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# SEMI-ALGEBRAIC GROUPS AND THE LOCAL CLOSURE OF AN ORBIT IN A HOMOGENEOUS SPACE

BY

MORIKUNI GOTO

ABSTRACT. Let $L$ be a topological group acting on a locally compact Hausdorff space $M$ as a transformation group. Let $m$ be in $M$. A subset $Q$ of $M$ is called the *local closure* of the orbit $Lm$ if $Q$ is the smallest locally compact invariant subset of $M$ with $m \in Q$. A partition

$$M = \bigcup_{\lambda \in \Lambda} Q_\lambda, \quad Q_\lambda \cap Q_\mu = \varnothing \quad (\lambda \neq \mu)$$

is called an LC-*partition* of $M$ with respect to the $L$ action if each $Q_\lambda$ is the local closure of $Lm$ for any $m$ in $Q_\lambda$.

THEOREM. *Let $G$ be a connected Lie group, and let $A$ and $B$ be subgroups of $G$ with only finitely many connected components. Suppose that $B$ is closed. Then the factor space $G/B$ has an LC-partition with respect to the $A$ action.*

**1. Introduction.** Let $G$ be a locally compact topological group, and let $A$ and $B$ be subgroups of $G$. A subset $P$ of $G$ is said to be $(A, B)$-*invariant* if $APB = \{apb;\ a \in A,\ p \in P,\ b \in B\} = P$. In this case the direct product topological group $A \times B$ acts on the underlying space of $G$ as a transformation group by

$$(a, b)g = agb^{-1} \quad \text{for } g \in G, \qquad (a, b) \in A \times B,$$

and $P$ is $(A, B)$-invariant if and only if $P$ is invariant under $A \times B$. For $g$ in $G$, the double coset $AgB$ is the orbit of the transformation group, passing through $g$. In this setting, we shall give some definitions.

DEFINITION 1. Let $g$ be in $G$, and suppose that there exists a minimal, locally compact, $(A, B)$-invariant subset $P$ containing $g$. Then the set $P$ is said to be the *local closure* of the double coset $AgB$.

Since the intersection of two locally compact sets is locally compact, the local closure of $AgB$ is unique, if it exists.

DEFINITION 2. Suppose that $G$ has a partition

$$G = \bigcup_{\lambda \in \Lambda} P_\lambda, \quad P_\lambda \cap P_\mu = \varnothing \qquad (\lambda \neq \mu),$$

where $\Lambda$ is a set of indices, such that for each $\lambda \in \Lambda$, the set $P_\lambda \neq \varnothing$ is the

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local closure of $AgB$ for every $g$ in $P_\lambda$. Then the partition (clearly unique) is called the LC-*partition* of $G$ with respect to the pair $(A, B)$.

Throughout the paper, the identity element of a group in question is always denoted by $e$; for a topological group $L$, the identity component, i.e. the connected component containing $e$, will be denoted by $L_e$.

A topological group $L$ is said to be *compactly*, *finitely*, or *countably connected* if the factor group $L/L_e$ is compact, finite, or countable, respectively.

The following theorem will be established in this paper:

THEOREM I. *Let $G$ be a connected Lie group, and let $A$ and $B$ be finitely connected subgroups of $G$. Then $G$ has an LC-partition with respect to $(A, B)$.*

Next, we shall consider transformation groups.

DEFINITION 3. Let $L$ be a topological group acting on a locally compact Hausdorff space $M$ as a transformation group. Let $m$ be in $M$. A subset $Q$ of $M$ is called the *local closure* of the orbit $Lm$ if $Q$ is the smallest locally compact, invariant subset of $M$ with $m \in Q$. Also, a partition

$$M = \bigcup_{\lambda \in \Lambda} Q_\lambda, \quad Q_\lambda \cap Q_\mu = \emptyset \qquad (\lambda \neq \mu),$$

is called the LC-*partition*, of $M$ with respect to the $L$ action, if each $Q_\lambda$ is the local closure of $Lm$ for any $m$ in $Q_\lambda$.

Let $G$ be a locally compact group, and $B$ a closed subgroup of $G$. Let $M = G/B = \{gB;\ g \in G\}$ denote the factor space, and $\pi\colon G \ni g \mapsto gB \in M$ the natural map. Let $A$ be a subgroup of $G$. Then $A$ acts on the homogeneous space $M$ by $a\pi(g) = \pi(ag)$ for $a \in A$ and $g \in G$. Now, it is easy to see that for any $g \in G$, a subset $Q$ of $M$ is the local closure of the orbit $A\pi(g) = \pi(Ag)$ if and only if $\pi^{-1}Q$ is the local closure of the double coset $AgB$. Therefore from Theorem I we have

THEOREM I′. *Let $G$ be a connected Lie group, and let $A$ and $B$ be finitely connected subgroups. Suppose that $B$ is closed, and let $M = G/B$ be the factor space. Then $M$ has an LC-partition with respect to the $A$ action.*

Throughout the paper, for a subset $X$ of a topological space, we let $\overline{X}$ denote the closure of $X$.

Let $L$ be a topological group, and let $A$ and $B$ be subgroups of $L$. If a locally compact subset $P$ of $L$ is $(A, B)$-invariant, then $P$ is $(\overline{A}, \overline{B})$-invariant, see (2.1) below. Also if $B$ is finitely connected, then so is $\overline{B}$; see (2.3). Therefore Theorem I′ implies Theorem I, i.e. *Theorem* I *and Theorem* I′ *are equivalent to each other*.

Next, in Theorem I′ we let $A$ be an abelian group and $B$ a connected group. In this case, each local closure has an extremely simple topological structure.

THEOREM II. *Let $G$ be a connected Lie group. Let $A$ be a finitely connected, abelian subgroup and $B$ a closed connected subgroup of $G$. Then the local closure of any orbit of the transformation group $A$ acting on $M = G/B$ is homeomorphic with the underlying space of a certain abelian Lie group.*

*If, in particular, $A$ is a one-parameter subgroup, then for any $m \in M$, the local closure of the orbit $Am$ is homeomorphic with either the real line* $\mathbf{R}$ *or a toral group.*

In order to explain the outline of our proof, we shall introduce a notation of an LC-family of subgroups. When a topological space $X$ is a union of countably many compact subsets, $X$ is said to be *$\sigma$-compact*.

DEFINITION 4. Let $G$ be a locally compact, $\sigma$-compact topological group. A set $\mathfrak{F}$ of subgroups of $G$ is said to form an LC-*family* of $G$ if the following conditions are satisfied:

(1) Any member of $\mathfrak{F}$ is closed.

(2) $G \in \mathfrak{F}$, and for any nonempty subset $\mathfrak{F}'$ of $\mathfrak{F}$, the intersection of all members of $\mathfrak{F}'$ is contained in $\mathfrak{F}$.

