SOME NEW CONSTRUCTIONS AND ESTIMATES IN THE PROBLEM OF LEAST AREA¹

BY

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ABSTRACT. Surfaces of least k dimensional area in $\mathbf{R}^n$ are constructed by minimization of the n dimensional volume of suitably thickened sets subject to a homological constraint. Specifically, let $1 < k < n$ be integers and $B \subset \mathbf{R}^n$ be compact and $k-1$ rectifiable. Let G be a compact abelian group and L be a subgroup of the Čech homology group $H_{k-1}(B; G)$ (in case $k = 1$, suppose, additionally, L is contained in the kernel of the usual augmentation map). J. F. Adams has defined what it means for a compact set $X \subset \mathbf{R}^n$ to *span* L . Using also a natural notion of what it means for a compact set to be ϵ -thick, we show that, for each $\epsilon > 0$, there exists an ϵ -thick set which minimizes n dimensional volume subject to the requirement that it span L . Our main result is that as ϵ approaches 0 a subsequence of the above volume minimizing sets converges in the Hausdorff distance topology to a set, X , which minimizes k dimensional area subject to the requirement that it span L . It follows, of course, from the regularity results of Reifenberg or Almgren that, except for a compact singular set of zero k dimensional measure, X is a real analytic minimal submanifold of $\mathbf{R}^n$.

1. Introduction. Consider a compact $k-1$ rectifiable set $B \subset \mathbf{R}^n$ ($1 < k \leq n$), a compact abelian group G , and a subgroup L of the Čech homology group $H_{k-1}(B; G)$ (in case $k = 1$, one supposes, additionally, L is contained in the kernel of the usual augmentation map). A compact set $X \subset \mathbf{R}^n$ is said to *span* L if and only if $B \subset X$ and $\iota_*(L) = 0$ where $\iota: B \rightarrow X$ is the inclusion map. Typically one seeks a compact set $X \subset \mathbf{R}^n$ spanning L such that

$$\mathcal{H}^k(X) = \inf\{\mathcal{H}^k(Y) : Y \subset \mathbf{R}^n \text{ is compact and spans } L\};$$

on the other hand, it is usually considerably easier to find a compact set $X \subset \mathbf{R}^n$ spanning L such that

$$\mathcal{L}^n\{x : \text{dist}(x, X) \leq \epsilon\} = \inf\{\mathcal{L}^n\{y : \text{dist}(y, Y) \leq \epsilon\} :$$

$$Y \subset \mathbf{R}^n \text{ is compact and spans } L\}; \quad (*)$$

here $\epsilon > 0$ and $\mathcal{H}^k$ and $\mathcal{L}^n$ denote Hausdorff k dimensional measure on $\mathbf{R}^n$ and Lebesgue n dimensional measure, respectively. The following theorem

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relates these two procedures. Let $\epsilon_1, \epsilon_2, \epsilon_3, \dots > 0$ with $\lim_i \epsilon_i = 0$.

THEOREM (1) *For each $i = 1, 2, 3, \dots$ there exists a compact set $X_i \subset \mathbf{R}^n$ which spans L and satisfies $(*)$ with ϵ replaced by ϵ_i .*

(2) For $X_1, X_2, X_3, \dots$ chosen as in (1) there exists a subsequence $i(1), i(2), i(3), \dots$ of $1, 2, 3, \dots$ and a compact set $X \subset \mathbf{R}^n$ such that $\lim_j X_{i(j)} = X$ in the Hausdorff distance topology, X spans L , and

$$\mathcal{H}^k(X) = \inf \{ \mathcal{H}^k(Y) : Y \subset \mathbf{R}^n \text{ is compact and spans } L \}.$$

It then follows from the regularity results of Reifenberg or Almgren that, except for a compact singular set of zero $\mathcal{H}^k$ measure, X is a real analytic minimal submanifold of $\mathbf{R}^n$.

The central tool used in proving the above theorem is a correspondence between ϵ -thick sets and polyhedral complexes (as defined in §5.1) based on Besicovitch's covering theorem. Polyhedral complexes, in particular, admit useful estimates on $\mathcal{L}^n \{x: \text{dist}(x, \cdot) \leq \epsilon\}$ and also admit various intricate geometric constructions (see, for example, the Isoperimetric Theorem 5.3).

2. Definitions. Except when otherwise stated, we will follow the notation and terminology of [FH]. Denote by n and k integers with $1 < k < n$ and by G a compact, abelian group.

(1) Set $\mathbf{R}^+ = \mathbf{R} \cap \{t: t > 0\}$.

(2) For each positive integer m , each $a \in \mathbf{R}^m$, and each $r \in \mathbf{R}^+$ define the closed [respectively, open] ball of radius r centered at a , $\mathbf{B}^m(a, r)$ [respectively, $\mathbf{U}^m(a, r)$], by setting

$$\mathbf{B}^m(a, r) = \mathbf{R}^m \cap \{x: |x - a| \leq r\}$$

[respectively,

$$\mathbf{U}^m(a, r) = \mathbf{R}^m \cap \{x: |x - a| < r\}]$$

and the sphere of radius r centered at a , $S^{m-1}(a, r)$, by setting

$$S^{m-1}(a, r) = \mathbf{R}^m \cap \{x: |x - a| = r\};$$

also, set

$$\text{Ct}[\mathbf{B}^m(a, r)] = \text{Ct}[\mathbf{U}^m(a, r)] = a,$$

and for each $A \subset \mathbf{R}^m$ and each (closed or open) ball $B \subset \mathbf{R}^m$, say B is centered in A if and only if $\text{Ct}(B) \in A$.

(3) Set

$$\mathcal{C}_0 = \{C: C \text{ is a compact subset of } \mathbf{R}^n\} \quad \text{and} \quad \mathcal{C} = \mathcal{C}_0 \sim \{\emptyset\}.$$

(4) Define the Hausdorff distance $d: \mathcal{C} \times \mathcal{C} \rightarrow \mathbf{R}$ by setting

$$d(C_1, C_2) = \sup \{ \text{dist}(x, C_i): i = 1 \text{ and } x \in C_2, \\ \text{or } i = 2 \text{ and } x \in C_1 \}$$

for each $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$.

(5) For each $\varepsilon \in \mathbf{R}^+$, define $c_\varepsilon, e_\varepsilon: \mathcal{C}_0 \rightarrow \mathcal{C}_0$ by setting

$$c_\varepsilon(C) = \mathbf{R}^n \cap \{x: \text{dist}(x, \mathbf{R}^n \sim C) \geq \varepsilon\} \subset C$$

and

$$e_\varepsilon(C) = \mathbf{R}^n \cap \{x: \text{dist}(x, C) \leq \varepsilon\} \supset C$$

for each $C \in \mathcal{C}_0$.

(6) For each $\varepsilon \in \mathbf{R}^+$, set

$$\mathcal{C}_\varepsilon = \mathcal{C} \cap \{C: e_\varepsilon \circ c_\varepsilon(C) = C\};$$

the elements of $\mathcal{C}_\varepsilon$ are called ε -thick sets.

(7) For each positive integer m , each $A \subset \mathbf{R}^m$, and each $B \subset \mathbf{R}^m$ set

$$A \otimes B = \{ta + (1-t)b: a \in A, b \in B, 0 \leq t \leq 1\}$$

and set $\text{Cv}(A)$ equal to the convex hull of A .

(8) For each positive integer m and each sequence $x_0, x_1, x_2, \dots, x_m \in \mathbf{R}^n$ set

$$\begin{aligned} \langle x_0, x_1, x_2, \dots, x_m \rangle \\ = \left\{ \sum_i \lambda_i x_i: 0 < \lambda_i \text{ for } i = 0, 1, 2, \dots, m \text{ and } \sum_i \lambda_i = 1 \right\} \end{aligned}$$

and set $\langle x_0 \rangle = \{x_0\}$.

(9) For each $A \subset \mathbf{R}^n$ and each

$$\mathcal{Q} \subset \{\mathbf{U}^n(x, r): x \in \mathbf{R}^n, r \in \mathbf{R}^+\}$$

with $A \subset \bigcup \mathcal{Q}$, set

$$\begin{aligned} \mathbf{N}(\mathcal{Q}, A) = \bigcup_{m \geq 0} \bigcup \left\{ \langle \text{Ct}(U_0), \text{Ct}(U_1), \text{Ct}(U_2), \dots, \text{Ct}(U_m) \rangle: \right. \\ \left. U_i \in \mathcal{Q} \text{ for } i = 0, 1, 2, \dots, m \text{ and } A \cap \bigcap_i U_i \neq \emptyset \right\}. \end{aligned}$$

(10) For each $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbf{R}^n$ set

$$\mathbf{m}(x) = \sup\{|x_i|: i = 1, 2, 3, \dots, n\}.$$

(11) Whenever $X, A \in \mathcal{C}_0$ with $A \subset X$ and q is a nonnegative integer, we denote by $H_q(X, A)$ the q dimensional Čech homology group of X relative to A with coefficients in G (see [ES, IX]). Each continuous mapping

$$f: (X, A) \rightarrow (Y, B),$$

$Y, B \in \mathcal{C}_0, B \subset Y$, induces a homomorphism

$$f_*: H_q(X, A) \rightarrow H_q(Y, B)$$

in the usual way. Set $H_q(X) = H_q(X, \emptyset)$ and let

$$\text{aug}(X): H_0(X) \rightarrow G$$

be the *augmentation map* induced by the map from X to the one point space.

(12) For each $B \in \mathcal{C}$ and each subgroup $L \subset H_{k-1}(B)$ we say that $X \in \mathcal{C}$ spans L if and only if $B \subset X$ and $\dot{\iota}_*(L) = 0$ where $\dot{\iota} : B \rightarrow X$ is the inclusion map. Denote by $\mathcal{C}(B, L)$ the collection of sets in $\mathcal{C}$ which span L and, for each $\varepsilon \in \mathbf{R}^+$, set

$$\mu_\varepsilon(B, L) = \inf\{\mathcal{L}^n \circ e_\varepsilon(X) : X \in \mathcal{C}(B, L)\},$$

$$\mathfrak{N}_\varepsilon(B, L) = \{X \in \mathcal{C}(B, L) : \mathcal{L}^n \circ e_\varepsilon(X) = \mu_\varepsilon(B, L)\}.$$

3. An existence theorem.

THEOREM. (1) For each $B \in \mathcal{C}$ and each subgroup L of $H_{k-1}(B)$, which in case $k = 1$ satisfies $L \subset \ker[\text{aug}(B)]$, $\mathcal{C}(B, L)$ is nonempty.

(2) For each $r \in \mathbf{R}^+$, $\mathcal{C} \cap \{C : C \subset \mathbf{B}^n(0, r)\}$ is compact in the Hausdorff distance topology.

(3) For each $\varepsilon \in \mathbf{R}^+$, $e_\varepsilon|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}_\varepsilon$ is continuous in the Hausdorff distance topology.

(4) For each $\varepsilon \in \mathbf{R}^+$, $\mathcal{L}^n : \mathcal{C}_\varepsilon \rightarrow \mathbf{R}^+$ is continuous in the Hausdorff distance topology.

(5) For each $B \in \mathcal{C}$ and each subgroup L of $H_{k-1}(B)$, $\mathcal{C}(B, L)$ is a closed subset of $\mathcal{C}$ in the Hausdorff distance topology.

(6) For each $\varepsilon \in \mathbf{R}^+$, each $B \in \mathcal{C}$, and each subgroup L of $H_{k-1}(B)$, which in case $k = 1$ satisfies $L \subset \ker[\text{aug}(B)]$, $\mathfrak{N}_\varepsilon(B, L)$ is nonempty.

PROOF. (1) Conclusion (1) follows from [RE, Lemma 2A] applied to $\{0\} \times B$.

(2) Conclusion (2) is contained in [FH, 2.10.21].

(3) First, we check that $e_\varepsilon(\mathcal{C}) \subset \mathcal{C}_\varepsilon$. Let $C \in \mathcal{C}$ be arbitrary. We must show that

$$e_\varepsilon \circ c_\varepsilon[e_\varepsilon(C)] = e_\varepsilon(C).$$

If $x \in e_\varepsilon \circ c_\varepsilon[e_\varepsilon(C)]$, then there is $y \in c_\varepsilon[e_\varepsilon(C)]$ with $|x - y| \leq \varepsilon$, but, since $y \in c_\varepsilon[e_\varepsilon(C)]$, we have $\mathbf{B}^n(y, \varepsilon) \subset e_\varepsilon(C)$, so $x \in e_\varepsilon(C)$; on the other hand, if $x \in e_\varepsilon(C)$, then there is $y \in C$ with $|x - y| \leq \varepsilon$, so $\mathbf{B}^n(y, \varepsilon) \subset e_\varepsilon(C)$ and thus $y \in c_\varepsilon[e_\varepsilon(C)]$, so $x \in e_\varepsilon \circ c_\varepsilon[e_\varepsilon(C)]$.

Next, we check the continuity of $e_\varepsilon|_{\mathcal{C}}$. Assume $C, D \in \mathcal{C}$ and $b \in e_\varepsilon(C)$. Then there exist $c \in C$ and $d \in D$ such that $|b - c| \leq \varepsilon$ and $|c - d| \leq d(C, D)$. In the line segment bd one can clearly choose $e \in e_\varepsilon(D)$ with $|b - e| \leq d(C, D)$. Similarly, for each $e' \in e_\varepsilon(D)$ there exists $b' \in e_\varepsilon(C)$ with $|b' - e'| \leq d(C, D)$. Thus we have

$$d[e_\varepsilon(C), e_\varepsilon(D)] \leq d(C, D).$$

(4) Assume $C \in \mathcal{C}_\varepsilon$ and let $0 < \lambda < 1$ be arbitrary. Set $\tau_1 = \lambda^{1/3}$ and $\tau_2 = 2^{-1}\tau_1$. Choose an integer m such that $\sum_{i=0}^m (\tau_1/2)^i > 2\lambda^{1/3}$, τ_3 such that

$\tau_2 < \tau_3 < 2^{-1}$, and $\sigma_0 \in \mathbf{R}^+$ such that for each $(p, q) \in \mathbf{R}^n \times \mathbf{R}^n$, with $|p - q| = \varepsilon$, and each $0 < \sigma < \sigma_0$ one has

$$\mathcal{L}^n[\mathbf{B}^n(p, \sigma) \cap \mathbf{B}^n(q, \varepsilon)] \geq \tau_3 \alpha(n) \sigma^n.$$

Set

$$U^+ = U^n(0, 1) \cap \{x: x \cdot \mathbf{e}_n > 0\}$$

and

$$U^- = U^n(0, 1) \cap \{x: x \cdot \mathbf{e}_n < 0\}.$$

By [FH, 2.8.15] there exists a finite disjointed family F of closed balls contained in U^+ such that

$$\mathcal{L}^n(\cup F) \geq \tau_1 \mathcal{L}^n(U^+).$$

Set $\alpha = \inf\{2^{-1} \text{diam } S: S \in F\}$; note that $\alpha < 2^{-1}$.

