SMALL ZEROS OF ADDITIVE FORMS IN MANY VARIABLES¹

BY

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ABSTRACT. It is shown that if s is large as a function of k and of $\varepsilon > 0$, then the diophantine equation $a_1x_1^k + \cdots + a_sx_s^k = b_1y_1^k + \cdots + b_sy_s^k$ with positive coefficients $a_1, \dots, a_s, b_1, \dots, b_s$ has a nontrivial solution in nonnegative integers $x_1, \dots, x_s, y_1, \dots, y_s$ not exceeding $m^{(1/k)+\varepsilon}$, where m is the maximum of the coefficients.

1. Introduction. A fairly direct application of the circle method shows that an equation

$$a_1x_1^k + \cdots + a_sx_s^k = 0 \quad (1.1)$$

where the coefficients $a_1, \dots, a_s$ are not all of the same sign has a nontrivial solution in nonnegative integers $x_1, \dots, x_s$, provided only that $s \geq c_1(k)$. (See, e.g., Davenport and Lewis [3], or Davenport [2].) As for the *size* of these solutions, it was shown by Pitman [6] that if the coefficients are as above, and each nonzero, and if $s \geq c_2(k)$ where $c_2(k)$ is explicitly given, then for given $\varepsilon > 0$ there is a nontrivial solution in nonnegative integers with

$$|a_1x_1^k| + \cdots + |a_sx_s^k| < c_3(k, \varepsilon)|a_1 \cdots a_s|^{k+\varepsilon}.$$

(Actually Pitman does not require the solutions to be nonnegative, hence for odd k allows the coefficients to be of arbitrary signs. But the result quoted is an immediate outcome of her method.) In particular, for $s \geq c_2(k)$ there is a solution with

$$\max(x_1, \dots, x_s) < c_4(k)m^{c_s(k)} \quad (1.2)$$

where $m = \max(|a_1|, \dots, |a_s|)$.

Under suitable conditions, and if s is very large, the estimate (1.2) may be considerably improved. Birch [1] combined Pitman's results with ideas contained in Linnik's elementary solution [4], [5] of Waring's problem to show that if k is *odd* and if $s \geq c_6(k, \varepsilon)$, then (1.1) has a nontrivial solution in

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integers $x_1, \dots, x_s$, which may be of arbitrary sign, and which have

$$\max(|x_1|, \dots, |x_s|) < c_7(k, \epsilon) m^{(1/k)+\epsilon}. \quad (1.3)$$

This estimate is probably not the best possible. If the right-hand side of (1.3) could be improved to $c_8(k, \epsilon) m^\epsilon$, it would have the important consequence that a form of odd degree k with real coefficients in enough variables can be made arbitrarily small for suitable (not all zero) integer values of the variables (see the remark in Birch [1]). For certain other applications in diophantine approximation, it is desirable to have a version where k may be even as well as odd, and where each variable is of a prescribed sign.

THEOREM. *Suppose $s \geq c_9(k, \epsilon)$, and suppose $a_1, \dots, a_s, b_1, \dots, b_s$ are positive integers. Then the equation*

$$a_1 x_1^k + \dots + a_s x_s^k = b_1 y_1^k + \dots + b_s y_s^k \quad (1.4)$$

has a nontrivial solution in nonnegative integers $x_1, \dots, x_s, y_1, \dots, y_s$ having

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \leq m^{(1/k)+\epsilon},$$

where

$$m = \max(a_1, \dots, a_s, b_1, \dots, b_s).$$

This is stronger than Birch's result even if k is odd. The estimate is essentially best possible, for every nontrivial solution of

$$a(x_1^k + \dots + x_s^k) = b(y_1^k + \dots + y_s^k)$$

with coprime positive a, b has $x_1^k + \dots + x_s^k \geq b$ and $y_1^k + \dots + y_s^k \geq a$, whence

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \geq c_{10}(k, s) m^{1/k}$$

where $m = \max(a, b)$. But it is conceivable that the Theorem holds with $c_9(k, \epsilon)$ replaced by some $c'_9(k)$, and the conclusion replaced by $\max(x_1, \dots, x_s, y_1, \dots, y_s) \leq c_{11}(k, \epsilon) m^{(1/k)+\epsilon}$. The constant $c_9(k, \epsilon)$ obtainable by our method is computable but very large.

Our proof is similar to Birch's in that we reduce the problem to that of finding solutions of

$$a_1 x_1^k + \dots + a_s x_s^k - (b_1 y_1^k + \dots + b_s y_s^k) = z$$

with very small z . But we shall employ the circle method instead of elementary estimates à la Linnik.

Our Theorem is applied by Schlickewei [7] to obtain a result about small values of indefinite diagonal forms with real coefficients.