By (2), to any subgroup $H$ of $G$, we can associate the *$\mathfrak{F}$-hull* $\mathfrak{F}(H)$, which is the smallest member of $\mathfrak{F}$ containing $H$. By (1), $\mathfrak{F}(\overline{H}) = \mathfrak{F}(H)$.

(3) If $F \in \mathfrak{F}$ and $g \in G$, then $gFg^{-1} \in \mathfrak{F}$.

(4) For $F_1$ and $F_2$ in $\mathfrak{F}$, the double coset $F_1F_2$ is a locally compact set.

(5) Suppose $F_1$ and $F_2$ are closed subgroups of $G$ such that $F_1 \supset F_2$ and the factor space $F_1/F_2$ is compact and totally disconnected. If one of $F_1$ and $F_2$ is in $\mathfrak{F}$, then so is the other one.

(6) If $H$ is a closed connected subgroup of $G$, then $\mathfrak{F}(H)$ is connected, and the commutator subgroup of $H$ coincides with the commutator subgroup of $\mathfrak{F}(H)$.

(7) If $H$ is a compactly connected, abelian subgroup of $G$, then $\mathfrak{F}(H)$ is abelian.

Suppose that $G$ has an LC-family $\mathfrak{F}$. Let $A$ and $B$ be subgroups of $G$, and put $A^* = \mathfrak{F}(A)$ and $B^* = \mathfrak{F}(B)$. By (3) and (4), every double coset $A^*gB^*$ is locally compact, and $G = \bigcup A^*gB^*$ gives a partition of $G$ into locally compact, $(A, B)$-invariant subsets. Hence we only have to study the action of $A \times B$ in each $A^*gB^*$. On the other hand, since $A^* \times B^*$ is a locally compact, $\sigma$-compact group, the map

$$\xi_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} \in A^*gB^*$$

is (continuous and) open, see (2.4), and induces a homeomorphism between the factor space $A^* \times B^*/D(g)$ and $A^*gB^*$, where $D(g)$ is the isotropy subgroup of $A^* \times B^*$ at $g$. We see easily that a subset $P$ of $A^*gB^*$ is locally compact if and only if $\xi_g^{-1}P$ is so, and $P$ is $(A, B)$-invariant if and only if $\xi_g^{-1}P$ is $(A \times B, D(g))$-invariant. Thus the existence of an LC-partition of $G$ with respect to $(A, B)$, is reduced to the existence of an LC-partition of $A^* \times B^*$ with respect to $(A \times B, D(g))$ for every $g \in G$.

Next, let us consider the double coset decomposition

$$A^* \times B^* = \bigcup (A \times B)xD(g).$$

When $A$ and $B$ are closed connected subgroups, $A \times B$ is a normal subgroup of $A^* \times B^*$, and $(A \times B)D(g)$ is a group. Therefore each double coset is a coset. It is easy to see that

$$A^* \times B^* = \bigcup x\,\overline{(A \times B)D(g)}$$

gives the LC-partition. When $A$ and $B$ are compactly connected or when $A$ is abelian, etc., we can modify the connected case and can study the $(A \times B) \times D(g)$ action on $A^* \times B^*$ in details using (5), (6) and (7). The results following from the existence of an LC-family are given in Theorem III, §2.

Now the problem boils down to the following:

Which groups have an LC-family?

First we consider $\mathrm{GL}(n, \mathbf{R})$. Let $\mathcal{Q}^*$ denote the set of all algebraic groups in $\mathrm{GL}(n, \mathbf{R})$. $\mathcal{Q}^*$ is almost an LC-family, but does not satisfy (5), (6). We shall call a subgroup $A$ of $\mathrm{GL}(n, \mathbf{R})$ *pre-algebraic* if $A$ is an open subgroup of a suitable algebraic group. Let $\mathcal{Q}$ denote the set of all pre-algebraic groups in $\mathrm{GL}(n, \mathbf{R})$. Then $\mathcal{Q}$ is known to be an LC-family. For a subgroup $H$ of $\mathrm{GL}(n, \mathbf{R})$, let $\mathcal{Q}(H)$ denote the pre-algebraic hull ($= \mathcal{Q}$-hull) of $H$.

A notion of (connected) semi-algebraic groups was introduced by the author in [**1**]. Here we shall extend it to nonconnected groups. A closed subgroup $S$ of $\mathrm{GL}(n, \mathbf{R})$ is said to be *semi-algebraic* if the factor space $\mathcal{Q}(S)/S$ is homeomorphic with a euclidean space $\mathbf{R}^k$. In §3, we shall see that the set $\mathcal{S}$ of all semi-algebraic groups forms an LC-family.

Let $G$ be a connected Lie group, and $\mathcal{G}$ its Lie algebra. Then for $g \in G$, the inner automorphism $G \ni x \mapsto gxg^{-1} \in G$ gives rise to an automorphism $\mathrm{Ad}(g)$ of $\mathcal{G}$, and we have a representation (the adjoint representation)

$$G \ni g \mapsto \mathrm{Ad}(g) \in \mathrm{GL}(\mathcal{G}).$$

If the adjoint group $\mathrm{Ad}(G)$ is semi-algebraic, $G$ is said to be *adjoint semi-algebraic*. In [**1**], the author proved that for any connected Lie group $G$, there exists an adjoint semi-algebraic group $S$ containing $G$ as a closed normal subgroup. Then by the nature of our problems, it suffices to consider only adjoint semi-algebraic groups instead of considering general connected Lie groups.

Let $G$ be an adjoint semi-algebraic group. We shall denote $\rho(g) = \mathrm{Ad}(g)$

for $g \in G$. A subgroup $H$ of $G$ is called an sa-*group* if

(1) $\rho(H)$ is semi-algebraic, and

(2) $H$ is an open subgroup of $\rho^{-1}\rho(H)$.

In §4, we shall see that the set of sa-groups in $G$ forms an LC-family.

Special cases of our results have been studied in Pukanszky [6] and Goto [1]. In particular, the last part of Theorem II was given in [1], in a slightly weaker form.

**2. Locally compact groups and LC-families of subgroups.** First let us recall some results on locally compact Hausdorff spaces and locally compact groups (= locally compact topological groups).

Let $M$ be a Hausdorff space, and let $Q$ be a subset of $M$. If $Q$ is locally compact, then $Q$ is the intersection of a closed set and an open set. If, in particular, $M$ is locally compact, then the converse is also true. Notice that the intersection, but not the union, of finitely many locally compact sets is again locally compact.

Let $G$ be a locally compact group. Then a subgroup $H$ of $G$ is closed if and only if $H$ is locally compact. Suppose that $H$ is a closed subgroup of $G$. Let $M = G/H$ denote the factor space, and $\pi\colon G \ni g \mapsto gH \in M$ the natural map. Then a subset $Q$ of $M$ is open, closed, or locally compact if and only if $\pi^{-1}Q$ is so.