Since $C = e_\varepsilon \circ c_\varepsilon(C)$, by [FH, 3.2.34], one has

$$\begin{aligned} \mathcal{L}^n(C) &\leq \mathcal{L}^n[c_\varepsilon(C)] + \lim_{t \rightarrow \varepsilon^+} \int_0^t \mathcal{H}^{n-1}\{x: \text{dist}[x, c_\varepsilon(C)] = s\} d\mathcal{L}_s^1 \\ &= \mathcal{L}^n[c_\varepsilon(C)] + \int_0^\varepsilon \mathcal{H}^{n-1}\{x: \text{dist}[x, c_\varepsilon(C)] = s\} d\mathcal{L}_s^1 \\ &= \mathcal{L}^n(\text{Int } C) \leq \mathcal{L}^n(C). \end{aligned}$$

Thus $\mathcal{L}^n(C) = \mathcal{L}^n(\text{Int } C)$ and, applying [FH, 2.8.15], one obtains a finite disjointed family, G_0 , of closed balls contained in $\text{Int } C$ of radius less than σ_0 such that

$$\mathcal{L}^n(\cup G_0) \geq \tau_1 \mathcal{L}^n(C).$$

Set $\beta_0 = \inf\{2^{-1} \text{diam } S: S \in G_0\}$. Choose $\delta_0 \in \mathbf{R}^+$ such that for each $p \in \mathbf{R}^n$, each $q \in \mathbf{R}^n$, each σ with $\alpha^m \beta_0 \leq \sigma < \sigma_0$, and each δ with $0 < \delta < \delta_0$: If $|p - q| = \varepsilon + \delta$, then

$$\mathcal{L}^n[\mathbf{B}^n(p, \sigma) \cap \mathbf{B}^n(q, \varepsilon)] \geq \tau_2 \alpha(n) \sigma^n.$$

Now, fix $D \in \mathcal{C}_\varepsilon$ with $d(C, D) < \delta_0$. We will construct inductively certain families, G_i , of closed balls contained in $\text{Int } C$ and certain Borel subsets, $E(i)$, of $D \cap \text{Int } C$, for $0 \leq i \leq m+1$, such that

- (a) G_i is disjointed;
- (b) if $S \in G_i$ and $r = 2^{-1} \text{diam } S$, then $\alpha^i \beta_0 \leq r < \sigma_0$;
- (c) if $1 \leq i \leq m+1$, then

$$\cup G_i \subset \cup G_{i-1} \quad \text{and} \quad \mathcal{L}^n(\cup G_i) \geq \tau_2 \mathcal{L}^n(\cup G_{i-1});$$

- (d) $E(0) = \emptyset$ and if $1 \leq i \leq m+1$, then

$$E(i) \subset \cup G_{i-1} \sim \cup G_i \quad \text{and} \quad \mathcal{L}^n(E(i)) \geq \tau_2 \mathcal{L}^n(\cup G_{i-1}).$$

Suppose G_i and $E(i)$ have been constructed and $0 \leq i \leq m$. For each $S \in G_i$, because $\text{Ct}(S) \in C$, $D \in \mathcal{C}_\varepsilon$, $d(C, D) < \delta_0$, and

$$\alpha^m \beta_0 \leq 2^{-1} \text{diam } S < \sigma_0,$$

there exists an isometry h_S of $\mathbf{R}^n$ so that

$$h_S \circ \mu_r[\mathbf{B}^n(0, 1)] = S \quad \text{and} \quad \mathcal{L}^n(E_S) \geq \tau_2 \alpha(n) r^n,$$

where

$$E_S = h_S \circ \mu_r(U^-) \cap D$$

and $r = 2^{-1} \text{diam } S$. Set

$$E(i+1) = \bigcup \{E_S : S \in G_i\}$$

and

$$G_{i+1} = \bigcup \{ \{h_S \circ \mu_r(T) : T \in F\} : S \in G_i \}.$$

One sees easily that (a)–(d) above are satisfied. Since, in particular, the sets $E(i)$ are pairwise disjoint, one has

$$\begin{aligned} \mathcal{L}^n(D \cap \text{Int } C) &\geq \sum_{i=1}^{m+1} \mathcal{L}^n(E(i)) \geq \tau_2 \sum_{i=0}^m \left(\frac{\tau_1}{2}\right)^i \mathcal{L}^n(\bigcup G_0) \\ &\geq \mathcal{L}^n(C) \tau_1 \tau_2 \sum_{i=0}^m \left(\frac{\tau_1}{2}\right)^i > \lambda \mathcal{L}^n(C). \end{aligned}$$

The above shows that if $C_1, C_2, C_3, \dots \in \mathcal{C}_\epsilon$ with $\lim_i C_i = C$ in the Hausdorff distance topology, then

$$\mathcal{L}^n(C) \leq \liminf_i \mathcal{L}^n(C_i).$$

Since $\mathcal{L}^n(C) = \lim_{\sigma \rightarrow 0+} \mathcal{L}^n \circ e_\sigma(C)$ and $d(C, C_i) < \sigma$ implies $C_i \subset e_\sigma(C)$, one sees that

$$\mathcal{L}^n(C) \geq \limsup_i \mathcal{L}^n(C_i).$$

(5) Assume $C \in \mathcal{C}$ and $C_1, C_2, C_3, \dots \in \mathcal{C}(B, L)$ with $\lim_i C_i = C$ in the Hausdorff distance topology. One checks that, for $j = 1, 2, 3, \dots$,

$$D_j = C \cup \bigcup_{i \geq j} C_i \in \mathcal{C}$$

and notes, by [RE, Lemma 7A], that $D_j \in \mathcal{C}(B, L)$. Also one has

$$D_1 \supset D_2 \supset D_3 \supset \dots \quad \text{and} \quad \bigcap_j D_j = C,$$

so it follows from [RE, Lemma 21A] that $C \in \mathcal{C}(B, L)$.

(6) Select $Y \in \mathcal{C}(B, L)$ with $\mathcal{L}^n \circ e_\epsilon(Y) \leq 2\mu_\epsilon(B, L)$, which by (1) we can do. Then there exists an integer i with

$$1 \leq i \leq 1 + 2\alpha(n)^{-1} \epsilon^{-n} \mu_\epsilon(B, L)$$

such that

$$Y \cap \mathbf{S}^{n-1}(0, r + 2\epsilon i) = \emptyset.$$

where $r = d(\{0\}, B)$.

By a direct sum theorem for Čech homology (see [ES, I, 13.2 and X]) one has

$$Y \cap \mathbf{B}^n(0, r + 2\epsilon i) \in \mathcal{C}(B, L).$$

Conclusion (6) now follows from (2)–(5).

4. The one dimensional case.

4.1. LEMMA. Assume m is a positive integer $\gamma \in \mathbf{R}^+$, $\{C, C_1, C_2, C_3, \dots\} \subset \mathcal{C}$, $\{\epsilon(1), \epsilon(2), \epsilon(3), \dots\} \subset \mathbf{R}^+$ with $\lim_i \epsilon(i) = 0$, $\{\delta(1), \delta(2), \delta(3), \dots\} \subset \mathbf{R}^+ \cup \{0\}$ with $\lim_i \delta(i) = 0$, and that, for each $i = 1, 2, 3, \dots$, C_i consists of no more than m connected components and $\delta(i) = d(C, C_i)$.

(1) There are no more than m connected components in C .

(2) If one has

$$\mathcal{L}^n \circ e_{\epsilon(i)}(C_i) \leq \gamma [\epsilon(i)]^{n-1}$$

for $i = 1, 2, 3, \dots$, then $\mathcal{H}^1(C)$ is finite and C is 1-rectifiable.

(3) If $\mathcal{H}^1(C)$ is finite, then one has

$$\mathcal{H}^1(C) \leq \liminf_i \mathcal{L}^n \circ e_{\epsilon(i)}(C_i) \alpha(n-1)^{-1} \epsilon(i)^{1-n}.$$

PROOF. (1) Conclusion (1) is clear.

(2) By [FH, 2.8.14] there exists a positive integer b such that, for each $i = 1, 2, 3, \dots$, there exist disjointed families, $F_1, F_2, F_3, \dots, F_b$, of closed balls of radius $\epsilon(i)$ centered in C_i so that $C_i \subset \bigcup (\bigcup_j F_j)$. The hypothesis of (2) implies

$$\text{card}(F_j) \leq \gamma \alpha(n)^{-1} \epsilon(i)^{-1},$$

for $j = 1, 2, 3, \dots, b$. Set L_i equal to the union of all the line segments the endpoints of which are the centers of a pair of intersecting balls in $\bigcup_j F_j$. One observes that

$$\mathcal{H}^1(L_i) \leq 2 \cdot 3^n b^2 \gamma \alpha(n)^{-1}$$

because, for each $r \in \mathbf{R}^+$, each closed ball in $\mathbf{R}^n$ of radius r intersects no more than 3^n disjointed balls of radius r . Observe also that each L_i consists of no more than m connected components, $L_{i,1}, L_{i,2}, L_{i,3}, \dots, L_{i,m(i)}$, and that the L_i converge to C in the Hausdorff distance topology.

Set

$$K = \bigcup_{j=1}^m \{t: 2j \leq t \leq 2j+1\}.$$

For $i = 1, 2, 3, \dots$ and $j = 1, 2, 3, \dots, m(i)$, by [EH, Theorem 2] there exists a function $f_{i,j}: \{t: 2j \leq t \leq 2j+1\} \rightarrow \mathbf{R}^n$ with $f_{i,j}(\{t: 2j \leq t \leq 2j+1\}) = L_{i,j}$ and $\text{Lip}(f_{i,j}) \leq 2\mathcal{H}^1(L_{i,j})$. We then define $f_i: K \rightarrow \mathbf{R}^n$ by setting

$$f_i(t) = f_{i,j}(t), \quad \text{if } j = 1, 2, 3, \dots, m(i) \text{ and } 2j \leq t \leq 2j+1,$$

and

$$f_i(t) = f_{i,m(i)}(2m(i) + 1), \quad \text{if } 2m(i) + 1 \leq t.$$

We have

$$f_i(K) = L_i$$

and

$$\begin{aligned} \text{Lip}(f_i) &\leq \sup\{\text{diam } L_i\} \cup \{2\mathcal{H}^1(L_{i,j}): j = 1, 2, 3, \dots, m(i)\} \\ &\leq \sup\{\text{diam } L_i, 2\mathcal{H}^1(L_i)\}. \end{aligned}$$

The above estimate for $\mathcal{H}^1(L_i)$ gives a uniform bound for $\text{Lip}(f_i)$. Since $\bigcup_i L_i \cup C$ is compact, we see that a subsequence of $f_1, f_2, f_3, \dots$ converges to a Lipschitzian function f . It is clear then that $f(K) = C$.

(3) By (1) and [EH, Theorem 2], C is 1-rectifiable. Let C' be the set of $c \in C$ such that $\Theta^1(\mathcal{H}^1 \llcorner C, c) = 1$ and $\text{Tan}^1(\mathcal{H}^1 \llcorner C, c)$ is a 1 dimensional vectorsubspace of $\mathbf{R}^n$. By [FH, 3.2.19] we have $\mathcal{H}^1(C) = \mathcal{H}^1(C')$.

Let $0 < \sigma < 3^{-1}$ be arbitrary. We will show that for each $c \in C'$ there exists $r_0(c) > 0$ so that if $0 < r < r_0(c)$, then

$$\begin{aligned} \liminf_i \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(c, r)] \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ > 2(2 + 6^{-1}\sigma)^{-1}(1 - 2\sigma - 3\sigma^2)^{1/2} \mathcal{H}^1[C' \cap \mathbf{B}^n(c, r)] \end{aligned} \quad (\#)$$

holds. Once this is established, conclusion (3) is obtained as follows: By [FH, 2.8.15], we can find a disjointed family of closed balls, $\{\mathbf{B}^n(c_j, r_j): j = 1, 2, 3, \dots\}$, such that $c_j \in C'$, $0 < r_j < r_0(c_j)$, for $j = 1, 2, 3, \dots$, and

$$\mathcal{H}^1\left[C' \cap \bigcup_j \mathbf{B}^n(c_j, r_j)\right] = \mathcal{H}^1(C');$$

then we have

$$\begin{aligned} \liminf_i \mathcal{L}^n \circ e_{\varepsilon(i)}(C_i) \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ &> \liminf_i \mathcal{L}^n\left[e_{\varepsilon(i)}(C_i) \cap \bigcup_j \mathbf{B}^n(c_j, r_j)\right] \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ &= \liminf_i \sum_j \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(c_j, r_j)] \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ &> \sum_j \liminf_i \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(c_j, r_j)] \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ &> 2(2 + 6^{-1}\sigma)^{-1}(1 - 2\sigma - 3\sigma^2)^{1/2} \sum_j \mathcal{H}^1[C' \cap \mathbf{B}^n(c_j, r_j)] \\ &= 2(2 + 6^{-1}\sigma)^{-1}(1 - 2\sigma - 3\sigma^2)^{1/2} \mathcal{H}^1(C). \end{aligned}$$

Now, fix $c \in C'$. To simplify the notation we suppose $c = 0$ and that $\text{Tan}^1(\mathcal{H}^1 \llcorner C, c)$ is spanned by $\mathbf{e}_1$. Set

$$C'' = C \sim \{x: \{x\} \text{ is a connected component of } C\},$$

$$r_1 = \text{dist}(C'', C \sim C''),$$

$$r_2 = \inf\{2^{-1}\text{diam } K: K \text{ is a connected component of } C''\}.$$

Note that $C' \subset C''$, and if $x \in C''$ and $0 < r < r_2$, then, by [FH, 2.10.12],

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(x, r)] \geq r.$$

It is clear that we can pick $r_3 > 0$ so that $0 < r < r_3$ implies

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(0, r)] \leq (2 + 6^{-1}\sigma)r$$

and

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(0, r) \cap \{x: |\mathbf{q}(x)| \leq (2 + \sigma)^{-1}\sigma r\}] \geq (2 - 6^{-1}\sigma)r,$$

where $\mathbf{q}: \mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$ is defined by

$$\mathbf{q}(x_1, x_2, x_3, \dots, x_n) = (x_2, x_3, \dots, x_n).$$

Set

$$r_4 = \inf\{r_1, 2r_2\sigma^{-1}, 2r_3(1 + \sigma)^{-1}\}.$$

We claim that if $0 < r < r_4$, then

$$C \cap \mathbf{B}^n(0, r) \cap \{x: |\mathbf{q}(x)| > \sigma r\} = \emptyset.$$

To see this, suppose on the contrary that

$$x \in C \cap \mathbf{B}^n(0, r) \cap \{x: |\mathbf{q}(x)| > \sigma r\}.$$

Since $r < r_1$, we have $x \in C''$, and thus, since $2^{-1}\sigma r < r_2$,

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(x, 2^{-1}\sigma r)] \geq 2^{-1}\sigma r.$$

Also, we have

$$\mathbf{B}^n(x, 2^{-1}\sigma r) \subset \mathbf{B}^n[0, 2^{-1}r(2 + \sigma)] \cap \{x: |\mathbf{q}(x)| > 2^{-1}\sigma r\}.$$

It follows, since $2^{-1}r(2 + \sigma) < r_3$, that

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(0, 2^{-1}r(2 + \sigma))] \geq 2^{-1}\sigma r + (2 - 6^{-1}\sigma)2^{-1}r(2 + \sigma)$$

and

$$\mathcal{H}^1[C' \cap \mathbf{B}^n(0, 2^{-1}r(2 + \sigma))] \leq (2 + 6^{-1}\sigma)2^{-1}r(2 + \sigma),$$

which is a contradiction.

