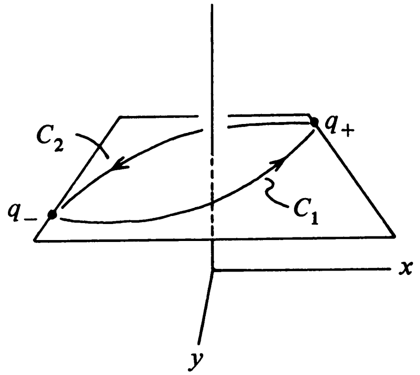


FIGURE 1

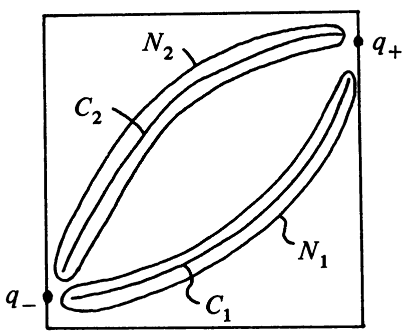


FIGURE 2

One now determines the image of $R$ under the Poincaré map of the flow determined by the Lorenz equations. Upon doing this, one gets Figure 3 for the neighborhood $N_1 \cup N_2$ and its image. Actually one of the sets contained in $N_1$ comes from the Poincaré map of $N_2$, but, by the symmetry of the Lorenz equation, $N_2$ will contain a set from the image of $N_1$. Therefore for simplicity assume the sets in $N_1$ form the image of $N_1$ under the Poincaré map.

Suppose that the strong stable manifolds of $C_1 \cup C_2$ foliate the neighborhood $N_1 \cup N_2$. In $N_1 \cup N_2$, define the equivalence relation $z_1 \sim z_2$ iff $z_1$ and $z_2$ belong to the same stable manifold. This equivalance relation is the canonical "collapsing" map. With this map, one can construct a mapping of the interval into itself by looking at, say, the image of $C_1$ under the Poincaré map and assuming the two sets in $N_1$ form the image of $N_1$. Doing this, one obtains a map, say $f$. (See Figure 4.) At the point $s$, one has a vertical tangency.

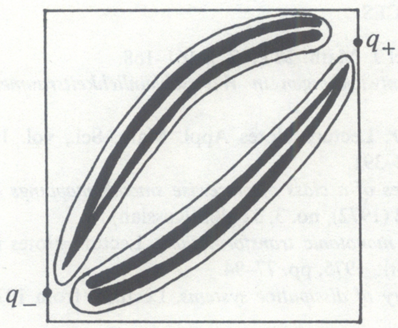


FIGURE 3

(The neighborhood has been exaggerated to permit better detail of image of $N_1 \cup N_2$)

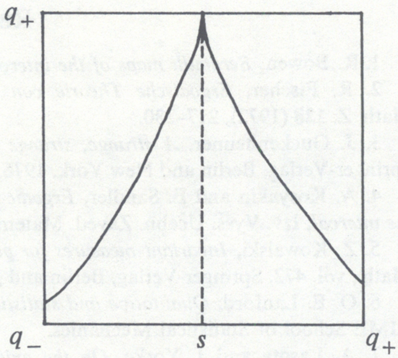


FIGURE 4

From conversations with Oscar Lanford, $f$ is at least piecewise $C^{1+\delta}$ with a vertical tangency at $s$. In fact, Lanford conjectures that $f$ is piecewise $C^2$ with the singularity at $s$. Using methods from statistical mechanics, he has been able to show, without knowing what type of piecewise continuous function is $f$, that the "natural" extension [**10**] of the dynamical system associated to the Poincaré map $f$ is a Kolmogorov automorphism [**10**]; hence $f$ possesses an ergodic measure and thus an invariant measure. As mentioned above Rufus Bowen has shown sufficient conditions under which the "natural" extension for a dynamical system associated to a piecewise $C^2$ mapping is measure isomorphic to a Bernoulli automorphism. However it is not known whether or not the "natural" extension associated to $f$ is measure isomorphic to a Bernoulli automorphism which in some sense is a "well-behaved" automorphism.

Knowing the existence of an invariant measure for $f$ leads one to ask what statistical properties does the system possess. For example, does a central limit theorem for Hölder functions exist?; does a law of the iterated logarithm for Hölder functions hold?; does the invariant measure mix measurable sets uniformly in some sense? These questions are all unanswered at this time. In fact all the statements about the actual Poincaré map for the Lorenz attractor are mere conjectures because the graph of $f$ is derived by making certain assumptions about the behavior of the attractor, and the assumptions are still under investigation by J. Guckenheimer (see e.g. [**3**]), O. E. Lanford [**6**], R. Williams (see e.g. [**13**]), and others in dynamical systems.

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