(2.1) Let $G$ be a topological group, and let $A$ and $B$ be subgroups of $G$. Let $P$ be a locally compact subset of $G$. If $P$ is $(A, B)$-invariant, then $P$ is $(\bar{A}, \bar{B})$-invariant.

PROOF. $AP = P$ implies $A\bar{P} = \bar{P}$, and we have that $A(\bar{P} - P) = \bar{P} - P$. But $\bar{P} - P$ is a closed set, since $P$ is locally compact. Hence $\bar{A}(\bar{P} - P) = \bar{P} - P$, and with $\bar{A}\bar{P} = \bar{P}$, we have that $\bar{A}P = P$. In a similar way, $P\bar{B} = P$. □

(2.2) Let $G$ be a locally compact group, and let $D$ and $E$ be subgroups of $G$. Suppose that $E$ is a compactly connected, closed subgroup and the identity component $E_e$ of $E$ is a normal subgroup of $G$. Then the double coset decomposition

$$G = \bigcup Eg\overline{DE_e}$$

is the LC-partition of $G$ with respect to $(E, D)$. Furthermore, each $Eg\overline{DE_e}$ is a closed subset of $G$.

PROOF. Let $\sigma\colon G \ni g \mapsto gE_e \in G/E_e$ be the natural homomorphism. We put $Eg\overline{DE_e} = P = P(g)$. Then $\sigma P = \sigma E \cdot \sigma g \cdot \overline{\sigma D}$ is closed, because $\sigma E$ is a compact group. Hence $P = \sigma^{-1}\sigma P$ is closed.

On the other hand, if $Q$ is a locally compact, $(E, D)$-invariant subset containing $g$, then $Q$ is $(E, DE_e)$-invariant, and $Q \supset P(g)$ by (2.1). Hence $P(g)$ is the local closure of $EgD$. □

(2.3) Let $G$ be a locally compact group, and $H$ a subgroup of $G$. If $H$ is compactly or finitely connected, then so is $\overline{H}$.

Proof. Suppose that $H$ is compactly connected. For each $h$ in $H$, we pick a compact neighborhood $U(h)$ of $h$ in the closure $\overline{H}$ of $H$. Then $V(h) = U(h) \cap H$ is a neighborhood of $h$ in $H$. Let $\varphi$ denote the natural homomorphism

$$H \ni h \mapsto hH_e \in H/H_e.$$

Then $\varphi(V(h))$ is a neighborhood of $hH_e$ in $H/H_e$. Since $H/H_e$ is compact, there exists a finite set $\{h_1, \ldots, h_k\} \subset H$ such that $\bigcup_{i=1}^k \varphi(V(h_i)) = H/H_e$, i.e. $\bigcup_{i=1}^k V(h_i)H_e = H$. We put $C = \bigcup_{i=1}^k U(h_i)$. Then $C$ is a compact set, and $C \cdot \overline{H_e}$ is closed. Since $\overline{H} \supset C\overline{H_e} \supset H$, we have $\overline{H} = C\overline{H_e}$. But since $\overline{H_e}$ is connected, we have $(\overline{H})_e \supset \overline{H_e}$ and $\overline{H} = C(\overline{H})_e$. Hence $\overline{H}$ is compactly connected.

Next, suppose that $H$ is finitely connected, and $H = \bigcup_{i=1}^j a_iH_e$ is the coset decomposition. Then $\overline{H} \supset \bigcup_{i=1}^j a_i\overline{H_e} \supset H$, and $\bigcup_{i=1}^j a_i\overline{H_e}$ is closed. Hence $\overline{H} = \bigcup_{i=1}^j a_i\overline{H_e}$. $\square$

Next, let $G$ be a locally compact, $\sigma$-compact group, and suppose that there exists an LC-family $\mathfrak{F}$ of subgroups of $G$, satisfying (1), ..., (7) in §1. Let $A$ and $B$ be subgroups of $G$. First we shall explain our method to study the $A \times B$ action on $G$.

We put $\mathfrak{F}(A) = A^*$ and $\mathfrak{F}(B) = B^*$. By (3) and (4), for any $g$ in $G$, we have that $A^*(gB^*g^{-1})$ is locally compact, and so is $A^*gB^*$. Hence the double coset decomposition

$$G = \bigcup_{g \in G} A^*gB^*$$

is a partition of $G$ into locally compact, $(A, B)$-invariant subsets. In order to study the $A \times B$ action on each $A^*gB^*$, we need the following known theorem, see e.g. Helgason [4].

(2.4) Let $L$ be a locally compact, $\sigma$-compact group, and let $M$ be a locally compact Hausdorff space. Suppose that $L$ acts on $M$ transitively. For $m$ in $M$, let $L_m$ denote the isotropy subgroup at $m$: $L_m = \{x \in L;\ xm = m\}$. Then the map $\xi$: $L \ni x \mapsto xm \in M$ is (continuous and) open and gives rise to a homeomorphism between the factor space $L/L_m$ and $M$.

Since $A^*$ and $B^*$ are closed subgroups of $G$, they are $\sigma$-compact, and so is $A^* \times B^*$. On the other hand, $A^* \times B^*$ acts transitively on $A^*gB^*$ and the isotropy group $D(g)$ at $g$ is given by

$$D(g) = \{(\gamma, g^{-1}\gamma g);\ \gamma \in A^* \cap gB^*g^{-1}\}.$$

By (2.4), the map

$$\xi = \xi_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} \in A^*gB^*$$

induces a homeomorphism between the factor space $(A^* \times B^*)/D(g)$ and $A^*gB^*$.

Let $P$ be a subset of $A^*gB^*$. If $P$ is locally compact, then so is $\xi^{-1}P$, and conversely. Suppose that $P$ is $(A, B)$-invariant. Then for $(\alpha, \beta) \in \xi^{-1}P$ and $(a, b) \in A \times B$, we have

$$\xi((a, b)(\alpha, \beta)) = a\alpha g\beta^{-1}b \in APB = P.$$

Hence $\xi^{-1}P$ is $(A \times B, D(g))$-invariant. Also it is easy to see that if $\xi^{-1}P$ is $(A \times B, D(g))$-invariant, then $P$ is $(A, B)$-invariant. Therefore, $\xi^{-1}$ gives a one-one correspondence between the totality of locally compact, $(A, B)$-invariant subsets of $A^*gB^*$, and the set of all locally compact, $(A \times B, D(g))$-invariant subsets of $A^* \times B^*$.