Next we note, using the fact that $\text{Tan}^1(\mathcal{H}^1 \llcorner C, 0)$ is the vector space spanned by $\mathbf{e}_1$, that there is $0 < r_0 < r_4$ so that if $0 < r < r_0$ and

$$-r(1 - \sigma^2)^{1/2} < t < r(1 - \sigma^2)^{1/2},$$

then

$$C \cap \mathbf{B}^n(0, r) \cap \{x: x \cdot \mathbf{e}_1 = t\} \neq \emptyset.$$

Now fix $0 < r < r_0$. Consider a positive integer i such that $\delta(i) < \sigma r$ and $\varepsilon(i) < \sigma r$. Set

$$a = r(1 - 2\sigma - 3\sigma^2)^{1/2}.$$

We claim that if

$$\begin{aligned} -a &\leq t_1 < t_2 \leq a, & t_2 - t_1 &> 2\delta(i), \\ C_i \cap \mathbf{B}^n(0, (1 - \sigma)r) \cap \{x: x \cdot \mathbf{e}_1 = t_1\} &= \emptyset, \end{aligned}$$

and

$$C_i \cap \mathbf{B}^n(0, (1 - \sigma)r) \cap \{x: x \cdot \mathbf{e}_1 = t_2\} = \emptyset$$

hold, then $\mathbf{B}^n(0, (1 - \sigma)r) \cap \{x: t_1 \leq x \cdot \mathbf{e}_1 \leq t_2\}$ will contain a connected component of C_i . To see this, it will suffice to show

$$C_i \cap \mathbf{S}^{n-1}(0, (1 - \sigma)r) \cap \{x: -a \leq x \cdot \mathbf{e}_1 \leq a\} = \emptyset$$

and

$$C_i \cap \mathbf{B}^n(0, (1 - \sigma)r) \cap \{x: t_1 \leq x \cdot \mathbf{e}_1 \leq t_2\} \neq \emptyset.$$

Suppose

$$x \in C_i \cap \mathbf{S}^{n-1}(0, (1 - \sigma)r) \cap \{x: -a \leq x \cdot \mathbf{e}_1 \leq a\},$$

then $|\mathbf{q}(x)| \geq 2\sigma r$. Also, there is $x' \in C \cap \mathbf{B}^n(x, \delta(i))$, so

$$x' \in C \cap \mathbf{B}^n(0, r) \cap \{x: |\mathbf{q}(x)| > \sigma r\},$$

a contradiction. On the other hand, there does exist

$$w \in C \cap \{x: |\mathbf{q}(x)| \leq \sigma r, x \cdot \mathbf{e}_1 = 2^{-1}(t_1 + t_2)\}$$

and, thus,

$$w' \in C_i \cap \mathbf{B}^n(w, \delta(i)).$$

One notes then that

$$|w'| \leq (1 - \sigma)r \quad \text{and} \quad t_1 < w' \cdot \mathbf{e}_1 < t_2.$$

It follows from the above that

$$\begin{aligned} \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(0, r)] \alpha(n-1)^{-1} \varepsilon(i)^{1-n} \\ \geq 2(1 - 2\sigma - 3\sigma^2)^{1/2} r - 2(m+1)\delta(i), \end{aligned}$$

and hence we have (#).

4.2. THEOREM. Assume $B \subset \mathbf{R}^n$ is finite, $L \subset H_0(B) \cap \ker[\text{aug}(B)]$ is a subgroup, $\{\varepsilon(1), \varepsilon(2), \varepsilon(3), \dots\} \subset \mathbf{R}^+$ with $\lim_i \varepsilon(i) = 0$, and $\{C_1, C_2, C_3, \dots\} \subset \mathcal{C}$ with $C_i \in \mathfrak{N}_{\varepsilon(i)}(B, L)$ for each $i = 1, 2, 3, \dots$.

(1) For each $i = 1, 2, 3, \dots$, there exists $C'_i \subset C_i$ such that $C'_i \in \mathcal{C}(B, L)$, $e_{\varepsilon(i)}(C'_i) = e_{\varepsilon(i)}(C_i)$, and each connected component of C'_i contains a point of B .

(2) For each $i = 1, 2, 3, \dots$, one has

$$\mathcal{L}^n \circ e_{\varepsilon(i)}(C_i) \leq \text{diam}(B) \cdot \text{card}(B) \cdot \alpha(n-1)\varepsilon(i)^{n-1} \\ + \text{card}(B) \cdot \alpha(n)\varepsilon(i)^n.$$

(3) There exists $r \in \mathbf{R}^+$ such that $C_i \subset \mathbf{B}(0, r)$ for each $i = 1, 2, 3, \dots$.

(4) There exist $C \in \mathcal{C}$, which is 1-rectifiable, and a subsequence $i(1), i(2), i(3), \dots$ of $1, 2, 3, \dots$ so that $C_{i(j)}$ converges to C in the Hausdorff distance topology as $j \rightarrow \infty$.

(5) For each C as in (4), one has $C \in \mathcal{C}(B, L)$ and

$$\mathcal{H}^1(C) = \inf\{\mathcal{H}^1(X) : X \in \mathcal{C}(B, L)\} < \infty.$$

PROOF. (1) One sets C'_i equal to the union of all connected components of C_i which contain a point of B . Conclusion (1) then follows from a direct sum theorem for Čech homology (see [ES, I, 13.2 and X] and [RE, Lemma 21A]).

(2) Conclusion (2) follows from [RE, Lemma 2A], applied as in the proof of 3(1), and an obvious estimate.

(3) Conclusion (3) follows from the construction in the proof of 3(6) where we use (2) to bound $\mu_{\varepsilon(i)}(B, L)$.

(4) It follows from (3) and 3(2) that a subsequence $C_{i(1)}, C_{i(2)}, C_{i(3)}, \dots$ converges to C . One notes also that $C'_{i(j)}$ converges to C in the Hausdorff distance topology, so (2) and 4.1(2) imply that C is 1-rectifiable.

(5) Let C be as in (4), then $C \in \mathcal{C}(B, L)$ by 3(5). Suppose $X \in \mathcal{C}(B, L)$ with $\mathcal{H}^1(X) < \infty$. As in the proof of (1), one sees that there exists $Y \subset X$ with $Y \in \mathcal{C}(B, L)$ such that Y has finitely many connected components. By [EH, Theorem 2] and [FH, 3.2.39], one sees that

$$\mathcal{H}^1(Y) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{L}^n \circ e_{\varepsilon}(Y) \alpha(n-1)^{-1} \varepsilon^{1-n}.$$

By (1) and 4.1(3), since $C'_i \in \mathcal{M}_{\varepsilon(i)}(B, L)$, one concludes

$$\mathcal{H}^1(C) \leq \mathcal{H}^1(Y) \leq \mathcal{H}^1(X).$$

5. Polyhedral complexes.

5.1. DEFINITIONS. (1) A 0 *polyhedron* is a point of $\mathbf{R}^n$; for each integer $m \geq 1$, an m *polyhedron* is a bounded, relatively open, nonempty subset of an m dimensional affine subspace of $\mathbf{R}^n$ the boundary of which is the closure of the union of finitely many $m-1$ polyhedra.

(2) For each nonnegative integer m , an m *polyhedral complex* is a subset, P , of $\mathbf{R}^n$ together with a decomposition of P into a finite union of $0, 1, 2, \dots, m$ polyhedra such that for each $l = 1, 2, 3, \dots, m$ the boundary of each l polyhedron of the decomposition is contained in the union of the $0, 1, 2, \dots, l-1$ polyhedra of the decomposition; for $l = 0, 1, 2, \dots, m$, denote by P_l the union of the $0, 1, 2, \dots, l$ polyhedra of the decomposition and set $\mathcal{P}^l(P) = \mathcal{H}^l(P_l)$; for each integer $l < 0$ or $l > m$ set $P_l = \emptyset$ and $\mathcal{P}^l(P) = 0$.

5.2. *Notation.* In 5.3 we will use the following notation. For each $c \in \mathbf{R}^+$, set

$$K(0, 2, c) = \sup\{2c^{-1}n^{1/2}, 1\} \quad \text{and} \quad K(n, m, c) = 2n^{1/2},$$

whenever $m = 3, 4, 5, \dots, n$. For each $c \in \mathbf{R}^+$, each $m = 3, 4, 5, \dots, n$, and each $l = n - 1, n - 2, n - 3, \dots, 0$, set

$$K(l, m, c) = w[1 + 2(u + 1)(m + 1)^{2(n+1)/2}]$$

where

$$u = K(0, m - 1, [(m + 1)2^{(n+2)/2}c^{1-m}]^{-1/(m-2)})$$

and

$$w = K(l + 1, m, c[u(m + 1)2^{(n+2)/2} + 1]^{-1/(m-1)}).$$

Finally, for each $c \in \mathbf{R}^+$ and each $m = 2, 3, 4, \dots, n$, set

$$K(m, c) = K(0, m, c).$$

5.3. THEOREM. (1) For each m polyhedral complex P ($0 \leq m \leq n$) and each $\epsilon \in \mathbf{R}^+$,

$$\mathcal{L}^n \circ e_\epsilon(P) \leq \sum_i \mathcal{P}^i(P) \alpha(n - i) \epsilon^{n-i}.$$

(2) For each $c \in \mathbf{R}^+$, each 1 polyhedral complex A , and each $d \in \mathbf{R}^+$ with

$$d \geq c\mathcal{P}^1(A),$$

there exists a 2 polyhedral complex P with

$$P \in \mathcal{C}[A, H_1(A)], \quad P \subset \text{Cv}(A) \cap e_{dK(2,c)}(A),$$

and, for each integer i ,

$$\mathcal{P}^i(P) \leq K(2, c) \sum_{j \geq -1} \mathcal{P}^{i+j}(A) d^{-j}.$$

(3) Let l and m be integers with $0 \leq l \leq n$ and $3 \leq m \leq n$. Fix a family $\{\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_l\}$ of pairwise orthogonal $n - 1$ dimensional affine subspaces of $\mathbf{R}^n$. For each $c \in \mathbf{R}^+$, each $m - 1$ polyhedral complex A , and each $d \in \mathbf{R}^+$ with

$$d \geq c[\mathcal{P}^{m-1}(A)]^{1/(m-1)}$$

and such that

$$\text{dist}(a, \Pi_i) \leq d,$$

for each $i = 1, 2, 3, \dots, l$ and each $a \in A$, there exists an m polyhedral complex P with

$$P \in \mathcal{C}[A, H_{m-1}(A)], \quad P \subset \text{Cv}(A) \cap e_{dK(l,m,c)}(A),$$

and, for each integer i ,

$$\mathcal{P}^i(P) \leq K(l, m, c) \sum_{j \geq -1} \mathcal{P}^{i+j}(A) d^{-j}.$$

(4) For each $m = 2, 3, 4, \dots, n$, each $c \in \mathbf{R}^+$, each $m - 1$ polyhedral complex A , and each $d \in \mathbf{R}^+$ with

$$d \geq c [\mathcal{P}^{m-1}(A)]^{1/(m-1)},$$

there exists an m polyhedral complex P with

$$P \in \mathcal{C}[A, H_{m-1}(A)], \quad P \subset \text{Cv}(A) \cap e_{dK(m,c)}(A),$$

and, for each integer i ,

$$\mathcal{P}^i(P) \leq K(m, c) \sum_{j \geq -1} \mathcal{P}^{i+j}(A) d^{-j}.$$

PROOF. (1) Conclusion (1) is clear.

(2) Let A be a nonempty 1 polyhedral complex. Let $\{\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_n\}$ be a family of pairwise orthogonal $n - 1$ dimensional affine subspaces of $\mathbf{R}^n$. For each $i = 1, 2, 3, \dots, n$ choose $v_i \in \mathbf{S}^{n-1}(0, 1)$ perpendicular to Π_i and, for each $0 \leq s \leq 2c^{-1}d$, set

$$\Pi_i(s) = \{x + (s + 2\lambda c^{-1}d)v_i : x \in \Pi_i, \lambda \in \mathbf{Z}\}.$$

For each $i = 1, 2, 3, \dots, n$, by [FH, 2.10.11] one has

$$\int_0^{2c^{-1}d} \mathcal{H}^0[A \cap \Pi_i(s)] d\mathcal{L}_s^1 \leq \mathcal{H}^1(A) = \mathcal{P}^1(A),$$

so one can choose $0 < s_i < 2c^{-1}d$ such that $A \cap \Pi_i(s_i) = \emptyset$. One sees that, for some integer r ,

$$A = A(1) \cup A(2) \cup A(3) \cup \dots \cup A(r)$$

where, for each $i = 1, 2, 3, \dots, r$, $A(i) \neq \emptyset$ is a 1 polyhedral complex, there is an open cube $C(i)$ with side length $2c^{-1}d$ such that $A(i) \subset C(i)$, and, further, the family $\{C(i)\}_{i=1,2,3,\dots,r}$ is disjointed. For each $i = 1, 2, 3, \dots, r$ choose $x_i \in A(i)_0$. One can make $\{x_i\} \ast A(i)$ a 2 polyhedral complex such that

$$\mathcal{P}^2[\{x_i\} \ast A(i)] \leq n^{1/2}c^{-1}d \mathcal{P}^1[A(i)],$$

$$\mathcal{P}^1[\{x_i\} \ast A(i)] \leq 2n^{1/2}c^{-1}d \mathcal{P}^0[A(i)] + \mathcal{P}^1[A(i)],$$

and

$$\mathcal{P}^0[\{x_i\} \ast A(i)] = \mathcal{P}^0[A(i)].$$

Set

$$P = \bigcup_i [\{x_i\} \ast A(i)]$$

and note that P can be made a 2 polyhedral complex such that

$$\mathcal{P}^2(P) \leq n^{1/2}c^{-1}d \mathcal{P}^1(A) \leq K(2, c) \mathcal{P}^1(A)d,$$

$$\mathcal{P}^1(P) \leq 2n^{1/2}c^{-1}d \mathcal{P}^0(A) + \mathcal{P}^1(A) \leq K(2, c) [\mathcal{P}^0(A)d + \mathcal{P}^1(A)],$$

and

$$\mathcal{P}^0(P) = \mathcal{P}^0(A) \leq K(2, c)\mathcal{P}^0(A).$$

By [RE, Lemma 3A], $P \in \mathcal{C}[A, H_1(A)]$ and it is clear that

$$P \subset \text{Cv}(A) \cap e_{dK(2,c)}(A).$$

(3) First we consider (3) in the case $l = n$. Let A be a nonempty $m - 1$ polyhedral complex. Set $\{y\} = \bigcap_i \Pi_i$ and choose $x \in A_0$. Notice $A \subset \mathbf{B}^n(y, n^{1/2}d)$ so $A \subset \mathbf{B}^n(x, 2n^{1/2}d)$. Set $P = \{x\} * A$ and notice that P can be made an m polyhedral complex such that for each integer $i > 0$,

$$\begin{aligned} \mathcal{P}^i(P) &\leq 2n^{1/2i-1}\mathcal{P}^{i-1}(A)d + \mathcal{P}^i(A) \\ &\leq K(n, m, c)[\mathcal{P}^{i-1}(A)d + \mathcal{P}^i(A)] \end{aligned}$$

and

$$\mathcal{P}^0(P) = \mathcal{P}^0(A) \leq K(n, m, c)\mathcal{P}^0(A).$$

By [RE, Lemma 2A], the conclusion of (3) follows in case $l = n$.

Set

$$N = \{(2, 0)\} \cup \{(m, l): 3 \leq m \leq n, 0 \leq l \leq n\}$$

and define the relation $<$ on N by setting $(m', l') < (m, l)$ if and only if

- (a) $m' < m$, or
- (b) $m' = m$ and $l' > l$.

The set N is well ordered by $<$ and $(2, 0)$ is the smallest element. Say that $(m, l) \in N'$ if $(m, l) \in N$ and the conclusion of (3) holds for that pair of integers. By (2) we have $(2, 0) \in N'$ and, by the preceding paragraph, $(m, n) \in N'$ for each $m = 3, 4, 5, \dots, n$. We will prove (3) by induction: Fix $(m, l) \in N$ and suppose $(m', l') \in N'$ for each $(m', l') < (m, l)$. We must show $(m, l) \in N'$. We may assume $3 \leq m$ and $l \leq n - 1$.