2. An inductive argument.

PROPOSITION 1. *Let $\lambda \geq 1/k$, $\epsilon > 0$ and $s \geq c_{12}(k, \lambda, \epsilon)$. Let $a_1, \dots, a_s, b_1, \dots, b_s$ be as in the Theorem. Then (1.4) has a nontrivial solution in*

nonnegative integers $x_1, \dots, x_s, y_1, \dots, y_s$ with

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \leq m^{\lambda+\varepsilon}. \quad (2.1)$$

The case $\lambda = 1/k$ is the Theorem. Moreover, since the truth of the proposition for a particular value of λ implies its truth for every $\lambda' > \lambda$, the proposition is in fact equivalent to the Theorem.

It will suffice to prove the proposition when m is large, say $m \geq c_{13}(k, \lambda, \varepsilon)$. For if $m < c_{13}$ and if s is large, then the a_i will assume the same value a at least m times, and the b_i will assume the same value b at least m times, so that a occurs at least b times and b occurs at least a times, and from this one can construct a solution of the equation consisting of zeros and ones only. Proposition 1 is true for *some* values of λ : By Pitman's estimate (1.2) it is true for $\lambda > c_5(k)$. Because of the $\varepsilon > 0$ in the formulation of the proposition, the set of numbers λ (this set depends only on k) for which the proposition holds is closed. Thus to prove Proposition 1 (and hence the Theorem), it will suffice to prove the following²

"INDUCTIVE ASSERTION." *If $\lambda > 1/k$ and if the proposition is true for λ , then it is true for some $\lambda' < \lambda$.*

In what follows, λ will be a fixed number $> 1/k$ for which the proposition holds. Pick μ so small that

$$1/k + 6c_5(k)\mu + 20\mu < \lambda \quad \text{and} \quad 22k\mu < 1, \quad (2.2)$$

and put

$$\lambda' = \max\left(\lambda\left(1 - \frac{1}{2}\mu\right) + \mu/2k, 1/k + 6c_5(k)\mu + 20\mu\right), \quad (2.3)$$

so that indeed $\lambda' < \lambda$. We proceed to prove the proposition for λ' .

Write

$$\delta = \min(\varepsilon/8\lambda', \varepsilon/4)$$

and divide the interval $0 \leq x \leq 1$ into a finite number of subintervals I of length not exceeding δ . If s is large, one of these intervals I will be such that many of the coefficients $a_1, \dots, a_s$ are of the type $a_i = m^{\alpha_i}$ with $\alpha_i \in I$. We may therefore suppose without loss of generality that $a_i/a_j \leq m^\delta$ ($1 \leq i, j \leq s$). Similarly we may suppose that $b_i/b_j \leq m^\delta$ ($1 \leq i, j \leq s$). Put $a = m^\delta \max(a_1, \dots, a_s)$ and $b = m^\delta \max(b_1, \dots, b_s)$. Let p_i, q_i , respectively, be the largest integers with

$$a_i p_i^k \leq a \quad \text{and} \quad b_i q_i^k \leq b \quad (i = 1, \dots, s).$$

Now $a/a_i \geq m^\delta$, and if m is large (which we may suppose), then $p_i \geq 2^{-1/k}(a/a_i)^{1/k}$, so that $a_i p_i^k \geq \frac{1}{2}a$. Similarly, $b_i q_i^k \geq \frac{1}{2}b$.

With $a'_i = a_i p_i^k$, $b'_i = b_i q_i^k$ and $x_i = p_i x'_i$, $y_i = q_i y'_i$ ($i = 1, \dots, s$), (1.4)

²A reader who finds our nonconstructive argument distasteful should be able to replace it by a constructive one.

becomes

$$a'_1 x_1'^k + \cdots + a'_s x_s'^k = b'_1 y_1'^k + \cdots + b'_s y_s'^k. \quad (2.4)$$

If Proposition 1 holds for λ' and for the particular equation (2.4), then we have a nontrivial nonnegative solution with

$$\begin{aligned} \max(x'_1, \dots, x'_s, y'_1, \dots, y'_s) &\leq (\max(a, b))^{\lambda' + (\varepsilon/4)} \\ &\leq m^{(1+\delta)(\lambda' + (\varepsilon/4))} \leq m^{\lambda' + (\varepsilon/2)}. \end{aligned}$$

But clearly $a_i \geq am^{-2\delta}$, so that $p_i \leq p_i^k \leq m^{2\delta} \leq m^{\varepsilon/2}$, whence $x_i \leq m^{\lambda' + \varepsilon}$ ($i = 1, \dots, s$), and similarly, $y_i \leq m^{\lambda' + \varepsilon}$, as desired.