Suppose that $B$ is closed and consider the map

$$\eta = \eta_g : A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g\beta^{-1}B \in A^*gB^*/B \subset G/B = M.$$

The map $\eta$ is also continuous and open, and gives rise to a homeomorphism between the totality of double cosets $(\{e\} \times B)\backslash A^* \times B^*/D(g)$ and $A^*gB^*/B$.

Before stating the theorem, we shall prove one lemma. For a subgroup $H$ of $G$, we shall denote by $\mathfrak{F}_e(H)$ the identity component of $\mathfrak{F}(H)$.

(2.5) Let $H$ be a compactly connected, closed subgroup of $G$. Then

$$\mathfrak{F}(H) = H \cdot \mathfrak{F}(H_e), \qquad \mathfrak{F}(H_e) = \mathfrak{F}_e(H).$$

$\mathfrak{F}(H)$ is compactly connected, and $H_e$ is a normal subgroup of $\mathfrak{F}(H)$.

Proof. Let $N$ denote the normalizer of $H_e$ in $G$: $N = \{g \in G;\ gH_eg^{-1} = H_e\}$. Then $\mathfrak{F}(H_e) \subset N$ by (6). For $g \in N$, by (3) we have

$$\mathfrak{F}(H_e) = \mathfrak{F}(gH_eg^{-1}) = g\mathfrak{F}(H_e)g^{-1},$$

and accordingly $\mathfrak{F}(H_e)$ is a normal subgroup of $N$. Since $H \subset N$, we have that $H \cdot \mathfrak{F}(H_e)$ is a subgroup of $N$.

Since $H/H_e$ is compact, there exists a compact subset $C$ of $H$ such that $H = CH_e$. Hence $H \cdot \mathfrak{F}(H_e) = C \cdot \mathfrak{F}(H_e)$ is a closed subgroup, and we have

$$H \cdot \mathfrak{F}(H_e)/\mathfrak{F}(H_e) \cong H/H \cap \mathfrak{F}(H_e).$$

Since $H \cap \mathfrak{F}(H_e) \supset H_e$, the factor group $H/H \cap \mathfrak{F}(H_e)$ is compact and totally disconnected. Since $\mathfrak{F}(H_e)$ is connected by (6), we see that $H \cdot \mathfrak{F}(H_e)$ is compactly connected, and the identity component of $H \cdot \mathfrak{F}(H_e)$ is $\mathfrak{F}(H_e)$. By (5), $H \cdot \mathfrak{F}(H_e) \in \mathfrak{F}$, and we have $\mathfrak{F}(H) = H \cdot \mathfrak{F}(H_e)$. □

Theorem III. *Let $G$ be a locally compact, σ-compact group, with an LC-family $\mathfrak{F}$. Let $A$ and $B$ be subgroups of $G$.*

(a) *If $A$ and $B$ are compactly connected, then $G$ has an LC-partition with respect to $(A, B)$.*

(b) *Suppose that $A$ is a compactly connected, abelian group and $B$ is a closed, connected group. Then for each $g \in G$, we can find a locally compact abelian group $L(g)$, a closed subgroup $L'(g)$ of $L(g)$, and a homeomorphism $\varphi$ from $\mathfrak{F}(A)g\mathfrak{F}(B)/B$ onto $L(g)$, such that the image of the local closure of each orbit $Am$, $m \in \mathfrak{F}(A)g\mathfrak{F}(B)/B$, is a coset of $L'(g)$.*

(c) *In* (b), *if in particular $A$ is a one-parameter subgroup, then the local closure of an orbit $Am$ is homeomorphic either with* $\mathbf{R}$ *or a certain compact connected abelian group.*

PROOF OF (a). By (2.3), $\bar{A}$ and $\bar{B}$ are compactly connected. Hence after this we can suppose that $A$ and $B$ are closed, by (2.1). Then by (2.5), $A_e$ and $B_e$ are normal subgroups of $A^* = \mathfrak{F}(A)$ and $B^* = \mathfrak{F}(B)$, respectively. Hence $(A \times B)_e = A_e \times B_e$ is a normal subgroup of $A^* \times B^*$. Since $A \times B$ is compactly connected, we can apply (2.2), and for each $g$ in $G$,

$$A^* \times B^* = \bigcup (A \times B)x\,\overline{D(g)(A_e \times B_e)}$$

is the LC-partition of $A^* \times B^*$ with respect to $(A \times B, D(g))$. Hence

$$A^*gB^* = \bigcup \xi_g\big((A \times B)x\,\overline{D(g)(A_e \times B_e)}\big)$$

is a partition of $A^*gB^*$ into minimal, locally compact, $(A, B)$-invariant subsets. This completes the proof of (a).

PROOF OF (b) AND (c). By (7) and (6), $A^*$ and $B^*/B$ are abelian groups. Hence $(A \times B)D(g)$ is a subgroup of $A^* \times B^*$, and the coset decomposition

$$A^* \times B^* = \bigcup x\,\overline{(A \times B)D(g)}$$

gives the LC-partition of $A^* \times B^*$ with respect to the pair $(A \times B, D(g))$. Hence $G$ has an LC-partition with respect to the pair $(A, B)$.

Since $(\{e\} \times B)D(g)$ is a closed normal subgroup of $A^* \times B^*$, the set $(\{e\} \times B) \backslash A^* \times B^*/D(g)$ can be identified with the abelian group $L(g) = A^* \times B^*/(\{e\} \times B)D(g)$. Thus the map

$$\eta_g\colon A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} B \in A^*gB^*/B$$

induces a homeomorphism $\eta'_g$ from $L(g)$ onto $A^*gB^*/B$.

Let us put $\overline{(A \times B)D(g)}/(\{e\} \times B)D(g) = L'(g)$. Then for each $x \in A^* \times B^*$, we put $x^\# = x(\{e\} \times B)D(g) \in L(g)$, and get that

$$\eta'_g\big(x^\# L'(g)\big) = \eta_g\overline{\big(x(A \times B)D(g)\big)}$$

is the local closure of the orbit $A\eta_g(x)$.

If, in particular, $A$ is a one-parameter subgroup, then $(A \times B)D(g)/(\{e\} \times B)D(g)$ is a one-parameter subgroup of $L(g)$, and is dense in $L'(g)$. Hence $L'(g)$ is either $\mathbf{R}$ or compact. This completes the proof of (b) and (c). □

REMARK. Let $L$ be a locally compact, $\sigma$-compact group, and $G$ a closed subgroup of $L$. Let $A$ and $B$ be subgroups of $G$. If $L$ has an LC-family $\mathfrak{F}$, then Theorem III, except for the part concerning $\mathfrak{F}$-hulls, holds for $G$. In fact, if $P_\lambda$ is the local closure of $AgB$ in $L$ for $g \in G$, then $P_\lambda \cap G$ is locally compact, $(A, B)$-invariant, and coincides with $P_\lambda$. Hence $P_\lambda \subset G$.