Let A be a nonempty $m - 1$ polyhedral complex. Let Π be an $n - 1$ dimensional affine subspace of $\mathbf{R}^n$ perpendicular to $\Pi_1, \Pi_2, \Pi_3, \dots$, and Π_l and choose $v \in S^{n-1}(0, 1)$ perpendicular to Π . Set

$$\Pi(s) = \{x + (s + \lambda d)v: x \in \Pi, \lambda \in \mathbf{Z}\}.$$

By [FH, 2.10.25 and 3.2.13] one has

$$\int_0^d \mathcal{H}^{-1}[A_i \cap \Pi(s)] d\mathcal{L}_s^1 \leq 2^{(n+2)/2}\mathcal{P}^i(A)$$

(one uses the fact

$$\begin{aligned} \alpha(r+2)\alpha(r+3)^{-1} &= [(r+3)(r+2)^{-1}]\alpha(r)\alpha(r+1)^{-1} \\ &< 2\alpha(r)\alpha(r+1)^{-1} \end{aligned}$$

for $r \in \mathbf{R}^+$), so one can choose $0 < s_0 < d$ such that no polyhedron of A is

contained in $\Pi(s_0)$ and

$$\mathcal{C}^{-1}[A_i \cap \Pi(s_0)] \leq (m+1)2^{(n+2)/2} \mathcal{P}^i(A) d^{-1}$$

for $i = 1, 2, 3, \dots, m-1$. For each integer r , set

$$D(r) = A \cap \{x + (s_0 + rd)v : x \in \Pi\}.$$

Notice that $D(r)$ can be made an $m-2$ polyhedral complex with

$$d \geq [(m+1)2^{(n+2)/2} c^{1-m}]^{-1/(m-2)} \{\mathcal{P}^{m-2}[D(r)]\}^{1/(m-2)}.$$

Notice also that $D(r) = \emptyset$ for all but finitely many $r \in \mathbb{Z}$.

By the induction hypothesis $((m-1, 0) \in N')$, for each integer r , there exists an $m-1$ polyhedral complex $B(r)$ such that

$$B(r) \in \mathcal{C}\{D(r), H_{m-2}[D(r)]\},$$

$$B(r) \subset \text{Cv}[D(r)] \cap e_{du}[D(r)],$$

and, for each integer i ,

$$\mathcal{P}^i[B(r)] \leq u \sum_{j \geq -1} \mathcal{P}^{i+j}[D(r)] d^{-j},$$

where

$$u = K(0, m-1, [(m+1)2^{(n+2)/2} c^{1-m}]^{-1/(m-2)}).$$

For each integer r set

$$A(r) = B(r) \cup B(r+1) \cup A$$

$$\cap \{x + [s_0 + (r+t)d]v : x \in \Pi, 0 \leq t \leq 1\};$$

notice that each point of $A(r)$ lies within d of $l+1$ pairwise orthogonal $n-1$ dimensional affine subspaces of $\mathbb{R}^n$ and that $A(r)$ can be made an $m-1$ polyhedral complex with

$$\begin{aligned} \mathcal{P}^{m-1}[A(r)] &\leq \mathcal{P}^{m-1}[B(r)] + \mathcal{P}^{m-1}[B(r+1)] + \mathcal{P}^{m-1}(A) \\ &\leq [u(m+1)2^{(n+2)/2} + 1] \mathcal{P}^{m-1}(A). \end{aligned}$$

By the induction hypothesis $((m, l+1) \in N')$, for each integer r , there exists $P(r)$ an m polyhedral complex such that

$$P(r) \in \mathcal{C}\{A(r), H_{m-1}[A(r)]\},$$

$$P(r) \subset \text{Cv}[A(r)] \cap e_{dw}[A(r)],$$

and, for each integer i ,

$$\mathcal{P}^i[P(r)] \leq w \sum_{j \geq -1} \mathcal{P}^{i+j}[A(r)] d^{-j},$$

where

$$w = K(l+1, m, c[u(m+1)2^{(n+2)/2} + 1]^{-1/(m-1)}).$$

Set $P = \bigcup_r P(r)$. Notice that by [RE, Lemma 13A] $P \in C[A, H_{m-1}(A)]$, and that $P \subset \text{Cv}(A) \cap e_{d(u+w)}(A)$. We can make P a polyhedral complex such that for each integer i ,

$$\begin{aligned}
 \mathcal{P}^i(P) &= \sum_r \mathcal{P}^i[P(r)] \leq w \sum_r \sum_{j \geq -1} \mathcal{P}^{i+j}[A(r)] d^{-j} \\
 &\leq w \sum_{j \geq -1} \left\{ \mathcal{P}^{i+j}(A) + 2 \sum_r \mathcal{P}^{i+j}[B(r)] + 2 \sum_r \mathcal{P}^{i+j}[D(r)] \right\} d^{-j} \\
 &\leq w \sum_{j \geq -1} \left\{ \mathcal{P}^{i+j}(A) + 2u \sum_r \sum_{h \geq -1} \mathcal{P}^{i+j+h}[D(r)] d^{-h} \right. \\
 &\quad \left. + 2 \sum_r \mathcal{P}^{i+j}[D(r)] \right\} d^{-j} \\
 &\leq w \sum_{j \geq -1} \mathcal{P}^{i+j}(A) d^{-j} \\
 &\quad + 2uw(m+1)2^{(n+2)/2} \sum_{j \geq -1} \sum_{h \geq -1} \mathcal{P}^{i+j+h+1}(A) d^{-j-h-1} \\
 &\quad + 2w(m+1)2^{(n+2)/2} \sum_{j \geq -1} \mathcal{P}^{i+j+1}(A) d^{-j-1} \\
 &\leq w[1 + 2(u+1)(m+1)2^{(n+2)/2}] \sum_{j \geq -1} \mathcal{P}^{i+j}(A) d^{-j},
 \end{aligned}$$

which proves (3).

(4) Conclusion (4) is a special case of (3).

5.4. Preliminaries. (1) Set $\mathbf{A} = \mathbf{R}^n \cap \{x: \mathbf{m}(x) \leq 1\}$.

(2) Fix $\theta \in \mathbf{R}^+$. For each $m \in \{0, 1, 2, \dots, n\}$, σ_m will be as in [FH, 4.2.6], and, for each $a \in \mathbf{A}$, we set

$$\begin{aligned}
 \Psi(m, a) &= \tau_a \circ \sigma_m \circ \tau_{-a}, \\
 \Psi(m, a, \theta) &= \mu_\theta \circ \Psi(m, a) \circ \mu_{1/\theta}.
 \end{aligned}$$

(3) Let P be a $k-1$ polyhedral complex. For $\mathbb{Q}^n$ almost every $a \in \mathbf{A}$ we have

$$\tau_{-a}(P) \cap \mathbf{W}_{n-k}'' = \emptyset \quad \text{and} \quad \tau_{-a} \circ \mu_{1/\theta}(P) \cap \mathbf{W}_{n-k}'' = \emptyset,$$

so $\Psi(k-1, a)$ and $\Psi(k-1, a, \theta)$ are defined on P . Restricting our attention to such $a \in \mathbf{A}$, we set

$$\begin{aligned}
 H(P, a) &= \bigcup_{m=k}^n \{t\Psi(m, a)(x) + (1-t)\Psi(m-1, a)(x): \\
 &\quad 0 \leq t \leq 1, x \in P\}
 \end{aligned}$$

and

$$\begin{aligned}
 H(P, a, \theta) &= \bigcup_{m=k}^n \{t\Psi(m, a, \theta)(x) + (1-t)\Psi(m-1, a, \theta)(x): \\
 &\quad 0 \leq t \leq 1, x \in P\}.
 \end{aligned}$$

We would like to pick a decomposition of P so that if L is an l polyhedron of the decomposition, then $\Psi(m, a)(L)$ is again an l polyhedron ($m = k - 1, k, k + 1, \dots, n$). Then $H(P, a)$ can be made a k polyhedral complex and estimates for $\mathcal{P}^i[H(P, a)]$ can be proved. To do this, the decomposition of P will have to depend on the choice of $a \in \mathbf{A}$. Thus we are lead to formulate the following definitions.

(4) Let r be an integer, $0 \leq r \leq n - 1$. Let Π be a family of r dimensional affine subspaces of $\mathbf{R}^n$; Π will be called an *affine r family* if the elements of Π are parallel to each other and only finitely many elements of Π intersect any compact subset of $\mathbf{R}^n$. Finally, $\{\emptyset\}$ will be the only *affine (-1) family*.

(5) Let $P \subset \mathbf{R}^n$ be an m polyhedral complex and let Π be an affine r family. Define the m polyhedral complex $P * \Pi$ by letting the l polyhedra of the decomposition of $P * \Pi$ consist of

(a) the connected components of $L \cap (\mathbf{R}^n \sim \cup \Pi)$ where L is an l polyhedron of the decomposition of P (each component is to be taken as a separate polyhedron),

(b) $L \cap \Pi$ where $\Pi \in \Pi$, L is a polyhedron of the decomposition of P , and $L \cap \Pi$ is l dimensional.

(6) If Π_1 is an affine r_1 family and Π_2 is an affine r_2 family, then $\Pi_1 \wedge \Pi_2$ will denote $\{\Pi_1 \cap \Pi_2: \Pi_1 \in \Pi_1 \text{ and } \Pi_2 \in \Pi_2\}$. We note that $\Pi_1 \wedge \Pi_2$ is an affine r_3 family for some $r_3 \leq \inf\{r_1, r_2\}$ and that if P is an m polyhedral complex, then, for each integer l ,

$$[(P * \Pi_1) * \Pi_2]_l \subset [P * \Pi_1]_l \cup [P * \Pi_2]_l \cup [P * (\Pi_1 \wedge \Pi_2)]_l.$$

(7) For each integer r with $0 \leq r \leq n - 1$, let $\mathbf{F}'_r$ be the collection of r dimensional affine subspaces each of which contains $r + 1$ affinely independent points of $\mathbf{A} \cap \mathbf{Z}^n$, set

$$\mathbf{F}'_r = \{\tau_z(\Pi): z \in \mathbf{Z}^n, \Pi \in \mathbf{F}'_r\},$$

and set $\mathbf{F} = \bigcup_{r=0}^{n-1} \mathbf{F}'_r$. Now, we can write

$$\mathbf{F} = \Pi'_1 \cup \Pi'_2 \cup \Pi'_3 \cup \dots \cup \Pi'_s$$

where Π'_i is an affine r_i family ($0 \leq r_i \leq n - 1$) for each i and the Π'_i are pairwise disjoint. Let $\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_s$ be an enumeration of

$$\{\Pi'_{\lambda(1)} \wedge \Pi'_{\lambda(2)} \wedge \Pi'_{\lambda(3)} \wedge \dots \wedge \Pi'_{\lambda(j)}: 1 \leq j \leq s', \lambda \in \Lambda(s', j)\}.$$

(8) Let P be a $k - 1$ polyhedral complex. For each $a \in \mathbf{A}$, set

$$P' = \tau_a[(\dots((\tau_{-a}(P) * \Pi'_1) * \Pi'_2) * \dots) * \Pi'_s];$$

P' is P with a new decomposition, and if L is an l polyhedron of P' , then $\Psi(m, a)(L)$ is an l polyhedron for $m = k - 1, k, k + 1, \dots, n$ (provided $\tau_{-a}(P) \cap \mathbf{W}''_{n-k} = \emptyset$). Thus $H(P, a) = H(P', a)$ can be made a k polyhedral complex for $\mathcal{L}^n$ almost every $a \in \mathbf{A}$.

5.5. THEOREM. (1) If V is an r dimensional linear subspace of $\mathbf{R}^n$, then there is $\lambda \in \Lambda(n, r)$ so that $\mathbf{p}_\lambda$ (see [FH, 1.7.4]) is one-to-one when restricted to V and

$$|(\mathbf{p}_\lambda|V)^{-1}(\mathbf{e}_i)| < \binom{n}{r}$$

holds for $i = 1, 2, 3, \dots, r$.

(2) There exists $\gamma \in \mathbf{R}^+$ such that if P is a $k - 1$ polyhedral complex, then one has

$$\int_{\mathbf{A}} \int_{(\tau_a(P) * \Pi_j)_l} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n < \gamma \sum_{i=l}^{k-1} \mathcal{P}^i(P)$$

for $0 \leq l \leq k - 1$ and $1 \leq j \leq s$; here u_{k-1} is as in [FH, 4.2.6].

(3) There exists $\gamma(n, k) \in \mathbf{R}^+$ such that, for each $k - 1$ polyhedral complex P ,

(a) $H(P, a)$ can be made a k polyhedral complex so as to satisfy

$$\int_{\mathbf{A}} \mathcal{P}^{l+1}[H(P, a)] d\mathcal{L}_a^n \leq \gamma(n, k) \sum_{i=l}^{k-1} \mathcal{P}^i(P),$$

(b) for each $0 < \varepsilon < 1$ one has

$$\int_{\mathbf{A}} \mathcal{L}^n \circ e_\varepsilon[H(P, a)] d\mathcal{L}_a^n \leq 6(k+1)\gamma(n, k) \sum_i \mathcal{P}^i(P) \varepsilon^{n-i-1},$$

(c) for each $0 < \varepsilon < \theta$ one has

$$\int_{\mathbf{A}} \mathcal{L}^n \circ e_\varepsilon[H(P, a, \theta)] d\mathcal{L}_a^n \leq 6(k+1)\gamma(n, k)\theta \sum_i \mathcal{P}^i(P) \varepsilon^{n-i-1}.$$

PROOF. (1) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_r$ be an orthonormal basis for V . Write

$$\mathbf{v}_i = (v_{i1}, v_{i2}, v_{i3}, \dots, v_{in})$$

for $i = 1, 2, 3, \dots, r$. By the Lagrange identity (see [BR, p. 49, Theorem 1]), we see there is $\lambda \in \Lambda(n, r)$ so that

$$|\det[(v_{i\lambda(j)})_{1 \leq i, j \leq r}]| \geq \binom{n}{r}^{-1}.$$

Define $f: \mathbf{R}^r \rightarrow V$ by setting $f(\mathbf{e}_i) = \mathbf{v}_i$ for $i = 1, 2, 3, \dots, r$. Then we have $|\det(\mathbf{p}_\lambda \circ f)| \geq \binom{n}{r}^{-1}$ and $|\mathbf{p}_\lambda \circ f(x)| \leq |x|$ for all $x \in \mathbf{R}^r$. It follows that, for $x \in \mathbf{R}^r$,

$$|x|^{-1} |\mathbf{p}_\lambda \circ f(x)| \geq |\det(\mathbf{p}_\lambda \circ f)| \geq \binom{n}{r}^{-1},$$

which implies conclusion (1), since $(\mathbf{p}_\lambda|V)^{-1} = f \circ (\mathbf{p}_\lambda \circ f)^{-1}$.

(2) Fix a $k - 1$ polyhedral complex P and integers l, j with $0 \leq l \leq k - 1$, $1 \leq j \leq s$.