(2.4) was special since $\frac{1}{2}a \leq a'_i \leq a$ and $\frac{1}{2}b \leq b'_i \leq b$. Thus we have the

REDUCTION. In proving Proposition 1 for λ' we may suppose that

$$\frac{1}{2}a \leq a_i \leq a \quad \text{and} \quad \frac{1}{2}b \leq b_i \leq b \quad (i = 1, \dots, s)$$

for certain a, b .

3. Two cases. In what follows, h will be the integer

$$h = c_{12}(k, \lambda, \varepsilon)$$

occurring in Proposition 1, and s will be assumed to be much larger than h . Write

$$\nu = \mu/2k. \quad (3.1)$$

We distinguish two cases.

A. There is a subset of h elements among $a_1, \dots, a_s$, say $a_1, \dots, a_h$, and there is a subset of h elements among $b_1, \dots, b_s$, say $b_1, \dots, b_h$, and there are natural integers

$$p_1, \dots, p_h, q_1, \dots, q_h \leq m^\nu, \quad (3.2)$$

such that

$$d = \text{g.c.d.}(a_1 p_1, \dots, a_h p_h, b_1 q_1, \dots, b_h q_h) \geq m^\mu.$$

In this case put $x_i = p_i x'_i$, $y_i = q_i y'_i$ ($i = 1, \dots, h$) and $x_{h+1} = y_{h+1} = \dots = x_s = y_s = 0$. After division by d , (1.4) becomes

$$a'_1 x_1'^k + \cdots + a'_h x_h'^k = b'_1 y_1'^k + \cdots + b'_h y_h'^k, \quad (3.3)$$

where $a'_i = a_i p_i^k / d$ and $b'_i = b_i q_i^k / d$ ($i = 1, \dots, h$). Because of the truth of the proposition for λ , and by our choice of h , (3.3) has a nontrivial nonnegative solution with

$$\max(x'_1, \dots, x'_h, y'_1, \dots, y'_h) \leq (m^{1+k\nu-\mu})^{\lambda+\varepsilon},$$

so that

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \leq m^{(1+k\nu-\mu)(\lambda+\varepsilon)+\nu} \leq m^{\lambda'+\varepsilon}.$$

We are thus reduced to case

B. For any h elements, say $a_1, \dots, a_h$, among $a_1, \dots, a_s$, and for any h elements, say $b_1, \dots, b_h$, among $b_1, \dots, b_s$, and given (3.2), we have

$$\text{g.c.d.}(a_1 p_1, \dots, a_h p_h, b_1 q_1, \dots, b_h q_h) < m^\mu. \quad (3.4)$$

Condition B depends on h, m, μ, ν , and if ν is given by (3.1), it is a condition $B(k, h, m, \mu)$.

PROPOSITION 2. Let $h \geq 1, k \geq 1$, and

$$0 < \mu < 1/22k. \quad (3.5)$$

Let $0 < a, b \leq m$ and let $a_1, \dots, a_s, b_1, \dots, b_s$ be integers with

$$\frac{1}{2}a \leq a_i \leq a, \quad \frac{1}{2}b \leq b_i \leq b \quad (i = 1, \dots, s)$$

with property $B(k, h, m, \mu)$. Then if $s \geq c_{14}(k, h, \mu)$, the equation

$$a_1 x_1^k + \dots + a_s x_s^k - (b_1 y_1^k + \dots + b_s y_s^k) = z \quad (3.6)$$

has a solution in nonnegative integers $x_1, \dots, x_s, y_1, \dots, y_s, z$ with

$$\max(x_1, \dots, x_s, y_1, \dots, y_s) \leq m^{(1/k)+20\mu}, \quad z \leq m^{6\mu}.$$

This proposition implies the Inductive Assertion, as we now proceed to show. For let $\lambda, \mu, \lambda', \nu$ be as above, in particular, with (2.2) (whence (3.5)), (2.3), (3.1). We may suppose that we are in the case $B = B(k, h, m, \mu)$ with $h = c_{12}(k, \lambda, \varepsilon)$ where $\varepsilon > 0$. Suppose that

$$s = c_2(k)c_{14}(k, h, \mu) = nu,$$

say. After a change of notation, (1.4) becomes

$$\sum_{i=1}^n (a_{i1}x_{i1}^k + \dots + a_{iu}x_{iu}^k - b_{i1}y_{i1}^k - \dots - b_{iu}y_{iu}^k) = 0. \quad (3.7)$$