**3. Semi-algebraic groups.** In this section we shall study subgroups of $\mathrm{GL}(n, \mathbf{R})$. A subgroup $H$ of $\mathrm{GL}(n, \mathbf{R})$ is said to be *pre-algebraic* if $H$ is an open subgroup of a suitable algebraic group. Since an algebraic group is finitely connected, so is any pre-algebraic group. Let $\mathcal{Q}$ denote the set of all pre-algebraic groups in $\mathrm{GL}(n, \mathbf{R})$. The following theorem is known; in particular, the proof of (4) for $\mathcal{Q}$ can be found in [**2**].

(3.1) THEOREM. *$\mathcal{Q}$ is an LC-family in* $\mathrm{GL}(n, \mathbf{R})$. *Any member of $\mathcal{Q}$ is finitely connected.*

For a subgroup $H$ of $\mathrm{GL}(n, \mathbf{R})$, let $\mathcal{Q}_e(H)$ denote the identity component of $\mathcal{Q}(H)$.

(3.2) (1) For any subgroup $H$ of $\mathrm{GL}(n, \mathbf{R})$, $H \cdot \mathcal{Q}_e(H) = \mathcal{Q}(H)$.

(2) If in particular $H$ is finitely connected, then $\mathcal{Q}_e(H) = \mathcal{Q}(H_e)$.

PROOF. (1) Since $\mathcal{Q}_e(H)$ is a normal subgroup of $\mathcal{Q}(H)$, we have that $H \cdot \mathcal{Q}_e(H)$ is a subgroup of $\mathcal{Q}(H)$. Since $\mathcal{Q}_e(H)$ is contained in $H \cdot \mathcal{Q}_e(H)$, we see that $H \cdot \mathcal{Q}_e(H)$ is open in $\mathcal{Q}(H)$, and is pre-algebraic. Hence $H \cdot \mathcal{Q}_e(H) = \mathcal{Q}(H)$.

(2) It is obvious that $\overline{H}_e$ is the identity component of $\overline{H}$. Hence it reduces to (2.5). □

Let $H$ be a closed connected subgroup of $\mathrm{GL}(n, \mathbf{R})$. Then $H$ is normal in $\mathcal{Q}(H)$, and the factor group $\mathcal{Q}(H)/H$ is a connected abelian group by (6). Since $\mathcal{Q}(H)/H$ is a Lie group, there exist nonnegative integers $k$ and $h$ such that $\mathcal{Q}(H)/H = \mathbf{R}^k \times (\mathbf{R}/\mathbf{Z})^h$. Let $K$ be a maximal compact subgroup of $\mathcal{Q}(H)$. Then $KH/H$ is a maximal compact subgroup of $\mathcal{Q}(H)/H$; see Iwasawa [**5**].

In [**2**], the author defined $H$ to be *semi-algebraic* if $H$ contains all compact subgroups of $\mathcal{Q}(H)$. This is equivalent to saying that the factor group $\mathcal{Q}(H)/H$ is isomorphic with the vector group $\mathbf{R}^k$, or that $\mathcal{Q}(H)/H$ is homeomorphic with the euclidean space $\mathbf{R}^k$. Now let us extend the definition to nonconnected groups.

DEFINITION 5. A closed subgroup $H$ of $\mathrm{GL}(n, \mathbf{R})$ is said to be *semi-algebraic* if the factor space $\mathcal{Q}(H)/H$ is homeomorphic with a euclidean space.

REMARK AND CORRECTION. In [**2**], the author defined nonconnected semi-algebraic groups in a more restrictive manner. However, he later found the new definition more convenient. In the Proposition, p. 72 in [**2**], "and conversely." must be removed.

(3.3) Let $G$ be a Lie group, and $H$ a closed subgroup of $G$. If the factor space $G/H$ is connected and simply connected, then

$$H \cap G_e = H_e \quad \text{and} \quad G/G_e \cong H/H_e.$$

PROOF. Since $G/H$ is connected, we have that $G_eH = G$ and $G/H \sim G_e/H \cap G_e$, where $\sim$ denotes the existence of a homeomorphism. Since $G/H$ is simply connected, $H \cap G_e$ is connected and coincides with $H_e$. Hence $G/G_e = HG_e/G_e \cong H/H \cap G_e = H/H_e$. $\square$

(3.4) If $S$ is semi-algebraic, then $S$ is finitely connected and $S_e$ is semi-algebraic, and vice versa.

PROOF. If $S$ is semi-algebraic, then by (3.3), $S/S_e \cong \mathcal{Q}(S)/\mathcal{Q}_e(S)$ is finite and $\mathcal{Q}_e(S) \cap S = S_e$. Hence by (3.2),

$$\mathcal{Q}(S)/S = S \cdot \mathcal{Q}_e(S)/S \sim \mathcal{Q}_e(S)/\mathcal{Q}_e(S) \cap S = \mathcal{Q}(S_e)/S_e,$$

and $\mathcal{Q}(S_e)/S_e$ is homeomorphic with a euclidean space.

Conversely, suppose that $S$ is finitely connected and $S_e$ is semi-algebraic. Then $(\mathcal{Q}_e(S) \cap S)/S_e$ is a subgroup of $\mathcal{Q}_e(S)/S_e = \mathcal{Q}(S_e)/S_e \cong \mathbf{R}^k$. Since $(\mathcal{Q}_e(S) \cap S)/S_e$ is a finite group, it must reduce to the identity and $\mathcal{Q}_e(S) \cap S = S_e$. Then

$$\mathcal{Q}(S)/S = S \cdot \mathcal{Q}_e(S)/S_e = S \cdot \mathcal{Q}(S_e)/S_e \sim \mathcal{Q}(S_e)/\left(\mathcal{Q}(S_e) \cap S\right)$$
$$= \mathcal{Q}(S_e)/S_e \cong \mathbf{R}^k. \quad \square$$

Let $G$ be a countably connected Lie group. A subgroup $H$ of $G$ is said to be a *Lie subgroup* if there exists a countably connected Lie group $H^*$ and a continuous one-one homomorphism $f$ from $H^*$ into $G$ such that $f(H^*) = H$. A closed subgroup is a Lie subgroup. Let $H$ be a Lie subgroup of $G$. Then the Lie group $H^*$ is uniquely determined up to topological isomorphisms. $H$ is called a *connected*, or a *finitely connected*, *Lie subgroup* if $H^*$ is connected, or finitely connected, respectively. $H$ is a connected Lie subgroup if and only if $H$ is arcwise connected. If $H_1$ and $H_2$ are Lie subgroups and if $H_1H_2$ is a subgroup, then $H_1H_2$ is a Lie subgroup.