Let L be an l polyhedron of P . Using [FH, 4.2.7(1)], we have

$$\begin{aligned}
& \int_{\mathbf{A}} \int_{\tau_{-a}(L) \cap (\mathbf{R}^n \sim \cup \Pi_j)} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \\
& \leq \int_{\mathbf{A}} \int_{\tau_{-a}(L)} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \leq \int_{\mathbf{A}} \int_{\tau_{-a}(L)} (u_{k-1})^{1-k} d\mathcal{H}^l d\mathcal{L}_a^n \\
& = \int_{\mathbf{A}} \int_L (u_{k-1} \circ \tau_{-a})^{1-k} d\mathcal{H}^l d\mathcal{L}_a^n \leq 2^n \binom{n}{k-1} \mathcal{H}^l(L).
\end{aligned}$$

To proceed with the proof we must consider the following situation: Let V be an r dimensional linear subspace of $\mathbf{R}^n$ where $r \geq n + l + 1 - k$, let Π be a translate of V , and let X be an $(\mathcal{H}^l, l)$ rectifiable Suslin set contained in Π . Suppose also that $\mathbf{p}: \mathbf{R}^n \rightarrow \mathbf{R}^r$ defined by

$$\mathbf{p}(x_1, x_2, x_3, \dots, x_n) = (x_1, x_2, x_3, \dots, x_r)$$

is one-to-one when restricted to V and

$$c \geq \prod_{i=1}^r |(\mathbf{p}|V)^{-1}(\mathbf{e}_i)|.$$

We have

$$c \geq [\text{ap } J_r(\mathbf{p}|V)(v)]^{-1}, \quad \text{for } v \in V,$$

and

$$c \geq [\text{ap } J_l(\mathbf{p}|X)(x)]^{-1}, \quad \text{for } \mathcal{H}^l \text{ almost all } x \in X.$$

Set $m = k + r - n - 1$. Let u'_m be defined as u_m is in [FH, 4.2.6] but with n replaced by r . Using [FH, 3.2.22] and the fact that $u_{k-1}(x) \geq u'_m \circ \mathbf{p}(x)$, we obtain

$$\begin{aligned}
& \int_{V \cap \mathbf{B}^n(0, \sqrt{n})} \int_X (u_{k-1} \circ \tau_{-v})^{-l} d\mathcal{H}^l d\mathcal{H}^r_v \\
& \leq c^2 \int_{\mathbf{p}[V \cap \mathbf{B}^n(0, \sqrt{n})]} \int_{\mathbf{p}(X)} (u'_m \circ \tau_{-w})^{-l} d\mathcal{H}^l d\mathcal{H}^r_w.
\end{aligned}$$

Now using [FH, 4.2.7(1)], we have

$$\begin{aligned}
& \int_{\mathbf{p}[V \cap \mathbf{B}^n(0, \sqrt{n})]} \int_{\mathbf{p}(X)} (u'_m \circ \tau_{-w})^{-l} d\mathcal{H}^l d\mathcal{H}^r_w \\
& \leq \int_{\mathbf{p}[V \cap \mathbf{B}^n(0, \sqrt{n})]} \int_{\mathbf{p}(X)} (u'_m \circ \tau_{-w})^{-m} d\mathcal{H}^l d\mathcal{H}^r_w \\
& \leq (n+1)^r \int_{\mathbf{p}(\mathbf{A})} \int_{\mathbf{p}(X)} (u'_m \circ \tau_{-w})^{-m} d\mathcal{H}^l d\mathcal{H}^r_w \\
& \leq (n+1)^r 2^r \binom{r}{m} \mathcal{H}^l(X).
\end{aligned}$$

Thus we conclude

$$\begin{aligned} & \int_{V \cap \mathbf{B}^n(0, \sqrt{n})} \int_X (u_{k-1} \circ \tau_{-v})^{-l} d\mathcal{H}^l d\mathcal{H}_v^r \\ & \leq c^2(n+1)^r 2^r \binom{r}{k+r-n-1} \mathcal{H}^l(X). \end{aligned}$$

Let L be an l' polyhedron of P and suppose Π_j is an affine r family. Fix $\Pi \in \Pi_j$. Let V be the r dimensional linear subspace of $\mathbf{R}^n$ parallel to Π and let W be the orthogonal complement of V . If $l' + r - n < 0$, then $\tau_{-a}(L) \cap \Pi = \emptyset$ for $\mathcal{L}^n$ almost every $a \in \mathbf{A}$. If $l' + r - n \geq 0$ then, for $\mathcal{L}^n$ almost every $a \in \mathbf{A}$, either $\tau_{-a}(L) \cap \Pi = \emptyset$ or $\tau_{-a}(L) \cap \Pi$ is an $l' + r - n$ polyhedron. We are trying to estimate the integral of $(u_{k-1})^{-l}$ over the l polyhedra of $\tau_{-a}(P) * \Pi_j$, thus only r such that $l = l' + r - n$ are of interest. So we assume $r = n + l - l'$ and notice $n + l - l' \geq n + l + 1 - k$.

Both sides of the inequality in (2) are invariant under a linear isometry of $\mathbf{R}^n$ which permutes the standard basis vectors. So, by (1), we may assume that $\mathbf{p}$, as above, is one-to-one when restricted to the V of the preceding paragraph and we may take

$$c = \binom{n}{r}^r.$$

Then, by the above and [FH, 2.10.25], we have

$$\begin{aligned} & \int_{\mathbf{A}} \int_{\tau_{-a}(L) \cap \Pi} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \\ & \leq \int_{W \cap \mathbf{B}^n(0, \sqrt{n})} \int_{V \cap \mathbf{B}^n(0, \sqrt{n})} \int_{\tau_{-w}(L) \cap \Pi} (u_{k-1} \circ \tau_{-v})^{-l} d\mathcal{H}^l d\mathcal{H}_v^r d\mathcal{H}_w^{n-r} \\ & \leq \binom{n}{r}^{2r} (n+1)^r 2^r \binom{r}{k+r-n-1} \\ & \quad \cdot \int_{W \cap \mathbf{B}^n(0, \sqrt{n})} \mathcal{H}^l[\tau_{-w}(L) \cap \Pi] d\mathcal{H}_w^{n-r} \\ & \leq \binom{n}{r}^{2r} (n+1)^r 2^r \binom{r}{k+r-n-1} \alpha(l) \alpha(n-r) \alpha(l')^{-1} \\ & \quad \cdot \mathcal{H}^{l'} \left[L \cap \bigcup_{w \in W \cap \mathbf{B}^n(0, \sqrt{n})} \tau_w(\Pi) \right]. \end{aligned}$$

Now, define an equivalence relation on Π_j by

$$\Pi_1 \approx \Pi_2 \text{ if and only if } \Pi_1 = \tau_z \Pi_2 \text{ for some } z \in \mathbf{Z}_n^n.$$

Then Π_j consists of finitely many, say c_j , equivalence classes. It follows that

$$\begin{aligned} & \int_{\mathbf{A}} \int_{\tau_{-a}(L) \cap \Pi_j} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \\ & \leq \binom{n}{r}^{2r} (n+1)^r 2^r \binom{r}{k+r-n-1} \alpha(l) \alpha(n-r) \alpha(l')^{-1} \\ & \quad \cdot c_j \text{ card} \{ z \in \mathbf{Z}_n^n : |z| \leq 2\sqrt{n} \} \mathcal{H}^{l'}(L). \end{aligned}$$

Conclusion (2) now follows.

(3a) We use the decomposition of $H(P, a)$ indicated in 5.4(8). Assuming that $\tau_{-a}(P) \cap \mathbf{W}_{n-k}'' = \emptyset$, we see that any $l+1$ polyhedron is either of the form

(i) $\Psi(m, a)(L)$, for some $m = n, n-1, n-2, \dots, k-1$ and some $l+1$ polyhedron L of P' (here $0 \leq l+1 \leq k-1$), or of the form

(ii) $\{t\Psi(m, a)(x) + (1-t)\Psi(m-1, a)(x): 0 < t < 1, x \in L\}$, for some $m = n, n-1, n-2, \dots, k$ and some l polyhedron L of P' (here $1 \leq l+1 \leq k$).

Now, we have, for $n \geq m \geq k-1$,

$$\begin{aligned} \mathcal{H}^{l+1}[\Psi(m, a)(L)] &\leq \int_{\tau_{-a}(L)} \|D\sigma_m\|^{l+1} d\mathcal{H}^{l+1} \\ &\leq (1+m)^{l+1} \int_{\tau_{-a}(L)} (u_m)^{-(l+1)} d\mathcal{H}^{l+1} \\ &\leq (1+m)^{l+1} \int_{\tau_{-a}(L)} (u_{k-1})^{-(l+1)} d\mathcal{H}^{l+1}, \end{aligned}$$

where we have used [FH, 4.2.6]. Next, notice that

$$\begin{aligned} \mathcal{H}^{l+1}[\{t\Psi(m, a)(x) + (1-t)\Psi(m-1, a)(x): 0 < t < 1, x \in L\}] \\ &\leq (l+1)^{-1} \cdot n^{1/2} \cdot \mathcal{H}^l[\Psi(m-1, a)(L)] \\ &\leq (l+1)^{-1} \cdot n^{1/2} \cdot m^l \int_{\tau_{-a}(L)} (u_{k-1})^{-l} d\mathcal{H}^l, \end{aligned}$$

where we have used the preceding observation with $l+1$ replaced by l and m replaced by $m-1$. So, by (2), we have

$$\begin{aligned} \int_{\mathbf{A}} \mathcal{H}^{l+1}[H(P, a)] d\mathcal{L}_a^n \\ &\leq \sum_{m=k-1}^n (1+m)^{l+1} \int_{\mathbf{A}} \int_{\tau_{-a}(P'_{l+1})} (u_{k-1})^{-(l+1)} d\mathcal{H}^{l+1} d\mathcal{L}_a^n \\ &\quad + \sum_{m=k}^n (l+1)^{-1} \cdot n^{1/2} \cdot m^l \int_{\mathbf{A}} \int_{\tau_{-a}(P'_l)} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \\ &\leq n(n+1)^{k-1} \sum_{j=1}^s \int_{\mathbf{A}} \int_{(\tau_{-a}(P) \bullet \Pi_j)_{l+1}} (u_{k-1})^{-(l+1)} d\mathcal{H}^{l+1} d\mathcal{L}_a^n \\ &\quad + n^{(2k+1)/2} \sum_{j=1}^s \int_{\mathbf{A}} \int_{(\tau_{-a}(P) \bullet \Pi_j)_{l+1}} (u_{k-1})^{-l} d\mathcal{H}^l d\mathcal{L}_a^n \\ &\leq n(n+1)^{k-1} \cdot s \cdot \gamma \sum_{i=l+1}^{k-1} \mathcal{P}^i(P) + n^{(2k+1)/2} \cdot s \cdot \gamma \sum_{i=l}^{k-1} \mathcal{P}^i(P) \\ &\leq [n(n+1)^{k-1} + n^{(2k+1)/2}] \cdot s \cdot \gamma \sum_{i=l}^{k-1} \mathcal{P}^i(P), \end{aligned}$$

where we have also used

$$P'_{l+1} \subset \bigcup_{j=1}^s (P * \Pi_j)_{l+1} \quad \text{and} \quad P'_l \subset \bigcup_{j=1}^s (P * \Pi_j)_l.$$

(3b) By 5.3(1), (3a) above, and the fact that $\alpha(i) < 6$ for each positive integer i , one has

$$\begin{aligned} \int_A \mathbb{L}^n \circ e_\epsilon [H(P, a)] d\mathbb{L}_a^n &\leq 6 \sum_{l=-1}^{k-1} \epsilon^{n-l-1} \int_A \mathfrak{P}^{l+1} [H(P, a)] d\mathbb{L}_a^n \\ &\leq 6 \sum_{l=-1}^{k-1} \gamma(n, k) \epsilon^{n-l-1} \sum_{i=l}^{k-1} \mathfrak{P}^i(P) \\ &\leq 6\gamma(n, k) \sum_{l=-1}^{k-1} \sum_{i=l}^{k-1} \epsilon^{n-i-1} \mathfrak{P}^i(P) \\ &\leq 6(k+1)\gamma(n, k) \sum_i \mathfrak{P}^i(P) \epsilon^{n-i-1}. \end{aligned}$$

(3c) Conclusion (3c) follows from (3b) since $Q = \mu_{1/\theta}(P)$ is a $k-1$ polyhedral complex with

$$\mathfrak{P}^i(Q) = \theta^{-i} \mathfrak{P}^i(P),$$

for $i = 0, 1, 2, \dots, k-1$, and

$$e_\epsilon [H(p, a, \theta)] = \mu_\theta \circ e_{\epsilon/\theta} [H(Q, a)].$$

6. Lower bound on density.

6.1. *Preliminaries.* For use in 6.2 we assume

- (1) $k \geq 2$,
- (2) $B \in \mathcal{C}$ and $L \subset H_{k-1}(B)$ is a subgroup,
- (3) $\{\epsilon(1), \epsilon(2), \epsilon(3), \dots\} \subset \mathbf{R}^+$ with $\lim_i \epsilon(i) = 0$,
- (4) $C_i \in \mathfrak{M}_{\epsilon(i)}(B, L)$ for each $i = 1, 2, 3, \dots$.

For $i = 1, 2, 3, \dots$, define the measure $\mathcal{G}_i$ over $\mathbf{R}^n$ by setting

$$\mathcal{G}_i(A) = [\alpha(n-k)\epsilon(i)^{n-k}]^{-1} \cdot \mathbb{L}^n[e_{\epsilon(i)}(C_i) \cap A]$$

for each $A \subset \mathbf{R}^n$ and the function

$$l_i: \mathbf{R}^n \times \mathbf{R}^+ \rightarrow \mathbf{R}$$

by setting

$$l_i(x, r) = [\alpha(n-k)\epsilon(i)^{n-k}]^{-1} \cdot \mathfrak{H}^{n-1}[e_{\epsilon(i)}(C_i) \cap \mathbf{S}^{n-1}(x, r)].$$

6.2. **THEOREM.** *There exists $\Gamma_1 \in \mathbf{R}^+$, which does not depend on $B, L, \{\epsilon(i)\}$, or $\{C_i\}$, such that the following hold:*

- (1) *For each $i = 1, 2, 3, \dots$, if $x \in \mathbf{R}^n$ and $r > \epsilon(i)$ with $\mathbf{B}^n(x, r) \cap B = \emptyset$ and $C_i \cap \mathbf{U}^n(x, r) \neq \emptyset$, then one has*

$$\mathcal{G}_i[\mathbf{B}^n(x, r)] \leq \Gamma_1 [l_i(x, r)]^{k/(k-1)}.$$

(2) For each $i = 1, 2, 3, \dots$, if $x \in C_i$ and $r \geq \varepsilon(i)$ with $\mathbf{B}^n(x, r) \cap B = \emptyset$, then one has

$$\mathcal{G}_i[\mathbf{B}^n(x, r)] \geq k^{-k} \Gamma_1^{-k} [r - \varepsilon(i)]^k.$$

(3) If B is $k - 1$ rectifiable, then one has

$$\limsup_i \mathcal{G}_i(\mathbf{R}^n) \leq k^{-1} \cdot \text{diam}(B) \cdot \mathcal{H}^{k-1}(B).$$

(4) If B is $k - 1$ rectifiable, then there exists a subsequence $i(1), i(2), i(3), \dots$ of $1, 2, 3, \dots$, a compact set $C \subset \mathbf{R}^n$, and a Radon measure $\mathcal{G}$ over $\mathbf{R}^n$ with compact support such that $\lim_j C_{i(j)} = C$ in the Hausdorff distance topology and $\mathcal{G}$ is the weak limit of $\mathcal{G}_{i(j)}$ as $j \rightarrow \infty$. For any such subsequence, $C \subset \mathbf{R}^n$, and measure $\mathcal{G}$ one has

(a) $C \in \mathcal{C}(B, L)$,

(b) if $x \in C, r \in \mathbf{R}^+$, with $\mathbf{B}^n(x, r) \cap B = \emptyset$, then

$$\mathcal{G}[\mathbf{B}^n(x, r)] \geq k^{-k} \Gamma_1^{-k} r^k,$$

(c) $C \sim B \subset \text{spt}(\mathcal{G}) \subset C$,

(d) $\mathcal{H}^k(C) < \infty$.