For each $i, 1 \leq i \leq n$, the coefficients $a_{i1}, \dots, a_{iu}, b_{i1}, \dots, b_{iu}$ satisfy the conditions of Proposition 2. Hence there are nonnegative $x'_{i1}, \dots, x'_{iu}, y'_{i1}, \dots, y'_{iu}$, not all zero, having

$$a_{i1}x'_{i1}^k + \dots + a_{iu}x'_{iu}^k - b_{i1}y'_{i1}^k - \dots - b_{iu}y'_{iu}^k = z_i \quad (3.8)$$

with $\max(x'_{i1}, \dots, x'_{iu}, y'_{i1}, \dots, y'_{iu}) \leq m^{(1/k)+20\mu}$ and $0 \leq z_i \leq m^{6\mu}$. No. Hold it! Keep $0 \leq z_i \leq m^{6\mu}$ for $i = 1, \dots, n-1$, but ask for $-m^{6\mu} \leq z_n \leq 0$. This is not asking for too much, in view of the symmetry in the + and - terms in (3.8). If some $z_i = 0$, we get a small solution of (3.7) straightaway. If $z_1, \dots, z_n$ are each nonzero, then Pitman's estimate (1.2) gives nonnegative $w_1, \dots, w_n$, not all zero, with

$$z_1 w_1^k + \dots + z_n w_n^k = 0$$

having $\max(w_1, \dots, w_n) \leq c_4(k)m^{6\mu c_5(k)}$. Putting $x_{ij} = w_i x'_{ij}, y_{ij} = w_i y'_{ij}$ ($1 \leq i \leq n, 1 \leq j \leq u$) we obtain a nontrivial solution of (3.7) with

$$\max(x_{ij}, y_{ij}) \leq c_4(k)m^{(1/k)+20\mu+6c_5(k)\mu} \leq m^\lambda$$

if m is large. Thus Proposition 1 is true for λ' .

4. Weyl's inequality. Write $e(x) = e^{2\pi i x}$.

LEMMA 1. *Suppose that*

$$|\alpha - u/q| < 1/q^2 \quad \text{where } q > 0, (u, q) = 1.$$

Then for $\eta > 0$,

$$\left| \sum_{x=1}^N e(\alpha x^k) \right| \leq c_{15}(k, \eta) N^{1+\eta} (N^{-1/K} + q^{-1/K} + (q/N^k)^{1/K})$$

where $K = 2^{k-1}$.

PROOF. This is the well-known "Weyl Inequality." See, e.g., [2, Lemma 1].

COROLLARY. *Suppose that $N \geq c_{16}(k, \eta)$, $C \geq N^{1-(1/K)+\eta}$ and*

$$\left| \sum_{x=1}^N e(\alpha x^k) \right| \geq C.$$

Then there is a natural

$$q \leq (N/C)^K N^\eta \quad \text{with } \|\alpha q\| \leq (N/C)^K N^{\eta-k},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

PROOF. We have $N^{k-\eta}(C/N)^K \geq N^{k-\eta-1+K\eta} \geq 1$. According to Dirichlet we may pick coprime q, u with

$$0 < q \leq N^{k-\eta} (C/N)^K$$

and

$$|\alpha q - u| = \|\alpha q\| \leq (N/C)^K N^{\eta-k}.$$

Now

$$N^{1+(\eta/2K)} (N^{-1/K} + (q/N^k)^{1/K}) \leq N^{1-(1/K)+(\eta/2K)} + CN^{-\eta/2K}$$

is of smaller order of magnitude than C if N is large. Thus by Lemma 1 (with $\eta/2K$ in place of η) we obtain that $N^{1+(\eta/2K)} q^{-1/K} \geq c_{17}(k, \eta) C$, whence that $q \leq (N/C)^K N^\eta$ if N is large.

5. Application of the Circle Method. Note that it will suffice to prove Proposition 2 for large m , say for $m \geq c_{18}(k, h, \mu)$. Put

$$A = [b^{1/k} m^{20\mu}], \quad B = [a^{1/k} m^{20\mu}], \quad H = [m^{6\mu}], \quad (5.1)$$

where $[\cdot]$ denotes the integer part. If m is sufficiently large, then

$$A \geq 2^{-1/k} b^{1/k} m^{20\mu}, \quad B \geq 2^{-1/k} a^{1/k} m^{20\mu}. \quad (5.2)$$

Write Z for the number of solutions of (3.6) in integers $x_1, \dots, x_s, y_1, \dots, y_s$,

z subject to

$$1 \leq x_1, \dots, x_s \leq A, \quad 1 \leq y_1, \dots, y_s \leq B, \quad 1 \leq z \leq H. \quad (5.3)$$

We will show that under the assumptions made in the proposition, Z is positive; in fact we will show that Z is at least of the order of magnitude of $HA^s B^s a^{-1} b^{-1} m^{-20k\mu}$.