Let us denote the set of all semi-algebraic groups in $\mathrm{GL}(n, \mathbf{R})$ by $\mathbb{S}$.

(3.5) Let $\{S_\lambda\}$ be a subset of $\mathbb{S}$. Then the intersection $\bigcap S_\lambda$ is semi-algebraic.

PROOF. Since any semi-algebraic group is finitely connected, $\mathbb{S}$ satisfies the descending chain condition. Hence it suffices to prove that $A \cap B \in \mathbb{S}$ for $A$ and $B$ in $\mathbb{S}$. We put

$$A_1 = \mathcal{Q}(A_e), \quad B_1 = \mathcal{Q}(B_e) \quad \text{and} \quad C_1 = A_1 \cap B_1.$$

Then $C_1$ is pre-algebraic and is finitely connected. Therefore $A_eC_1$ is a finitely connected Lie subgroup of $A_1$, and so is $A_eC_1/A_e$ in $A_1/A_e$. Since $A_1/A_e$ is a vector group, so is $A_eC_1/A_e$. Thus $C_1/C_1 \cap A_e \cong A_eC_1/A_e$ is a vector group. By (3.3), $C_1 \cap A_e$ is finitely connected since $C_1$ is. In a similar way, $C_1/C_1 \cap$

$B_e$ is a vector group and $C_1 \cap B_e$ is finitely connected. It, then, follows that $(C_1 \cap A_e)(C_1 \cap B_e)$ is a finitely connected Lie subgroup in $C_1$, and so is $(C_1 \cap A_e)(C_1 \cap B_e)/C_1 \cap B_e$ in $C_1/C_1 \cap B_e = \mathbf{R}^k$. Hence

$$(C_1 \cap A_e)(C_1 \cap B_e)/C_1 \cap B_e \cong C_1 \cap A_e/C$$

is a vector group, where $C = A_e \cap B_e = (C_1 \cap A_e) \cap (C_1 \cap B_e)$. $C_1 \cap A_e$ being finitely connected, so is $C$ by (3.3). Moreover, $C$ is of finite index in $A \cap B$ since $A$ and $B$ are finitely connected. Therefore $A \cap B$ is finitely connected.

Next, we shall prove that $(A \cap B)_e$ is semi-algebraic. Let $K$ be a compact connected subgroup of $C_1$. Then $K \subset A_1$, and $K \subset A$. Similarly $K \subset B$. Hence $K \subset (A \cap B)_e$. That is, any maximal compact subgroup of $(C_1)_e$ is contained in $(A \cap B)_e$. It follows from $(C_1)_e$ being pre-algebraic that $(A \cap B)_e$ is semi-algebraic. □

(3.6) Let $A$ and $B$ be in $\mathbb{S}$. Then the double coset $AB$ is locally compact.[1]

PROOF. As before, we put $\mathcal{Q}(A) = A^*$, $\mathcal{Q}(B) = B^*$ and

$$D = \{(\gamma, \gamma); \gamma \in A^* \cap B^*\} \subset A^* \times B^*.$$

Since $A^*B^*$ is locally compact, the map

$$\xi: A^* \times B^* \ni (\alpha, \beta) \to \alpha\beta^{-1} \in A^*B^*$$

is continuous and open, and gives rise to a homeomorphism between the factor space $A^* \times B^*/D$ and $A^*B^*$. Hence it suffices to prove that $\xi^{-1}(AB) = (A \times B)D$ is closed in $A^*B^*$.

Since $A$, $B$ and $D$ are all finitely connected, $(A \times B)D$ is a finite union of the sets of the form

$$\varepsilon(A_e \times B_e)D_e\delta, \qquad \varepsilon \in A \times B, \delta \in D.$$

On the other hand, $(A_e \times B_e)D_e$ is a connected Lie subgroup of $(A^* \times B^*)_e$ containing all compact subgroups, and is closed. □

From (3.1), (3.4), (3.5) and (3.6), we have the following theorem:

(3.7) THEOREM. *$\mathbb{S}$ is an LC-family in* GL($n$, $\mathbf{R}$), *and any member of $\mathbb{S}$ is finitely connected.*

(3.8) Let $\varphi$ be a rational homomorphism from an algebraic group $A$ into GL($j$, $\mathbf{R}$). Let $S$ be a semi-algebraic group in $A$. Then $\varphi(S)$ is semi-algebraic.

PROOF. Since $S$ is finitely connected, so is $\varphi(S)$. Hence it suffices to prove that $\varphi(S_e)$ is semi-algebraic. Therefore, we may suppose that $S$ is connected.

Let $N$ be the kernel of $\varphi$ restricted to $\mathcal{Q}(S)$. Then $N$ is pre-algebraic, and is finitely connected. Hence $NS/S$ is a finitely connected Lie subgroup of $\mathcal{Q}(S)/S$. Since $\mathcal{Q}(S)/S$ is a vector group, so is $NS/S$, and $NS$ is closed and

[1](3.6) was proved in [2] under a slightly stronger condition.

connected. Hence $\varphi(S) = \varphi(NS)$ is closed in $\varphi\mathcal{A}(S)$. Since $\varphi\mathcal{A}(S) = \mathcal{A}\varphi(S)$, see [3], we have

$$\mathcal{A}\varphi(S)/\varphi(S) = \varphi\mathcal{A}(S)/\varphi(S) \cong \mathcal{A}(S)/NS.$$

Thus, recalling $\mathcal{A}(S)/NS = \mathcal{A}(S)/S/NS/S$ is a vector group, it follows that $\varphi(S)$ is semi-algebraic. $\square$

**4. sa-groups in an adjoint semi-algebraic group.** Let $G$ be a connected Lie group, $\mathcal{G}$ its Lie algebra, and let $\rho$ denote the adjoint representation of $G$:

$$G \ni g \mapsto \rho(g) = \mathrm{Ad}(g) \in \mathrm{Ad}(\mathcal{G}) \subset \mathrm{GL}(\mathcal{G}).$$

The kernel of $\rho$ is the center $Z$ of $G$. The connected Lie group $G$ is said to be *adjoint semi-algebraic* if the adjoint group $\mathrm{Ad}(G) = \mathrm{Ad}(\mathcal{G})$ is semi-algebraic. By (3.8), a connected semi-algebraic group is adjoint semi-algebraic, and the converse is given by the following:

(4.1) Let $G$ be an adjoint semi-algebraic group. Then there exists a semi-algebraic group $G' \subset \mathrm{GL}(n, \mathbf{R})$, for a sufficiently large $n$, such that $G$ is locally isomorphic with $G'$.

(4.1) was proved in [2], along with (4.2).

(4.2) Let $G$ be a connected Lie group. Then there exists an adjoint semi-algebraic group $S$ containing $G$ as a closed normal subgroup.