PROOF. (1) By [FH, 2.8.14] there exists a positive integer b such that for each triple (i, x, r) as in (1) there exist disjointed families, $F_1, F_2, F_3, \dots, F_b$, of closed balls of radius $2^{-1}\varepsilon(i)$ centered in $C_i \cap \mathbf{S}^{n-1}(x, r)$ so that

$$C_i \cap \mathbf{S}^{n-1}(x, r) \subset \bigcup_j F_j.$$

It is clear that there exists $c \in \mathbf{R}^+$ such that for each $y \in \mathbf{S}^{n-1}(x, r)$,

$$\mathcal{H}^{n-1}[\mathbf{S}^{n-1}(x, r) \cap \mathbf{B}^n(y, 2^{-1}\varepsilon(i))] \geq c\varepsilon(i)^{n-1}.$$

Hence, setting $\mathcal{F} = \bigcup_j F_j$ and $l = l_i(x, r)$, one sees that

$$\text{card}(\mathcal{F}) \leq \alpha(n - k)bc^{-1}l\varepsilon(i)^{1-k}.$$

Choose a collection of open balls, $\mathcal{G}$, such that there is a one-to-one and onto function $f: \mathcal{F} \rightarrow \mathcal{G}$ with $f[\mathbf{B}^n(y, 2^{-1}\varepsilon(i))] = \mathbf{U}^n(y, s)$ for some $2^{-1}\varepsilon(i) < s < \varepsilon(i)$ and

$$f[\mathbf{B}^n(y, 2^{-1}\varepsilon(i))] \cap f[\mathbf{B}^n(z, 2^{-1}\varepsilon(i))] = \emptyset$$

if and only if

$$\mathbf{B}^n(y, 2^{-1}\varepsilon(i)) \cap \mathbf{B}^n(z, 2^{-1}\varepsilon(i)) = \emptyset$$

whenever $[\mathbf{B}^n(y, 2^{-1}\varepsilon(i)), \mathbf{B}^n(z, 2^{-1}\varepsilon(i))] \in \mathcal{F} \times \mathcal{F}$. For each $D \in \mathcal{G}$ one sees, as in the proof of 4.1(2), that

$$\text{card}\{E: E \in \mathcal{G}, D \cap E \neq \emptyset\} \leq 3^nb.$$

Notice that $N[\mathcal{G}, C_i \cap S^{n-1}(x, r)]$ can obviously be considered as a polyhedral complex and set

$$A = N[\mathcal{G}, C_i \cap S^{n-1}(x, r)]_{k-1}.$$

Observe that, for $j = 0, 1, 2, \dots, k-1$,

$$\mathcal{P}^j(A) \leq \alpha(j)(3^n b)^{k-1} \alpha(n-k) b c^{-1} \cdot l \varepsilon(i)^{j+1-k},$$

so, in particular,

$$l^{1/(k-1)} \geq [\alpha(k-1)(3^n b)^{k-1} \alpha(n-k) b c^{-1}]^{-1/(k-1)} \cdot [\mathcal{P}^{k-1}(A)]^{1/(k-1)}.$$

One applies 5.3(4) with m replaced by k , c replaced by

$$[\alpha(k-1)(3^n b)^{k-1} b \alpha(n-k) c^{-1}]^{-1/(k-1)},$$

and $d = l^{1/(k-1)}$ to obtain $p \in \mathcal{C}[A, H_{k-1}(A)]$ a k polyhedral complex with

$$\begin{aligned} \mathcal{P}^m(P) &\leq \gamma_1 \sum_{j \geq -1} \mathcal{P}^{m+j}(A) l^{-j/(k-1)} \\ &\leq \gamma_2 \sum_{j=-1}^{k-1-m} l^{(k-1-j)/(k-1)} \varepsilon(i)^{m+j+1-k} \end{aligned}$$

for $m = 0, 1, 2, \dots, k$, where γ_1 and γ_2 are appropriate constants. By 5.3(1) one has

$$\mathcal{L}^n \circ e_{\varepsilon(i)}(P) \leq \gamma_2 \sum_{m=0}^k \sum_{j=-1}^{k-1-m} \alpha(n-m) l^{(k-1-j)/(k-1)} \varepsilon(i)^{n+j+1-k}.$$

Since $C_i \cap U^n(x, r) \neq \emptyset$, we conclude, by a direct sum theorem for Čech homology, that $C_i \cap S^{n-1}(x, r) \neq \emptyset$, thus

$$l \geq c \alpha(n-k)^{-1} \varepsilon(i)^{k-1},$$

and it follows that

$$\mathcal{L}^n \circ e_{\varepsilon(i)}(P) \leq \gamma_3 l^{k/(k-1)} \varepsilon(i)^{n-k},$$

for an appropriate $\gamma_3 \in \mathbf{R}^+$. Set

$$C = [C_i \sim \mathbf{B}^n(x, r)] \cup e_{2\varepsilon(i)}[C_i \cap S^{n-1}(x, r)] \cup P,$$

so, by 8.4(1), $C \in \mathcal{C}(B, L)$. Observe that

$$\begin{aligned} \mathcal{L}^n \circ e_{\varepsilon(i)} \circ e_{2\varepsilon(i)}[C_i \cap S^{n-1}(x, r)] \\ \leq \alpha(n) 4^n \alpha(n-k) b c^{-1} l \varepsilon(i)^{n-k+1} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^n \circ e_{\varepsilon(i)}[C_i \sim \mathbf{B}^n(x, r)] \\ \leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \alpha(n-k) l \varepsilon(i)^{n-k+1}, \end{aligned}$$

and estimate

$$\begin{aligned}
 & \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(x, r)] \\
 &= \mathcal{L}^n \circ e_{\varepsilon(i)}(C) \leq \mathcal{L}^n \circ e_{\varepsilon(i)}(C) \\
 &\leq \mathcal{L}^n \circ e_{\varepsilon(i)}[C_i \sim \mathbf{B}^n(x, r)] \\
 &\quad + \mathcal{L}^n \circ e_{\varepsilon(i)} \circ e_{2\varepsilon(i)}[C_i \cap \mathbf{S}^{n-1}(x, r)] + \mathcal{L}^n \circ e_{\varepsilon(i)}(P) \\
 &\leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \alpha(n-k)l\varepsilon(i)^{n-k+1} \\
 &\quad + \alpha(n)4^n\alpha(n-k)bc^{-1}l\varepsilon(i)^{n-k+1} + \gamma_3 l^{k/(k-1)}\varepsilon(i)^{n-k} \\
 &\leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \gamma_4 l^{k/(k-1)}\varepsilon(i)^{n-k},
 \end{aligned}$$

for an appropriate $\gamma_4 \in \mathbf{R}^+$. Conclusion (1) follows.

(2) Let i, x , and r be as in (2). Define $f: \mathbf{R}^+ \rightarrow \mathbf{R}$ by setting

$$f(s) = \mathcal{J}_i[\mathbf{B}^n(x, s)],$$

so f is an increasing function with $f'(s) \geq l_i(x, s)$ for $\mathcal{L}^1$ almost all positive s . For each s with $\text{dist}(x, B) > s \geq \varepsilon(i)$, we have

$$f(s) \leq \Gamma_1 f'(s)^{k/(k-1)},$$

thus

$$[f(s)^{1/k}]' \geq k^{-1}\Gamma_1^{(1-k)/k}.$$

Conclusion (2) follows.

(3) Conclusion (3) follows from [FH, 3.2.22 and 3.2.39] and [RE, Lemma 2A].

(4) By (2) and (3) above we see that the C_i are uniformly bounded. Then the existence of a subsequence as in (4) is a consequence of 3(2) and [FH, 2.5.2] together with the fact that $\mathcal{H}(\mathbf{R}^n)$ has a countable dense set. Conclusion (a) follows from 3(5). Conclusion (b) follows from (2). Conclusion (c) is clear. Conclusion (d) follows from (b), (c), (3), and [FH, 2.10.19(3)].

7. Main theorems.

7.1. *Preliminaries.* Throughout §7 we make the assumptions of 6.1. Further, we assume B is $k-1$ rectifiable and that there exist $C \in \mathcal{C}$ and a Radon measure $\mathcal{J}$ over $\mathbf{R}^n$ with compact support such that $\lim_i C_i = C$ in the Hausdorff distance topology and $\mathcal{J}$ is the weak limit of $\mathcal{J}_i$ as $i \rightarrow \infty$.

7.2. **THEOREM.** *There exists $\Gamma_2 \in \mathbf{R}^+$ which does not depend on $B, L, \{\varepsilon(i)\}, \{C_i\}$, such that, for each $x \in C \sim B$ and $r \in \mathbf{R}^+$,*

$$\begin{aligned}
 & \mathcal{J}[\mathbf{B}^n(x, r)] \leq \alpha(k)r^k \\
 & \cdot \sup\{\Gamma_2, \alpha(k)^{-1}k^{-1} \cdot \text{diam}(B) \cdot \mathcal{H}^{k-1}(B) \cdot [\text{dist}(x, B)]^{-k}\}.
 \end{aligned}$$

PROOF. Let $r \in \mathbf{R}^+$ and $i \in \{1, 2, 3, \dots\}$ be such that $\mathbf{B}^n(x, r) \cap B = \emptyset$, $\mathbf{U}^n(x, r) \cap C_i \neq \emptyset$, and $r > \varepsilon(i)$. Proceed as in the proof of 6.2(1) to obtain the $k-1$ polyhedral complex A with

$$\mathcal{P}^j(A) \leq \alpha(j)(3^n b)^{k-1} \alpha(n-k) b c^{-1} \cdot l \varepsilon(i)^{j+1-k}$$

for each integer j , where $l = l_i(x, r)$. We will apply 5.5(3c) with $\theta = \tau r$ where $\tau \leq 1$ has yet to be determined. By 5.5(3c), provided $\varepsilon(i) < \theta$, we may choose $a \in A$ so that

$$\mathcal{L}^n \circ e_{\varepsilon(i)}[H(A, a, \theta)] \leq 2^{-n} \cdot 6(k+1)\gamma(n, k)\theta \sum_j \mathcal{P}^j(A) \varepsilon(i)^{n-j-1}.$$

Set $H = H(A, a, \theta)$ and

$$D = [C_i \sim \mathbf{B}^n(x, r)] \cup e_{2\varepsilon(i)}[C_i \cap \mathbf{S}^{n-1}(x, r)] \cup H \\ \cup [\cup \{\mu_\theta \circ \tau_a(\mathbf{W}'(z)): z \in \mathbf{Z}_k^n, \mu_\theta \circ \tau_a(\mathbf{W}'(z)) \subset \mathbf{B}^n(x, 2nr)\}],$$

so, by 8.4(1), $D \in \mathcal{C}(B, L)$. Hence one has

$$\begin{aligned} \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(x, r)] \\ \leq \mathcal{L}^n[e_{\varepsilon(i)}(D)] \\ \leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \alpha(n-k) l \varepsilon(i)^{n-k+1} \\ + \alpha(n) 4^n \alpha(n-k) b c^{-1} l \varepsilon(i)^{n-k+1} \\ + 2^{-n} \cdot 6(k+1)\gamma(n, k)\theta \sum_j \mathcal{P}^j(A) \varepsilon(i)^{n-j-1} \\ + (12n)^n \sum_{j=0}^k r^j \tau^{j-n} \alpha(n-j) \varepsilon(i)^{n-j} \end{aligned}$$

provided $\tau \leq 1$ and $\varepsilon(i) < \theta$. Setting

$$f(r) = \mathcal{J}_i[\mathbf{B}^n(x, r)],$$

one has

$$\beta f(r) \leq [\varepsilon(i) + \tau r] f'(r) + \tau^{-n} r^k$$

where $\beta < 1$ depends only on n and k . Set

$$\tau = 4^{-1} k^{-1} \beta \quad \text{and} \quad \Gamma_2 = 2\tau^{-n} \beta^{-1} \alpha(k)^{-1}.$$

Observe that, as a consequence of 6.2(3),

$$\mathcal{J}_i[\mathbf{B}^n(x, r)] \leq \alpha(k) r^k [\alpha(k)^{-1} k^{-1} \cdot \text{diam}(B) \cdot \mathcal{H}^{k-1}(B) [\text{dist}(x, B)]^{-k}]$$

whenever $r \geq \text{dist}(x, B)$. If, for $r < \text{dist}(x, B)$,

$$f(r) \alpha(k)^{-1} r^{-k} \geq \Gamma_2, \quad (*)$$

then one has

$$f(r) \leq k^{-1} r f'(r)$$

provided $\varepsilon(i) < \tau r$. Thus if $(*)$ holds, then one sees that $f(r) \alpha(k)^{-1} r^{-k}$ is increasing. The conclusion of 7.2 follows.

7.3. THEOREM. *The set C is $(\mathfrak{I}^k, k)$ rectifiable.*

PROOF. By 6.2(4d) and [FH 3.3.13] there exist Borel sets S and U such that $S \cap U = \emptyset$, $S \cup U = C$, S is $(\mathfrak{I}^k, k)$ rectifiable, U is purely $(\mathfrak{I}^k, k)$ unrectifiable, and $\mathfrak{I}_1^k(U) = 0$. Set

$$R = (U \sim B) \cap \{x: \Theta^k(\mathfrak{I}^k \llcorner S, x) = 0\}$$

and notice that by [FH, 2.10.19(4)] it suffices to show $R = \emptyset$. By [FH, 2.10.15] we may assume $\mathcal{L}^k[\mathbf{p}_\lambda(U)] = 0$ for each $\lambda \in \Lambda(n, k)$, where $\mathbf{p}_\lambda$ is as in [FH, 1.7.4].

Let $2^{-1} < \rho < 1$ be arbitrary and set

$$\sigma = 3^{-(k+1)} n^{-k/2} (1 - \rho)^k \text{card}[\Lambda(n, k)]^{-1}.$$

Let $x \in R$ be arbitrary. There exists $r_0 \in \mathbf{R}^+$, with $r_0 < \text{dist}(x, B)$, such that if $r \leq r_0$, then

$$\mathfrak{I}^k[\mathbf{B}^n(x, r) \cap S] < \sigma r^k.$$

Let $r \leq r_0$ be arbitrary and set $s = \rho r$, $\theta = (r - s)6^{-1}n^{-1/2}$.

For each $\lambda \in \Lambda(n, k)$ define $f_\lambda: \mathbf{R}^n \rightarrow \{0, 1\}$ by setting $f_\lambda(a) = 1$ if and only if

$$\mathbf{p}_\lambda^{-1}[\mathbf{p}_\lambda(z)] \cap \tau_{-a} \circ \mu_{1/\theta}[\mathbf{B}^n(x, r) \cap C] \neq \emptyset$$

for some $z \in \mathbf{Z}_n^n$. Set

$$f = \sum_{\lambda \in \Lambda(n, k)} f_\lambda \quad \text{and} \quad \mathbf{A} = \mathbf{R}^n \cap \{x: \mathbf{m}(x) < 1\}.$$

One estimates

$$\begin{aligned} \int_{\mathbf{A}} f(a) d\mathcal{L}_a^n &= \sum_{\lambda \in \Lambda(n, k)} \int_{\mathbf{A}} f_\lambda(a) d\mathcal{L}_a^n \\ &< \text{card}[\Lambda(n, k)] \sigma r^k \theta^{-k} 2^{n-k} = 3^{-1} \mathcal{L}^n(\mathbf{A}). \end{aligned}$$

Hence there exists a compact $\mathbf{A}' \subset \mathbf{A}$ with $\mathcal{L}^n(\mathbf{A}') = 2 \cdot 3^{-1} \mathcal{L}^n(\mathbf{A})$ such that $f(a) = 0$ for each $a \in \mathbf{A}'$.

Choose an arbitrary $i \in \{1, 2, 3, \dots\}$ which satisfies $\varepsilon(i) < \theta$. Proceed in a manner similar to the proof of 6.2(1) to construct a k polyhedral complex P from a covering of $C_i \cap [\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)]$ which satisfies

$$\mathcal{P}^j(P) \leq \gamma_1 \mathfrak{I}_i^j[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)] \varepsilon(i)^{j-k}$$

for each integer j , where $\gamma_1 \in \mathbf{R}^+$ depends only on n and k .