Recall the definition (3.1) of ν and pick $\eta > 0$ with

$$\eta \leq 1/2K, \quad \eta(1 + 20\mu) < \frac{1}{2}\nu, \quad (5.4)$$

and pick s so large that

$$s > (6K/\nu) + h. \quad (5.5)$$

All of the parameters h, μ, ν, η, s will be fixed from now on. We shall employ the O -notation or $\ll$ notation with the understanding that the implicit constants may depend on k, h, μ, ν, η, s , but they will be independent of $a_1, \dots, a_s, b_1, \dots, b_s, a, b, m$. We are going to show that the hypotheses of Proposition 2 imply

$$Z \gg HA^s B^s a^{-1} b^{-1} m^{-20k\mu}. \quad (5.6)$$

The number Z will be estimated by the Circle Method. Note that this method has already been used implicitly, via Pitman's estimate (1.2). We have

$$Z = \int_0^1 f(\alpha) d\alpha \quad (5.7)$$

where

$$f(\alpha) = \sum_{z=1}^H \sum_{x_1=1}^A \cdots \sum_{x_s=1}^A \sum_{y_1=1}^B \cdots \sum_{y_s=1}^B e(\alpha(a_1 x_1^k + \cdots + a_s x_s^k - b_1 y_1^k - \cdots - b_s y_s^k - z)). \quad (5.8)$$

We define the *major arcs* to be the intervals modulo 1 of the type

$$\mathfrak{M}_{qu}: |\alpha - u/q| < a^{-1} b^{-1} m^{-16k\mu}$$

where $q < m^\mu$ and $(q, u) = 1$.

LEMMA 2. Suppose that $|f(\alpha)| \geq HA^s B^s m^{-3}$. Then α lies in a major arc.

PROOF. The inequality of the hypothesis implies that

$$|S_1(\alpha)| \cdots |S_s(\alpha)| |T_1(\alpha)| \cdots |T_s(\alpha)| \geq A^s B^s m^{-3}, \quad (5.9)$$

where

$$S_i(\alpha) = \sum_{x=1}^A e(\alpha a_i x^k), \quad T_i(\alpha) = \sum_{y=1}^B e(\alpha b_i y^k). \quad (5.10)$$

If, say, $|S_1(\alpha)| \geq \dots \geq |S_s(\alpha)|$, then the left-hand side of (5.9) is

$$< |S_h(\alpha)|^{s-h+1} A^{h-1} B^s,$$

so that (5.9) yields

$$|S_i(\alpha)| \geq |S_h(\alpha)| \geq Am^{-3/(s-h+1)} = C, \quad \text{say} \quad (i = 1, \dots, h).$$

Observe that

$$m^{3/(s-h+1)} \leq A^{1/6\mu(s-h+1)} \leq A^{(1/2K)} \leq A^{(1/K)-\eta}$$

by (5.2), (5.4), (5.5). We may apply the Corollary to Lemma 1 to each of the sums $S_1(\alpha), \dots, S_h(\alpha)$ to obtain natural numbers $p_1, \dots, p_h$ with

$$p_i \leq m^{3K/(s-h+1)} A^\eta, \quad \|\alpha a_i p_i\| \leq m^{3K/(s-h+1)} A^{\eta-k} \quad (i = 1, \dots, h).$$

Using (5.4) and (5.5) again we get

$$p_i \leq m^\nu \quad (i = 1, \dots, h), \quad (5.11)$$

$$\|\alpha a_i p_i\| \leq m^\nu A^{-k} \quad (i = 1, \dots, h). \quad (5.12)$$

Similarly, after a possible reordering of $b_1, \dots, b_s$, there are natural $q_1, \dots, q_h$ having

$$q_j \leq m^\nu \quad (j = 1, \dots, h), \quad (5.13)$$

$$\|\alpha b_j q_j\| \leq m^\nu B^{-k} \quad (j = 1, \dots, h). \quad (5.14)$$

There are integers $u_1, \dots, u_h, v_1, \dots, v_h$ with

$$\|\alpha a_i p_i\| = |\alpha a_i p_i - u_i| \quad (i = 1, \dots, h)$$

and

$$\|\alpha b_j q_j\| = |\alpha b_j q_j - v_j| \quad (j = 1, \dots, h).$$

Subtracting $a_i p_i$ times (5.14) from $b_j q_j$ times (5.12) and observing (5.11), (5.13) and (5.2), we obtain

$$\begin{aligned} |u_i b_j q_j - v_j a_i p_i| &\leq b m^\nu m^\nu A^{-k} + a m^\nu m^\nu B^{-k} \\ &\leq m^{2\nu-18\mu k} + m^{2\nu-18\mu k} < 1 \end{aligned}$$