After this, we assume that $G$ is an adjoint semi-algebraic group, $\mathcal{G}$ its Lie algebra, $\rho$ the adjoint representation of $G$ and $Z$ is the center of $G$.

DEFINITION 6. A subgroup $H$ of $G$ is said to be an sa-*group* if

(i) $\rho(H)$ is semi-algebraic, and

(ii) $H$ is open in $\rho^{-1}\rho(H)$.

If $H$ is an sa-group, then

(ii′) $H$ is closed and $H \supset Z_e$.

Conversely, (i) and (ii′) imply (ii) obviously. Let $\mathfrak{S}$ denote the set of all sa-groups in $G$.

(4.3) Let $S$ be an sa-group.

(1) If $S'$ is an open subgroup of $S$, then $S'$ is sa.

(2) If $S''$ contains $S$ as a subgroup of finite index, then $S''$ is an sa-group.

PROOF. If $S$ is an sa-group, then $\rho(S)$ is semi-algebraic and

$$\rho(S'') \supset \rho(S) \supset \rho(S') \supset \rho(S_e) = \rho(S'')_e.$$

Hence $\rho(S')$ is open in $\rho(S)$, and is semi-algebraic. Also $\rho(S)$ is of finite index in $\rho(S'')$, and $\rho(S'')$ is semi-algebraic.

Next, $S'$ and $S''$ are closed subgroups and contain $S_e \supset Z_e$. Hence $S'$ and $S''$ are sa-groups. $\square$

(4.4) If $\{S_\lambda; \lambda \in \Lambda\}$ is a nonempty subset of $\mathfrak{S}$, then $\bigcap S_\lambda \in \mathfrak{S}$.

PROOF. We put $S'_\lambda = S_\lambda Z = \rho^{-1}\rho(S_\lambda)$ for $\lambda \in \Lambda$. Then it is obvious that

$$\bigcap \rho(S'_\lambda) = \rho(\bigcap S'_\lambda) \quad \text{and} \quad \rho^{-1}\rho(\bigcap S'_\lambda) = \bigcap S'_\lambda.$$

Since $\rho(S'_\lambda) = \rho(S_\lambda)$ is semi-algebraic, so is $\bigcap \rho(S'_\lambda)$. Hence $\bigcap S'_\lambda$ is an sa-group.

Next, since $S_\lambda$ is open in $S'_\lambda$, the identity components of the two groups coincide, and $(\bigcap S_\lambda)_e = (\bigcap S'_\lambda)_e$. Therefore $\bigcap S'_\lambda \supset \bigcap S_\lambda \supset (\bigcap S'_\lambda)_e$, and $\bigcap S_\lambda$ is an open subgroup of $\bigcap S'_\lambda$. By (4.3), $\bigcap S_\lambda$ is an sa-group. □

By (4.4), for any subgroup $H$ of $G$, there corresponds the $\mathfrak{s}$-hull $\mathfrak{s}(H)$, the smallest sa-group containing $H$. Let $\mathfrak{s}_e(H)$ denote the identity component of $\mathfrak{s}(H)$.

(4.5) (1) If $H$ is a connected subgroup of $G$, then

$$\mathfrak{s}(H) = \left(\rho^{-1}\mathfrak{S}\,\rho(H)\right)_e.$$

(2) If $H$ is a finitely connected, closed subgroup of $G$, then

$$\mathfrak{s}(H) = H \cdot \mathfrak{s}_e(H), \qquad \mathfrak{s}_e(H) = \mathfrak{s}(H_e),$$

$\mathfrak{s}(H)$ is finitely connected, and $H_e$ is a normal subgroup of $\mathfrak{s}(H)$.

PROOF. (1) We put $S = (\rho^{-1}\mathfrak{S}\,\rho(H))_e$. Then

$$\rho(S) = \mathfrak{S}\,\rho(H) \quad \text{and} \quad \rho^{-1}\rho(S)_e = S,$$

and $S$ is an sa-group.

Next, suppose that $T$ is an sa-group containing $H$. Then

$$\rho(T) \supset \mathfrak{S}\,\rho(H) \quad \text{and} \quad \rho^{-1}\rho(T) \supset \rho^{-1}\mathfrak{S}\,\rho(H) \supset S.$$

Since $T$ is open in $\rho^{-1}\rho(T)$ and $S$ is connected, we have that $T \supset S$.

(2) Let $\mathfrak{H}$ be the Lie algebra of $H$, and let $N$ be the normalizer of $H_e$ in $G$. Then $M = \{x \in \mathrm{GL}(\mathfrak{G});\ x\mathfrak{H} = \mathfrak{H}\}$ is an algebraic group, and $\rho(N) = \mathrm{Ad}(\mathfrak{G}) \cap M$ is a semi-algebraic group in $\mathrm{GL}(\mathfrak{G})$. Hence

$$\rho(N) \supset \mathfrak{S}\,\rho(H_e) \quad \text{and} \quad N = \rho^{-1}\rho(N) \supset \rho^{-1}\mathfrak{S}\,\rho(H_e).$$

Since $\rho(H_e)$ is a normal subgroup of $\rho(N)$, for $x \in \rho(N)$

$$\mathfrak{S}\,\rho(H_e) = \mathfrak{S}\left(x\rho(H_e)x^{-1}\right) = x\mathfrak{S}\,\rho(H_e)x^{-1},$$

and $\mathfrak{S}\,\rho(H_e)$ is a normal subgroup of $\rho(N)$. Hence $N$ normalizes $\rho^{-1}\mathfrak{S}\,\rho(H_e)$, and its identity component $\mathfrak{s}(H_e)$. Since $H$ is contained in $N$, $H \cdot \mathfrak{s}(H_e)$ is a subgroup of $N$. By $\mathfrak{s}(H_e) \supset H_e$, $\mathfrak{s}(H_e)$ is of finite index in $H \cdot \mathfrak{s}(H_e)$. Hence $H \cdot \mathfrak{s}(H_e)$ is an sa-group, by (4.3) (2), and $\mathfrak{s}(H) = H \cdot \mathfrak{s}(H_e)$. Thus $\mathfrak{s}(H_e)$ is a closed connected subgroup of finite index in $\mathfrak{s}(H)$, and so $\mathfrak{s}_e(H) = \mathfrak{s}(H_e)$. That $N \supset \mathfrak{s}(H)$ implies that $H_e$ is normal in $\mathfrak{s}(H)$. □

For a group $L$, let $[L, L]$ denote the commutator subgroup of $L$. If, in particular, $L$ is a connected Lie group with Lie algebra $\mathfrak{L}$, then $[L, L]$ is a connected Lie subgroup of $L$ and the Lie algebra of $[L, L]$ is $[\mathfrak{L}, \mathfrak{L}]$.