For each $a \in \mathbf{A}$, set

$$V'(a) = \cup \{ \text{Clos } \mathbf{W}'(z) : z \in \mathbf{Z}_n^n \text{ with} \\ \tau_{-a} \circ \mu_{1/\theta}[\mathbf{B}^n(x, s + \theta)] \cap \text{Clos } \mathbf{W}'(z) \neq \emptyset \}$$

and

$$V(a) = \mu_\theta \circ \tau_a[V'(a)].$$

Now, for $\mathbb{L}^n$ almost all $a \in \mathbf{A}$, $P \cap \text{Bdry } V(a)$ can be made a $k-1$ polyhedral complex so as to satisfy

$$\int_{\mathbf{A}} \mathcal{P}^j[P \cap \text{Bdry } V(a)] d\mathbb{L}_a^n \leq \gamma_2 \sum_{l=1}^{k-j} \theta^{-l} \mathcal{P}^{j+l}(P)$$

for $j = 0, 1, 2, \dots, k-1$, where $\gamma_2 \in \mathbf{R}^+$ depends only on n and k . Thus we may select $b \in \mathbf{A}$ so that

$$\mathcal{P}^j[P \cap \text{Bdry } V(b)] \leq (k+1)\gamma_2 \sum_{l=1}^{k-j} \theta^{-l} \mathcal{P}^{j+l}(P)$$

for $j = 0, 1, 2, \dots, k-1$. By 5.5(3c) one can choose $a_i \in \mathbf{A}'$ so that

$$\begin{aligned} \mathbb{L}^n \circ e_{\varepsilon(i)}[H(P \cap \text{Bdry } V(b), a_i, \theta)] \\ \leq 3 \cdot 2^{-n-1} \cdot 6(k+1)\gamma(n, k)\theta \\ \cdot \sum_j \mathcal{P}^j[P \cap \text{Bdry } V(b)] \varepsilon(i)^{n-j-1} \\ \leq \gamma_3 \theta \sum_{j=0}^{k-1} (k+1)\gamma_2 \sum_{l=1}^{k-j} \mathcal{P}^{j+l}(P) \theta^{-l} \varepsilon(i)^{n-j-1} \\ \leq (k+1)\gamma_1\gamma_2\gamma_3 \cdot \theta \mathcal{G}_i[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)] \\ \cdot \sum_{j=0}^{k-1} \sum_{l=1}^{k-j} \theta^{-l} \varepsilon(i)^{n-k+l-1} \\ \leq \gamma_4 \mathcal{G}_i[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)] \varepsilon(i)^{n-k}, \end{aligned}$$

where $\gamma_3, \gamma_4 \in \mathbf{R}^+$ depend only on n and k .

Set

$$C'_i = [C_i \sim (\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s))] \\ \cup e_{2\varepsilon(i)}[(C_i \cap \mathbf{S}^{n-1}(x, r)) \cup (C_i \cap \mathbf{S}^{n-1}(x, s))] \cup P.$$

By 8.4(2) we have $C'_i \in \mathcal{C}(B, L)$. Now, $V(b) \subset \mathbf{B}^n(x, r)$ so if $\mu_\theta \circ \tau_{a_i} \circ \sigma_{k-1} \circ \tau_{-a_i} \circ \mu_{1/\theta}(\sigma_{k-1}$ as in [FH, 4.2.6]) is not defined on $C'_i \cap V(b)$, then there is $y_i \in C'_i \cap \mathbf{B}^n(x, r)$ such that at least k coordinates of $\tau_{-a_i} \circ \mu_{1/\theta}(y_i)$ are even integers. Since C_i converges to C in the Hausdorff distance topology and $\mathbf{A}'$ is compact, it follows that

$\mu_\theta \circ \tau_{a_i} \circ \sigma_{k-1} \circ \tau_{-a_i} \circ \mu_{1/\theta}$ is defined on $C'_i \cap V(b)$ for all sufficiently large i . Also, for sufficiently large i ,

$$C'_i \cap \text{Bdry } V(b) = P \cap \text{Bdry } V(b)$$

holds. It then follows that

$$\begin{aligned} & [C'_i \sim V(b)] \cup H[P \cap \text{Bdry } V(b), a_i, \theta] \\ & \cup \mu_\theta \circ \tau_{a_i} \circ \sigma_{k-1} \circ \tau_{-a_i} \circ \mu_{1/\theta} [C'_i \cap V(b)] \end{aligned}$$

is in $\mathcal{C}(B, L)$ for all sufficiently large i , and, hence, by [RE, Lemma 17A],

$$[C'_i \sim V(b)] \cup H[P \cap \text{Bdry } V(b), a_i, \theta]$$

is in $\mathcal{C}(B, L)$ for all sufficiently large i . So one has, for all sufficiently large i ,

$$\begin{aligned} & \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \cap \mathbf{B}^n(x, r)] \\ & = \mathcal{L}^n \circ e_{\varepsilon(i)}(C_i) \\ & \leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] + \mathcal{L}^n \circ e_{4\varepsilon(i)}(P) \\ & \quad + \mathcal{L}^n \circ e_{\varepsilon(i)}[H(P \cap \text{Bdry } V(b), a_i, \theta)] \\ & \leq \mathcal{L}^n[e_{\varepsilon(i)}(C_i) \sim \mathbf{B}^n(x, r)] \\ & \quad + \gamma_1 \mathcal{J}_i[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)] \cdot \sum_{j=0}^k \alpha(n-j) 4^{n-j} \varepsilon(i)^{n-k} \\ & \quad + \gamma_4 \mathcal{J}_i[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)] \varepsilon(i)^{n-k}. \end{aligned}$$

Thus one has

$$\mathcal{J}[\mathbf{U}^n(x, r)] \leq \gamma_5 \mathcal{J}[\mathbf{B}^n(x, r) \sim \mathbf{U}^n(x, s)]$$

where $\gamma_5 \in \mathbf{R}^+$ depends only on n and k . But $2^{-1} < \rho < 1$ was arbitrary so, by 6.2(4b),

$$\mathcal{J}[\mathbf{S}^{n-1}(x, r)] \geq \gamma_5^{-1} k^{-k} \Gamma_1^{1-k} r^k,$$

and this contradicts 7.2.

7.4. THEOREM. (1) *There exists $\Gamma_3 \in \mathbf{R}^+$ such that for each $X \in \mathcal{C}$, each $\varepsilon \in \mathbf{R}^+$, and each Lipschitzian $f: X \rightarrow \mathbf{R}^n$ one has*

$$\mathcal{L}^n \circ e_\varepsilon[f(X)] \leq \Gamma_3 [1 + \text{Lip}(f)]^n \mathcal{L}^n \circ e_\varepsilon(X).$$

(2) *For each $(\mathcal{K}, k)$ rectifiable and $\mathcal{K}$ measurable set $X \subset \mathbf{R}^n$, if there exists $\beta \in \mathbf{R}^+$ such that, for each $x \in X$ and each $r \in \mathbf{R}^+$,*

$$\mathbf{B}^n(x, r) \cap X = \emptyset \text{ implies } \mathcal{K}[\mathbf{B}^n(x, r) \cap X] \geq \beta r^k,$$

then $\mathfrak{M}^k(X) = \mathcal{K}(X)$.

(3) *Let $X \in \mathcal{C}(B, L)$ satisfy*

$$\mathcal{K}(X) = \inf \{ \mathcal{K}(Y) : Y \in \mathcal{C}(B, L) \}$$

and set $Z = \text{spt}(\mathcal{H}^k \llcorner X) \cup B$; then $Z \in \mathcal{C}(B, L)$ and $\mathfrak{M}^k(Z) = \mathcal{H}^k(Z)$.

(4) If $x \in C \sim B$, $f \in \mathbf{O}^*(n, k)$, and $g \in \mathbf{O}^*(n, n - k)$ are such that

$$f^*(\mathbf{R}^k) = \text{Tan}^k(\mathcal{H}^k \llcorner C, x) \quad \text{and} \quad g^*(\mathbf{R}^{n-k}) = \ker(f),$$

then there exist arbitrarily small $r \in \mathbf{R}^+$ such that

$$f(f^{-1}[\mathbf{U}^k(0, r)] \cap g^{-1}[\mathbf{U}^{n-k}(0, r)] \cap \tau_{-x}(C_i)) = \mathbf{U}^k(0, r)$$

for all sufficiently large integers i .

(5) One has

- (a) $\lim_i \mathcal{H}_i(\mathbf{R}^n) \geq \mathcal{H}^k(C)$,
- (b) $\mathcal{H}^k(C) = \mathfrak{M}^k(C)$,
- (c) $\mathfrak{M}^k(C) = \inf\{\mathfrak{M}_*^k(Y): Y \in \mathcal{C}(B, L)\}$,
- (d) $\mathcal{H}^k(C) = \inf\{\mathcal{H}^k(Y): Y \in \mathcal{C}(B, L)\}$.

PROOF. (1) Let X, ε, f be as in (1). Set $l = \text{Lip}(f)$. By [FH, 2.8.14] there exists a positive integer b , depending only on n , such that there are disjointed families, $F_1, F_2, F_3, \dots, F_b$, of closed balls of radius ε centered in X with $X \subset \bigcup (\cup_j F_j)$. Now, any point of $f(X)$ is within $l\varepsilon$ of $f(c)$ for some $c \in \{\text{Cl}(B): B \in \cup_j F_j\}$, so we have

$$\mathcal{L}^n \circ e_\varepsilon[f(X)] \leq \text{card}\left(\bigcup_j F_j\right) \cdot (1 + l)^n \alpha(n) \varepsilon^n.$$

For each $j = 1, 2, 3, \dots, b$,

$$\text{card}(F_j) \leq [\alpha(n) \varepsilon^n]^{-1} \mathcal{L}^n \circ e_\varepsilon(X)$$

holds. Conclusion (1) follows with $\Gamma_3 = b$.

(2) Let X and β be as in (2). Let $\sigma \in \mathbf{R}^+$ be arbitrary. By [FH, 3.2.18] one can choose a compact, k rectifiable $W \subset X$ with $\mathcal{H}^k(X \sim W) < \sigma^{k+1}$. For each $\varepsilon \in \mathbf{R}^+$ set

$$A_\varepsilon = X \sim [e_{\sigma\varepsilon}(W) \cup e_{\sigma\varepsilon}(B)].$$

By [FH, 2.8.14] there exists a positive integer b such that, for each $\varepsilon \in \mathbf{R}^+$, there exist disjointed families, $F_1, F_2, F_3, \dots, F_b$, of closed balls of radius $\sigma\varepsilon$ centered in A_ε so that $A_\varepsilon \subset \bigcup (\cup_j F_j)$. The hypothesis of (2) implies $\text{card}(F_j) \leq \sigma\beta^{-1}\varepsilon^{-k}$ for $j = 1, 2, 3, \dots, b$, hence one has

$$\mathcal{L}^n \circ e_\varepsilon(A_\varepsilon) \leq (1 + \sigma)^n \alpha(n) b \sigma \beta^{-1} \varepsilon^{n-k}.$$

Since

$$\mathcal{L}^n \circ e_\varepsilon(X) \leq \mathcal{L}^n \circ e_{(1+\sigma)\varepsilon}(B) + \mathcal{L}^n \circ e_{(1+\sigma)\varepsilon}(W) + \mathcal{L}^n \circ e_\varepsilon(A_\varepsilon),$$

one concludes by [FH, 3.2.39] that

$$\mathfrak{M}^{*k}(X) \leq (1 + \sigma)^{n-k} \mathcal{H}^k(W) + (1 + \sigma)^n \alpha(n) b \sigma \beta^{-1} \alpha(n - k)^{-1}.$$

Because σ was arbitrary, conclusion (2) follows from [FH, 3.2.37].

(3) Let X and Z be as in (3). By [RE, Lemma 21A] there exists a minimal

$Y \subset X$ with $Y \in \mathcal{C}(B, L)$. Let $x \in X \sim Z$ be arbitrary. By [FH, 2.10.25], there exists $r \in \mathbf{R}^+$ such that $\mathbf{B}^n(x, r) \cap B = \emptyset$ and

$$\mathfrak{H}^{k-1}[X \cap \mathbf{S}^{n-1}(x, r)] = 0,$$

which implies

$$H_{k-1}[Y \cap \mathbf{S}^{n-1}(x, r)] = 0$$

by [RE, Lemma 17A]. By 8.2(2), $Y \sim \mathbf{U}^n(x, r) \in \mathcal{C}(B, L)$, which implies $x \notin Y$. Since $x \in X \sim Z$ was arbitrary, one has $Y \subset Z$, so $Z \in \mathcal{C}(B, L)$ by [RE, Lemma 7A]. That $\mathfrak{M}^k(Z) = \mathfrak{H}^k(Z)$ follows from (2) and the last paragraph on p. 37 of [RE].

(4) Without loss of generality assume $x = 0$. For each $r \in \mathbf{R}^+$ set

$$V(r) = f^{-1}[\mathbf{B}^k(0, r)] \cap g^{-1}[\mathbf{B}^{n-k}(0, r)]$$

and $U(r) = \text{Int } V(r)$. Assume, contrary to (4), that there exists $r_0 \in \mathbf{R}^+$, with $r_0 < 8^{-1} \text{dist}(0, B)$, such that for each $r \in \mathbf{R}^+$ with $r < r_0$ there exist infinitely many positive integers i such that

$$\mathbf{U}^k(0, r) \sim f[U(r) \cap C_i] \neq \emptyset.$$

By 6.2(4b), 7.2, and [FH, 2.10.19(1) and (3)] there exist $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4 \in \mathbf{R}^+$ such that if

$$y \in C \cap \mathbf{B}^n[0, 2^{-1} \text{dist}(0, B)]$$

and $0 < s < 4^{-1} \text{dist}(0, B)$, then

$$\begin{aligned} \gamma_1 s^k &\leq \mathcal{J}[\mathbf{B}^n(y, s)] \leq \gamma_2 \mathfrak{H}^k[\mathbf{B}^n(y, s) \cap C] \\ &\leq \gamma_3 \mathcal{J}[\mathbf{B}^n(y, s)] \leq \gamma_4 s^k. \end{aligned}$$

Let $2^{-1} < \rho < 1$ and $0 < \xi < 2^{-3k/2}(1-\rho)^k \gamma_1 \gamma_4^{-1}$ be arbitrary. Choose $0 < r_1 \leq r_0$ such that

$$\mathfrak{H}^k(C \cap V(t) \sim g^{-1}[\mathbf{B}^{n-k}(0, 2^{-1}(1-\rho)t)]) \leq \xi \mathfrak{H}^k[C \cap V(t)]$$

for each $t \leq r_1$. Let $0 < r \leq r_1$ be arbitrary and notice that

$$C \cap V[2^{-1}(1+\rho)r] \sim g^{-1}[\mathbf{B}^{n-k}(0, (1-\rho)r)] = \emptyset,$$

since if y is in the above set, then

$$\mathbf{B}^n[y, 2^{-1}(1-\rho)r] \subset V(r) \sim g^{-1}[\mathbf{B}^{n-k}(0, 2^{-1}(1-\rho)r)]$$

and, consequently,

$$\mathfrak{H}^k(C \cap V(r) \sim g^{-1}[\mathbf{B}^{n-k}(0, 2^{-1}(1-\rho)r)]) \geq 2^{-k}(1-\rho)^k \gamma_1 r^k \gamma_2^{-1}$$

while

$$\mathfrak{H}^k[V(r) \cap C] \leq \gamma_4 2^{k/2} r^k \gamma_2^{-1}$$

which is impossible by the choice of ξ and r .