if m is sufficiently large. Thus the $2h$ nonzero vectors $(a_i p_i, u_i)$ ($i = 1, \dots, h$) and $(b_j q_j, v_j)$ ($j = 1, \dots, h$) are proportional to each other. They are integer multiples of some vector (q, u) where $q > 0$ and q, u are coprime. Since q is a common divisor of $a_1 p_1, \dots, a_h p_h, b_1 q_1, \dots, b_h q_h$, condition (3.4) of case B yields $q < m^\mu$. If, say, the vector $(a_i p_i, u_i)$ is l_i times (q, u) , then $l_i \geq \frac{1}{2} a q^{-1}$, whence

$$\begin{aligned} |\alpha q - u| &= l_i^{-1} |\alpha a_i p_i - u_i| \leq 2a^{-1} q \|\alpha a_i p_i\| \leq 2a^{-1} q m^\nu A^{-k} \\ &\leq 2qa^{-1} b^{-1} m^{\nu-18\mu k} < qa^{-1} b^{-1} m^{-16\mu k} \end{aligned}$$

if m is large. Thus α lies in $\mathfrak{M}_{qu}$.

6. The major arcs. Since the major arcs do not overlap, and from Lemma 2, we obtain

$$Z = \sum_{q < m^\mu} \sum_{\substack{u=1 \\ (u,q)=1}}^q \int_{\mathfrak{M}_{qu}} f(\alpha) d\alpha + O(HA^s B^s m^{-3}). \quad (6.1)$$

LEMMA 3. For $\alpha = u/q + \beta \in \mathfrak{M}_{qu}$ we have

$$S_i(\alpha) = q^{-1} \hat{S}_i(u/q) I_i(\beta) + O(m^{5k_\mu}) \quad (i = 1, \dots, s) \quad (6.2)$$

where

$$\hat{S}_i(u/q) = \sum_{x=1}^q e\left(\frac{a_i u}{q} x^k\right) \quad \text{and} \quad I_i(\beta) = \int_0^A e(a_i \beta \xi^k) d\xi.$$

PROOF. Write $x = qy + z$. Then

$$S_i(\alpha) = \sum_{z=1}^q e\left(\frac{a_i u}{q} z^k\right) \sum_y e(a_i \beta (qy + z)^k), \quad (6.3)$$

where the sum over y is over the integers in $1 \leq qy + z \leq A$. There will be a certain error if we replace the sum over y by the integral of $e(a_i \beta (q\zeta + z)^k)$ with respect to ζ , with the range of integration given by $0 \leq q\zeta + z \leq A$. The function

$$g(\zeta) = e(a_i \beta (q\zeta + z)^k)$$

has

$$|g'(\zeta)| \leq qa_i |\beta| A^{k-1}, \quad |g(\zeta)| \leq 1$$

in this range, and this range is an interval of length A/q . Therefore

$$\begin{aligned} & \left| \sum_y e(a_i \beta (qy + z)^k) - \int e(a_i \beta (q\zeta + z)^k) d\zeta \right| \\ & \leq (A/q)(qa_i |\beta| A^{k-1}) + 3 \leq A^k a_i |\beta| + 3 \\ & < A^k a a^{-1} b^{-1} m^{-16k_\mu} + 3 < m^{4k_\mu} + 3. \end{aligned}$$

Taking the sum over z in (6.3) we get

$$S_i(\alpha) = \sum_{z=1}^q e\left(\frac{a_i u}{q} z^k\right) \int e(a_i \beta (q\zeta + z)^k) d\zeta + O(m^{5k_\mu}).$$

The change of variables $\xi = q\zeta + z$ yields the desired result.

In analogy to Lemma 3 we obtain

$$T_i(\alpha) = q^{-1} \hat{T}_i(u/q) J_i(\beta) + O(m^{5k_\mu}) \quad (i = 1, \dots, s) \quad (6.4)$$

where $\hat{T}_i, J_i$ are defined in the obvious way.

LEMMA 4. If $\mathfrak{M}$ is the totality of the major arcs, then

$$\int_{\mathfrak{M}} f(\alpha) d\alpha = A^s B^s a^{-1} b^{-1} m^{-20k\mu} \mathfrak{S}(m^\mu, H) \mathfrak{Z}(m^{4k\mu}) \\ + O(H A^s B^s a^{-1} b^{-1} m^{-22k\mu}),$$

where the “singular series”

$$\mathfrak{S}(m^\mu, H) = \sum_{z=1}^H \sum_{q < m^\mu} \sum_{\substack{u=1 \\ (q,u)=1}}^q q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \\ \hat{S}_s\left(\frac{u}{q}\right) \hat{T}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) e\left(-\frac{u}{q} z\right),$$

and the “singular integral”

$$\mathfrak{Z}(m^{4k\mu}) = \int_{|\beta| < m^{4k\mu}} \prod_{i=1}^s \left(\int_0^1 e(\rho_i \xi_i^k \beta) d\xi_i \right) \prod_{j=1}^s \left(\int_0^1 e(-\sigma_j \zeta_j^k \beta) d\zeta_j \right) d\beta,$$

for certain constants $\rho_1, \dots, \rho_s, \sigma_1, \dots, \sigma_s$ in the interval $[\frac{1}{4}, 1]$.