(4.6) Let $H$ be a connected Lie subgroup of $G$. Then $[\mathfrak{s}(H), \mathfrak{s}(H)] = [H, H]$.

PROOF. First suppose that $G$ is a semi-algebraic group in $GL(n, \mathbf{R})$. We put $S = \mathfrak{S}(H)Z$. Then $[S, S] = [\mathfrak{S}(H), \mathfrak{S}(H)] = [H, H]$. By (3.8), $\rho(S) = \rho\mathfrak{S}(H)$ is semi-algebraic. Hence $S = \rho^{-1}\rho(S)$ is an sa-group, and $S \supset \mathfrak{s}(H)$. Therefore $[S, S] \supset [\mathfrak{s}(H), \mathfrak{s}(H)] \supset [H, H]$ and $[\mathfrak{s}(H), \mathfrak{s}(H)] = [H, H]$.

Now, we shall consider the general case. By (4.1), there exists a connected semi-algebraic group $G'$ which is locally isomorphic with $G$. Let us identify $\mathcal{G}$ with the Lie algebra of $G'$. Let $\mathcal{H}$ be the Lie algebra of $H$, and let $H'$ be the connected Lie subgroup of $G'$ corresponding to the Lie algebra $\mathcal{H}$. Then by (4.5) (1), the Lie algebra of $\mathfrak{s}(H')$ coincides with the Lie algebra $\mathfrak{s}(\mathcal{H})$ of $\mathfrak{s}(H)$. Thus $[\mathfrak{s}(H'), \mathfrak{s}(H')] = [H', H']$ implies $[\mathfrak{s}(\mathcal{H}), \mathfrak{s}(\mathcal{H})] = [\mathcal{H}, \mathcal{H}]$, whence $[\mathfrak{s}(H), \mathfrak{s}(H)] = [H, H]$. □

(4.7) If $H$ is a finitely connected, abelian subgroup of $G$, then $\mathfrak{s}(H)$ is abelian.

PROOF. Because $\overline{HZ_e}$ is finitely connected and abelian, we can suppose that $H$ is closed and $H \supset Z_e$, without loss of generality. By (4.5) (2),

$$\mathfrak{s}(H) = H \cdot \mathfrak{s}_e(H), \qquad \mathfrak{s}_e(H) = \mathfrak{s}(H_e),$$

where $\mathfrak{s}_e(H)$ is abelian, by (4.6).

Let $A$ be an abelian group in $GL(n, \mathbf{R})$, and let $C$ be the center of the centralizer of $A$. Then $C$ is an abelian algebraic group and $C \supset A$. Therefore $\mathcal{Q}(A)$ is abelian, and so is $\mathfrak{S}(A)$. That is, *the $\mathfrak{S}$-hull of any abelian group is abelian*. Since $H$ is abelian, so is $\rho(H)$, and $\mathfrak{S}\rho(H)$ is an abelian group. Hence $[\mathfrak{s}(H), \mathfrak{s}(H)] \subset Z$.

Let $\mathcal{H}$ and $\mathfrak{s}(\mathcal{H})$ denote the Lie algebra of $H$ and $\mathfrak{s}(H)$, respectively. $\mathfrak{s}_e(H)$ being a normal subgroup of $\mathfrak{s}(H)$, $\rho(H)$ leaves $\mathfrak{s}(\mathcal{H})$ invariant. For $h \in H$, let $\mu(h)$ denote the restriction of $\rho(h)$ to $\mathfrak{s}(\mathcal{H})$. Since $\mathfrak{s}_e(H)$ is abelian and $H_e \subset \mathfrak{s}_e(H)$, the kernel of $\mu$ contains $H_e$, and $\mu$ induces a representation of the finite group $H/H_e$. Therefore, the representation $\mu$: $H \to GL(\mathfrak{s}(\mathcal{H}))$ is completely reducible. Owing to $\mu(H)\mathcal{H} = \mathcal{H}$, we can find a subspace (subalgebra) $\mathfrak{M}$ of $\mathfrak{s}(\mathcal{H})$ such that $\mathfrak{s}(\mathcal{H}) = \mathcal{H} \oplus \mathfrak{M}$, $\mu(H)\mathfrak{M} = \mathfrak{M}$. For any $h$ in $H$, we have $(\mu(h) - 1)\mathcal{H} = 0$ because $H$ is abelian, and $(\mu(h) - 1)\mathfrak{s}(\mathcal{H}) \subset \mathfrak{M}$.

On the other hand, for any $X \in \mathfrak{s}(\mathcal{H})$ and the real parameter $t$,

$$a(t) = h(\exp tX)h^{-1}(\exp(-tX)) = \exp\big(t(\mu(h) - 1)X + O(t^2)\big)$$

is a curve in $Z$, and belongs to $Z_e$. Since $Z_e \subset H$ and the tangent vector to $a(t)$ at $t = 0$ is $(\mu(h) - 1)X$, we have that $(\mu(h) - 1)X \in \mathcal{H}$, and $(\mu(h) - 1)\mathfrak{s}(\mathcal{H}) \subset \mathcal{H}$. It follows that $(\mu(h) - 1)\mathfrak{s}(\mathcal{H}) = 0$ since $\mathcal{H} \cap \mathfrak{M} = \{0\}$. Hence $h$ commutes with every element of $\mathfrak{s}_e(H)$. Therefore $\mathfrak{s}(H) = H \cdot \mathfrak{s}_e(H)$ is abelian. □

REMARK. The semi-algebraic hull of an abelian group is abelian as we saw in the proof above. But this is not true for the $\mathfrak{S}$-hull. For example,

$$G = \left\{ g(x, y, z) = \begin{bmatrix} e^x & 0 & 0 & 0 \\ 0 & \cos x & \sin x & y \\ 0 & -\sin x & \cos x & z \\ 0 & 0 & 0 & 1 \end{bmatrix}; x, y, z \in \mathbf{R} \right\}$$

is an adjoint (semi-)algebraic group, and

$$H = \{ g(x, y, z); x \in 2\pi\mathbf{Z}, y \in \mathbf{Z}, z \in \mathbf{Z} \}$$

is an abelian subgroup of $G$, but $\mathfrak{S}(H) = G$ is not abelian.

Thus we have

(4.8) THEOREM. *Let $G$ be an adjoint semi-algebraic group, and let $\mathfrak{S}$ be the totality of sa-groups in $G$. Then $\mathfrak{S}$ is an LC-family.*

ADDED IN PROOF. The author learned from Philip Green that he had proved in his unpublished paper that any connected Lie group can be embedded as a closed normal subgroup in a suitable Lie group whose adjoint group is pre-algebraic. By his theorem, the main results of this paper can be established without semi-algebraic groups. But the author thinks the theory of semi-algebraic groups itself has some significance, so he leaves the paper in the original form.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19174