By hypothesis, one can choose a subsequence $i(1), i(2), i(3), \dots$ of $1, 2, 3, \dots$ and for each $j = 1, 2, 3, \dots$ an $a_j \in U^k(0, \rho r)$ such that $a_j \notin f[U(\rho r) \cap C_{i(j)}]$. For each $j = 1, 2, 3, \dots$ set $\bar{C}_j = C_{i(j)}$, $\bar{\varepsilon}(j) = \varepsilon(i(j))$, and $\bar{f}_j = f_{i(j)}$. Define a map Φ_j as follows:

(a) Φ_j is the identity on $\mathbf{R}^n \sim U[3^{-1}(1 + 2\rho)r]$;

(b) for $x \in U[3^{-1}(1 + 2\rho)r] \sim V[4^{-1}(1 + 3\rho)r]$,

$$\Phi_j(x) = tx + (1 - t)f^* \circ f(x)$$

where

$$t = [|f(x)| - 4^{-1}(1 + 3\rho)r][3^{-1}(1 + 2\rho)r - 4^{-1}(1 + 3\rho)r]^{-1};$$

(c) on $V[4^{-1}(1 + 3\rho)r] \sim f^{-1}(f(a_j))$ Φ_j is $f^* \circ f$ followed by central projection from a_j onto $f^*[\mathbf{S}^{k-1}(0, 4^{-1}(1 + 3\rho)r)]$.

We restrict our attention to j such that

$$d(\bar{C}_j, C) < 6^{-1}(1 - \rho)r \quad \text{and} \quad \bar{\varepsilon}(j) < 4^{-1}(1 - \rho)r.$$

For such j , we have

$$\bar{C}_j \cap V[3^{-1}(1 + 2\rho)r] \sim g^{-1}[\mathbf{B}^{n-k}(0, 6^{-1} \cdot 5(1 - \rho)r)] = \emptyset,$$

so $\Phi_j(\bar{C}_j) \in \mathcal{C}(B, L)$. Also, we have

$$\begin{aligned} e_{\bar{\varepsilon}(j)}[\Phi_j(\bar{C}_j)] &\subset [e_{\bar{\varepsilon}(j)}(\bar{C}_j) \sim V(r)] \\ &\cup [e_{\bar{\varepsilon}(j)}(\bar{C}_j) \cap V(r) \sim V(\rho r)] \\ &\cup e_{\bar{\varepsilon}(j)}[\Phi_j(\bar{C}_j \cap V(3^{-1}(1 + 2\rho)r) \sim U(4^{-1}(1 + 3\rho)r))] \\ &\cup e_{\bar{\varepsilon}(j)}[f^*(\mathbf{S}^{k-1}(0, 4^{-1}(1 + 3\rho)r))]. \end{aligned}$$

Now, one can check that

$$\text{Lip}[\Phi_j \bar{C}_j \cap V(3^{-1}(1 + 2\rho)r) \sim U(4^{-1}(1 + 3\rho)r)] \leq 26$$

holds. So, using (1), we obtain

$$\begin{aligned} \mathcal{L}^n[e_{\bar{\varepsilon}(j)}(\bar{C}_j) \cap V(r)] &\leq \mathcal{L}^n[e_{\bar{\varepsilon}(j)}(\bar{C}_j) \cap V(r) \sim V(\rho r)] \\ &\quad + \Gamma_3(27)^n \mathcal{L}^n \circ e_{\bar{\varepsilon}(j)}[\bar{C}_j \cap V(3^{-1}(1 + 2\rho)r) \sim U(4^{-1}(1 + 3\rho)r)] \\ &\quad + \mathcal{L}^n \circ e_{\bar{\varepsilon}(j)}[f^*(\mathbf{S}^{k-1}(0, 4^{-1}(1 + 3\rho)r))] \\ &\leq [1 + \Gamma_3(27)^n] \mathcal{L}^n[e_{\bar{\varepsilon}(j)}(\bar{C}_j) \cap V(r) \sim V(\rho r)] \\ &\quad + 2k r^{k-1} \alpha(k) \bar{\varepsilon}(j)^{n-k+1}. \end{aligned}$$

It follows that

$$\mathcal{J}[V(r)] \leq [1 + \Gamma_3(27)^n] \mathcal{J}[V(r) \sim V(\rho r)]$$

holds for $0 < r \leq r_1$ and from that we obtain

$$\mathcal{J}[V(r) \sim V(\rho r)] \geq \gamma_1 [1 + \Gamma_3(27)^n]^{-1} r^k$$

for $0 < r \leq r_1$. Thus we have

$$\begin{aligned} \mathcal{J}[V(r)] &= \sum_{j=0}^{\infty} \mathcal{J}[V(\rho^j r) \sim V(\rho^{j+1} r)] \\ &\geq \gamma_1 [1 + \Gamma_3(27)^n]^{-1} \cdot r^k \cdot \sum_{j=0}^{\infty} \rho^{jk} \\ &= \gamma_1 [1 + \Gamma_3(27)^n]^{-1} (1 - \rho^k)^{-1} \cdot r^k \end{aligned}$$

for $0 < r \leq r_1$. Since

$$\mathcal{J}[V(r)] \leq \gamma_3^{-1} \gamma_4 2^{k/2} r^k$$

holds for $r \leq r_0$, we obtain a contradiction by choosing

$$1 > \rho > \left(1 - \gamma_1 [1 + \Gamma_3(27)^n]^{-1} \gamma_3 \gamma_4^{-1} 2^{-k/2}\right)^{1/k}.$$

(5a) Conclusion (5a) follows from [FH, 2.8.7 and 3.2.19] and (4).

(5b) Let $\sigma \in \mathbf{R}^+$ be arbitrary. By 6.2(4b), 6.2(4c), 7.2, and [FH, 2.10.19 (1,3)], $\mathcal{J}\mathcal{L}(\mathbf{R}^n \sim B)$ and $\mathcal{H}^k\mathcal{L}(C \sim B)$ have the same sets of measure zero. Thus [FH, 3.2.18] allows us to choose a compact, k -rectifiable $W \subset C \sim B$ with $\mathcal{J}[C \sim (B \cup W)] < \sigma^{k+1}$. The proof now proceeds in a manner similar to the proof of (2), but we use 6.2(4b) to conclude

$$\text{card}(F_j) \leq \sigma k^k \Gamma_1^{k-1} \varepsilon^{-k}.$$

(5c) From (5a), (5b) it follows that

$$\mathfrak{N}^k(C) = \inf \{ \mathfrak{N}^{*k}(Y) : Y \in \mathcal{C}(B, L) \} \quad (\dagger)$$

holds. Let $D \in \mathcal{C}(B, L)$ be arbitrary and choose a new sequence $\{\varepsilon(1), \varepsilon(2), \varepsilon(3), \dots\} \subset \mathbf{R}^+$, with $\lim_i \varepsilon(i) = 0$, so that

$$\lim_i \alpha(n-k)^{-1} \varepsilon(i)^{k-n} \mathcal{L}^n \circ e_{\varepsilon(i)}(D) = \mathfrak{N}_*^k(D).$$

Applying the previous results, one obtains $\bar{C} \in \mathcal{C}(B, L)$ such that, by (5a), (5b),

$$\mathfrak{N}^k(\bar{C}) < \mathfrak{N}_*^k(D)$$

while, by $(\dagger)$ applied to $\bar{C}$, $\mathfrak{N}^k(C) = \mathfrak{N}^k(\bar{C})$. This proves (5c).

(5d) Conclusion (5d) follows from (3), (5b), (5c), and the main theorem of [RE].

8. Topological lemmas.

8.1. *Preliminaries.* For use in 8.2 we fix the following terminology:

- (1) $B, C, D, E, F \in \mathcal{C}$;
- (2) $L \subset H_{k-1}(B)$ is a subgroup;

(3) $\varepsilon \in \mathbf{R}^+$ and $\mathcal{Q}$ is a finite covering of $D \cap C$ by open balls of radius no larger than ε ;

(4) set $H = e_\varepsilon[\text{Clos } \cup \mathcal{Q}]$ and $A = \mathbf{N}(\mathcal{Q}, D \cap C)_{k-1}$;

(5) assume $E \in \mathcal{C}(B, L)$, $F \cap B = \emptyset$, and $k \geq 2$;

(6) $i: D \cap C \rightarrow H$, $j: A \rightarrow H$, $l: B \rightarrow E \sim \text{Int } F$, $m: E \cap \text{Bdry } F \rightarrow E \sim \text{Int } F$ are inclusion maps.

8.2. LEMMA. *One has*

(1) $i_*[H_{k-1}(D \cap C)] \subset j_*[H_{k-1}(A)]$,

(2) $l_*(L) \subset m_*[H_{k-1}(E \cap \text{Bdry } F)]$.

PROOF. (1) Let $\mathfrak{D}$ be an arbitrary covering of $D \cap C$ by open balls. Then $\mathfrak{D}$ induces a covering of the space $D \cap C$, and the nerve, $\mathbf{N}_{\mathfrak{D}}$, of this covering can be mapped to $\mathbf{N}(\mathfrak{D}, D \cap C)$ in the obvious manner; we will denote this map by $f_{\mathfrak{D}}$.

Let $\sigma \in H_{k-1}(D \cap C)$ be arbitrary. Let $\mathfrak{B}$ be an arbitrary covering of $D \cap C$ by open balls which refines $\mathcal{Q}$ and let $p: \mathbf{N}_{\mathfrak{B}} \rightarrow \mathbf{N}_{\mathcal{Q}}$ be the projection (see [ES, IX, 2]). One notes that $r \circ f_{\mathfrak{B}}$ is homotopic to $q \circ f_{\mathcal{Q}} \circ p$, where $r: \mathbf{N}(\mathfrak{B}, D \cap C) \rightarrow H$ and $q: \mathbf{N}(\mathcal{Q}, D \cap C) \rightarrow H$ are the inclusion maps. Since $\mathfrak{B}$ refines $\mathcal{Q}$ we have $p_* \circ \pi_{\mathfrak{B}} = \pi_{\mathcal{Q}}$ and we see that

$$r_* \circ f_{\mathfrak{B}*} \circ \pi_{\mathfrak{B}}(\sigma) = q_* \circ f_{\mathcal{Q}*} \circ \pi_{\mathcal{Q}}(\sigma).$$

By the continuity of Čech homology [ES, X, Theorem 2.1], one concludes

$$i_*(\sigma) \in q_*[H_{k-1}(\mathbf{N}(\mathcal{Q}, D \cap C))].$$

But, clearly, one has

$$q_*[H_{k-1}(\mathbf{N}(\mathcal{Q}, D \cap C))] \subset j_*[H_{k-1}(A)].$$

(2) Consider the following commutative diagram in which all the homomorphisms are induced by inclusion maps.

$$\begin{array}{ccccc} H_{k-1}(E, E \cap F) & \xleftarrow{\theta_*} & H_{k-1}(E \sim \text{Int } F, E \cap \text{Bdry } F) & & \\ \phi_* \uparrow & & \delta_* \uparrow & & \\ H_{k-1}(E) & \xleftarrow{\gamma_*} & H_{k-1}(E \sim \text{Int } F) & \xleftarrow{l_*} & H_{k-1}(B) \\ & & m_* \uparrow & & \\ & & H_{k-1}(E \cap \text{Bdry } F) & & \end{array}$$

By [ES, IX, Theorem 7.6], $\ker(\delta_*) = \text{im}(m_*)$, while, by [ES, X, Theorem 5.4], θ_* is an isomorphism. Let $\sigma \in L$ be arbitrary. By hypothesis $\gamma_* \circ l_*(\sigma) = 0$, so

$$0 = \phi_* \circ \gamma_* \circ l_*(\sigma) = \theta_* \circ \delta_* \circ l_*(\sigma);$$

thus one has $\delta_* \circ l_*(\sigma) = 0$, which implies $l_*(\sigma) \in \text{im}(m_*)$.

8.3. *Preliminaries.* For use in 8.4 we fix the following terminology:

- (1) $B, C, D \in \mathcal{C}$;
- (2) $L \subset H_{k-1}(B)$ is a subgroup;
- (3) $\varepsilon \in \mathbf{R}^+$, $\mathcal{Q}$ is a finite covering of $D \cap C$ by open balls of radius no larger than ε , and $\mathcal{Q}_1 \subset \mathcal{Q}$ is a covering of $\text{Bdry}(D) \cap C$;
- (4) set $A = N(\mathcal{Q}, D \cap C)_k$ and $A_1 = N(\mathcal{Q}_1, \text{Bdry}(D) \cap C)_{k-1}$;
- (5) assume $C \in \mathcal{C}(B, L)$, $D \cap B = \emptyset$, and $k \geq 2$.

8.4. THEOREM. (1) For each $Y \in \mathcal{C}[A_1, H_{k-1}(A_1)]$, one has

$$(C \sim D) \cup e_{2\varepsilon}[\text{Bdry}(D) \cap C] \cup Y \in \mathcal{C}(B, L).$$

(2) One has

$$(C \sim D) \cup e_{2\varepsilon}[\text{Bdry}(D) \cap (C)] \cup A \in \mathcal{C}(B, L).$$

PROOF. (1) Set

$$U = (C \sim D) \cup e_{2\varepsilon}(C \cap \text{Bdry } D) \cup Y.$$

Consider the following commutative diagram in which all the homomorphisms are induced by inclusion maps.

$$\begin{array}{ccccc}
 H_{k-1}(A_1) & \xrightarrow{\phi} & H_{k-1}(Y) & & \\
 \delta \downarrow & & \downarrow & & \\
 H_{k-1}(e_\varepsilon(\text{Clos } \bigcup \mathcal{Q}_1)) & \xrightarrow{\quad} & H_{k-1}(U) & \xleftarrow{\theta} & \\
 \gamma \uparrow & & \uparrow & & \\
 H_{k-1}(C \cap \text{Bdry } D) & \xrightarrow{\beta} & H_{k-1}(C \sim \text{Int } D) & \xleftarrow{\alpha} & H_{k-1}(B)
 \end{array}$$

Let $\sigma \in L$ be arbitrary. By 8.2(2) there exists $\sigma' \in H_{k-1}(C \cap \text{Bdry } D)$ such that $\beta(\sigma') = \alpha(\sigma)$. By 8.2(1) there exists $\sigma'' \in H_{k-1}(A_1)$ such that $\delta(\sigma'') = \gamma(\sigma')$. One sees easily that $\phi(\sigma'') = 0$ implies $\theta(\sigma) = 0$.

(2) Set

$$\begin{aligned}
 C_1 &= C \sim \text{Int } D, \quad C_2 = C \cap D, \quad E = C \cap \text{Bdry } D, \\
 V &= (C \sim D) \cup e_{2\varepsilon}[C \cap \text{Bdry } D] \cup A.
 \end{aligned}$$

Consider the following commutative diagram in which all homomorphisms, except ∂_1 , ∂_2 , and ∂ are induced by inclusion maps.

Let $\sigma \in L$ be arbitrary. By 8.2(2) there exists $\sigma' \in H_{k-1}(E)$ such that $\beta_1(\sigma') = \alpha(\sigma)$. Since $C \in \mathcal{C}(B, L)$, one has $\gamma(\sigma') = 0$. Thus there exists $\tau' \in H_k(C, E)$ such that $\partial(\tau') = \sigma'$. The maps ϕ_1 and ϕ_2 provide an injective representation of $H_k(C, E)$ as a direct sum, so there exists $\tau'_i \in H_k(C_i, E)$, for $i = 1, 2$, such that $\tau' = \phi_1(\tau'_1) + \phi_2(\tau'_2)$. Set $\sigma'_i = \partial_i(\tau'_i)$ for $i = 1, 2$. One has $\sigma' = \sigma'_1 + \sigma'_2$ and $\beta_1(\sigma'_1) = 0$. Now by 8.2(1) there exists $\sigma''_2 \in H_{k-1}(A_1)$ so that $w_2(\sigma''_2) = w_1(\sigma'_2)$, and since $\beta_2(\sigma'_2) = 0$, one sees by the proof of 8.2(1) that $w_3(\sigma''_2) = 0$. It follows that $\theta \circ \beta_1(\sigma'_2) = 0$, hence that $\theta \circ \beta_1(\sigma') = 0$, and finally that $0 = \theta \circ \alpha(\sigma) = \mu(\sigma)$.

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