PROOF. The integral in question is

$$\sum_{z=1}^H \sum_{q < m^\mu} \sum_{\substack{u=1 \\ (q,u)=1}}^q \int_{\mathfrak{M}_{qu}} S_1(\alpha) \cdots S_s(\alpha) T_1(\alpha) \cdots T_s(\alpha) e(-z\alpha) d\alpha. \quad (6.5)$$

If $\alpha = u/q + \beta$ lies in $\mathfrak{M}_{qu}$, then Lemma 3 and the trivial estimates $|I_i(\beta)| < A$, $|J_i(\beta)| < B$ yield

$$S_1(\alpha) \cdots S_s(\alpha) T_1(\alpha) \cdots T_s(\alpha) \\ = q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{S}_s\left(\frac{u}{q}\right) \hat{T}_1\left(\frac{u}{q}\right) \cdots \\ \hat{T}_s\left(\frac{u}{q}\right) I_1(\beta) \cdots I_s(\beta) J_1(\beta) \cdots J_s(\beta) \\ + O(A^s B^s \max(m^{5k\mu} A^{-1}, m^{5k\mu} B^{-1})).$$

The error term here is $O(A^s B^s m^{-15k\mu})$, and since $\mathfrak{M}_{qu}$ is of length $2a^{-1}b^{-1}m^{-16k\mu}$, the integral over $\mathfrak{M}_{qu}$ in (6.5) is

$$q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) e\left(-\frac{u}{q} z\right) \int_{|\beta| < a^{-1}b^{-1}m^{-16k\mu}} I_1(\beta) \\ \cdots J_s(\beta) e(-\beta z) d\beta + O(A^s B^s a^{-1}b^{-1}m^{-30k\mu}). \quad (6.6)$$

In the integral in (6.6) we replace $e(-\beta z)$ by 1. The error is

$$\ll A^s B^s z (a^{-1} b^{-1} m^{-16k\mu})^2 \ll A^s B^s H a^{-2} b^{-2} m^{-32k\mu} \ll A^s B^s a^{-1} b^{-1} m^{-25k\mu}.$$

Thus the integral over $\mathfrak{M}_{qu}$ in (6.5) is

$$q^{-2s} \hat{S}_1\left(\frac{u}{q}\right) \cdots \hat{T}_s\left(\frac{u}{q}\right) \int_{|\beta| < a^{-1} b^{-1} m^{-16k\mu}} I_1(\beta) \cdots J_s(\beta) d\beta + O(A^s B^s a^{-1} b^{-1} m^{-25k\mu}). \quad (6.7)$$

To evaluate the integral in (6.7), put $\xi_i = A \xi'_i$ ($i = 1, \dots, s$), $\zeta_i = B \zeta'_i$ ($i = 1, \dots, s$) and $\beta = a^{-1} b^{-1} m^{-20k\mu} \beta'$. Then

$$a_i \beta \xi_i^k = (a_i A^k / abm^{20k\mu}) \beta' \xi_i'^k = \rho_i \beta' \xi_i'^k \quad (i = 1, \dots, s),$$

say, where by (5.2), $\frac{1}{4} \leq \rho_i \leq 1$. Similarly, $-b_i \beta \zeta_i^k = -\sigma_i \beta' \zeta_i'^k$. The integral in (6.7) becomes

$$A^s B^s a^{-1} b^{-1} m^{-20k\mu} \mathfrak{Z}(m^{4k\mu})$$

and the integral over $\mathfrak{M}_{qu}$ in (6.5) turns out to be

$$A^s B^s a^{-1} b^{-1} m^{-20k\mu} q^{-2s} \hat{S}_1(u/q) \cdots \hat{T}_s(u/q) e(-uz/q) \mathfrak{Z}(m^{4k\mu}) + O(A^s B^s a^{-1} b^{-1} m^{-25k\mu}).$$

Taking the sum over z, q, u in (6.5) we obtain Lemma 4.

7. The singular integral. We have

$$\begin{aligned} \int_0^1 e(\rho_i \xi_i^k \beta) d\xi_i &= k^{-1} \rho_i^{-1/k} \int_0^{\rho_i} \varphi_i^{-1+(1/k)} e(\varphi_i \beta) d\varphi_i \\ &= k^{-1} (\rho_i \beta)^{-1/k} \int_0^{\rho_i \beta} \varphi_i^{-1+(1/k)} e(\varphi_i) d\varphi_i. \end{aligned} \quad (7.1)$$

The last integral is bounded as a function of the upper limit of integration so that the integral on the left is $\ll \beta^{-1/k}$. It follows that as a function of m ,

$$\mathfrak{Z}(m^{4k\mu}) = \mathfrak{Z}(\infty) + o(1), \quad (7.2)$$

where $\mathfrak{Z}(\infty)$ is as $\mathfrak{Z}(m^{4k\mu})$, but with the integral over β extended over the real line. Using the middle expression in (7.1) we get

$$\begin{aligned} \mathfrak{Z}(\infty) &= k^{-2s} (\rho_1 \cdots \rho_s)^{-1/k} \\ &\quad \cdot \int_{-\infty}^{\infty} d\beta \int_0^{\rho_1} d\varphi_1 \cdots \int_0^{\sigma_s} d\psi_s (\varphi_1 \cdots \varphi_s \psi_1 \cdots \psi_s)^{-1+(1/k)} \\ &\quad \cdot e((\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_s) \beta) \\ &= k^{-2s} (\rho_1 \cdots \rho_s)^{-1/k} \\ &\quad \cdot \lim_{\omega \rightarrow \infty} \int_0^{\rho_1} d\varphi_1 \cdots \int_0^{\sigma_s} d\psi_s (\varphi_1 \cdots \varphi_s \psi_1 \cdots \psi_s)^{-1+(1/k)} \\ &\quad \cdot \frac{\sin 2\pi\omega(\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_s)}{\pi(\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_s)}, \end{aligned}$$

as in [2, p. 27]. Continuing as in [2] we get

$$\mathfrak{S}(\infty) = k^{-2s} (\rho_1 \cdots \sigma_s)^{-1/k} \lim_{\omega \rightarrow \infty} \int_{-s}^s \Omega(\omega) \frac{\sin 2\pi\omega u}{\pi u} d\omega, \quad (7.3)$$

where

$$\begin{aligned} \Omega(\omega) = & \int_0^{\rho_1} d\varphi_1 \cdots \int_0^{\rho_s} d\varphi_s \int_0^{\sigma_1} d\psi_1 \cdots \int_0^{\sigma_{s-1}} d\psi_{s-1} \\ & u < \varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_{s-1} < u + \sigma_s \\ & \cdot (\varphi_1 \cdots \varphi_s \psi_1 \cdots \psi_{s-1} (\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots \\ & \quad - \psi_{s-1} - u))^{-1+(1/k)}. \end{aligned}$$

The limit in (7.3) equals

$$\begin{aligned} \Omega(0) \geq & \int_0^{1/4} d\varphi_1 \cdots \int_0^{1/4} d\varphi_s \int_0^{1/4} d\psi_1 \cdots \int_0^{1/4} d\psi_{s-1} \\ & 0 < \varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_{s-1} < 1/4 \\ & \cdot (\varphi_1 \cdots \varphi_s \psi_1 \cdots \psi_{s-1} (\varphi_1 + \cdots + \varphi_s - \psi_1 - \cdots - \psi_{s-1}))^{-1+(1/k)} \\ & \gg 1. \end{aligned}$$

Combining our estimates we find that for m sufficiently large,

$$\mathfrak{S}(m^{4k\mu}) \gg 1. \quad (7.4)$$

8. The singular series.

$$\begin{aligned} \mathfrak{S}(m^\mu, H) = & \sum_{z=1}^H \sum_{q < m^\mu} \sum_{\substack{u=1 \\ (u,q)=1}}^q \sum_{x_1=1}^q \cdots \\ & \sum_{y_s=1}^q q^{-2s} e\left(\frac{u}{q} (a_1 x_1^k + \cdots - b_s y_s^k - z)\right). \end{aligned}$$

The summands with $q = 1$ give the contribution H .

When $q > 1$,

$$\sum_{z=1}^H e(-uz/q) \ll q,$$

so that the summands with fixed $q > 1$ contribute $\ll q^2$. Since $\sum q^2$ over $q < m^\mu$ is $\ll m^{3\mu}$, we obtain

$$\mathfrak{S}(m^\mu, H) = H + O(m^{3\mu}) \gg H. \quad (8.1)$$

9. Conclusion. Combining (6.1), Lemma 4, (7.4) and (8.1) we get

$$Z \gg HA^s B^s a^{-1} b^{-1} m^{-20k\mu} + O(HA^s B^s m^{-3} + HA^s B^s a^{-1} b^{-1} m^{-22k\mu}).$$

Since $m^3 \geq abm \geq abm^{22k\mu}$ by (3.5), the error term here is smaller than the main term, and (5.6) follows.

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