BALANCED COHEN-MACAULAY COMPLEXES

BY

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ABSTRACT. A balanced complex of type $(a_1, \dots, a_m)$ is a finite pure simplicial complex Δ together with an ordered partition $(V_1, \dots, V_m)$ of the vertices of Δ such that $\text{card}(V_i \cap F) = a_i$ for every maximal face F of Δ . If $\mathbf{b} = (b_1, \dots, b_m)$, then define $f_{\mathbf{b}}(\Delta)$ to be the number of $F \in \Delta$ satisfying $\text{card}(V_i \cap F) = b_i$. The formal properties of the numbers $f_{\mathbf{b}}(\Delta)$ are investigated in analogy to the f -vector of an arbitrary simplicial complex. For a special class of balanced complexes known as balanced Cohen-Macaulay complexes, simple techniques from commutative algebra lead to very strong conditions on the numbers $f_{\mathbf{b}}(\Delta)$. For a certain complex $\Delta(P)$ coming from a poset P , our results are intimately related to properties of the Möbius function of P .

1. Introduction. We are concerned with the problem of obtaining information on the number $f_i = f_i(\Delta)$ of i -dimensional faces of a finite simplicial complex Δ . (All terminology is defined below.) There are two significant classes of complexes Δ for which a complete characterization of the numbers f_i , $0 \leq i \leq \dim \Delta$, has been obtained, viz., the class of all complexes and the class of Cohen-Macaulay complexes. Here we introduce a new class which we call balanced complexes. Balanced complexes possess invariants $f_{\mathbf{b}}$ more discriminating than the numbers f_i , and the formal properties of these invariants will be investigated. In the case of balanced Cohen-Macaulay complexes Δ , simple techniques from commutative algebra lead to conditions on the invariants $f_{\mathbf{b}}$ which are considerably stronger than those obtained merely by assuming Δ is Cohen-Macaulay. For a certain complex $\Delta(P)$ coming from a poset P , our results are intimately related to properties of the Möbius function of P .

We now proceed to the basic definitions and terminology. We employ the following notation throughout:

$\mathbf{N}$, set of nonnegative integers,

$\mathbf{P}$, set of positive integers,

$[n]$, $\{1, 2, \dots, n\}$, where $n \in \mathbf{P}$,

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$T \subset S$, T is a subset of S , allowing $T = \emptyset$ or $T = S$,

$\mathbf{e}_i$, the i th unit coordinate vector in $\mathbf{N}^m$, i.e., $\mathbf{e}_i = (\varepsilon_1, \dots, \varepsilon_m)$, where $\varepsilon_j = \delta_{ij}$.

Now let Δ be a simplicial complex, or *complex* for short, on the vertex set $V = \{x_1, \dots, x_n\}$. Thus Δ is a collection of subsets of V satisfying the two conditions: (i) $\{x\} \in \Delta$ for all $x \in V$, and (ii) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. If $F \in \Delta$ and F has $i + 1$ elements, then we call F an i -face of Δ and write $\dim F = i$. If $\delta = \max\{\dim F: F \in \Delta\}$, then we call Δ a δ -complex and write $\dim \Delta = \delta$. If every maximal face of Δ is a δ -face, then we say that Δ is *pure*. Let $f_i = f_i(\Delta)$ denote the number of i -faces of Δ . Thus $f_0 = n$. The vector $\mathbf{f} = \mathbf{f}(\Delta) = (f_0, f_1, \dots, f_\delta)$ is called the f -vector of Δ . We will employ the notation $\Delta = \langle abc, bcd, bde \rangle$ to indicate that the maximal faces of Δ are $\{a, b, c\}$, $\{b, c, d\}$ and $\{b, d, e\}$. Hence the f -vector of this Δ is $(5, 7, 3)$.

The problem often arises of obtaining information about the f -vectors of various complexes Δ . The first significant result along these lines, essentially due to Joseph Kruskal [12] and G. Katona [11] (see [9, §8] for an exposition), is an explicit characterization of those vectors $\mathbf{f} = (f_0, f_1, \dots, f_\delta)$ which are the f -vectors of some complex Δ . We will call such vectors K -vectors. Kruskal and Katona actually only proved that the condition in Theorem 1.1 below is necessary; but the sufficiency of this condition is immediate from their proofs.

1.1 THEOREM. *Given positive integers f and i , write*

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \dots > n_j \geq j \geq 1$ (such a representation exists and is unique), and define

$$f^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \dots + \binom{n_j}{j+1}.$$

Then the vector $\mathbf{f} = (f_0, f_1, \dots, f_\delta)$ of positive integers is a K -vector if and only if $f_{i+1} \leq f_i^{(i+1)}$ for $0 \leq i \leq \delta - 1$. $\square$

One can now ask for special classes of complexes whose f -vectors have characterizations analogous to that of Theorem 1.1. One class for which such a characterization can be given consists of the so-called Cohen-Macaulay complexes [23]. These may be defined either algebraically or topologically; it is a basic result of G. Reisner [16] that the two definitions are equivalent. Both definitions will be of use to us here. First we define the notion of a Cohen-Macaulay ring in the case of interest to us. Let k be a field, and let m be a positive integer. Suppose R is a finitely-generated $\mathbf{N}^m$ -graded k -algebra. I.e., the additive group of R can be written as a direct sum $R = \sum_{\alpha \in \mathbf{N}^m} R_\alpha$,

where $R_\alpha R_\beta \subset R_{\alpha+\beta}$, $R_0 = k$, and R is finitely generated as a k -algebra. If $x \in R_\alpha$ we say that x is *homogeneous* of degree α , written $\deg x = \alpha$. If we wish to emphasize that $\alpha \in \mathbb{N}^m$, then we say that x is $\mathbb{N}^m$ -homogeneous. It follows from the fact that R is finitely-generated that $\dim_k R_\alpha < \infty$ for each $\alpha \in \mathbb{N}^m$, and we define the *Hilbert function* $H(R, \alpha) = \dim_k R_\alpha$, $\alpha \in \mathbb{N}^m$. If $\alpha = (\alpha_1, \dots, \alpha_m)$ let $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m}$, and define the *Poincaré series* $F(R, \lambda) = \sum_\alpha H(R, \alpha) \lambda^\alpha$. It is well known that the formal power series $F(R, \lambda)$ represents a rational function of $\lambda = (\lambda_1, \dots, \lambda_m)$. Let d be the *Krull dimension* $\dim R$ of R , i.e., the maximum number of elements of R which are algebraically independent over k . (Do not confuse the Krull dimension “dim” with the vector space dimension “ $\dim_k$ ”.) Suppose $\theta_1, \dots, \theta_d$ are homogeneous elements of R of nonzero degree such that $\dim_k R/(\theta_1, \dots, \theta_d) < \infty$, or equivalently, such that $\dim R/(\theta_1, \dots, \theta_d) = 0$. Then $\theta_1, \dots, \theta_d$ are called a *homogeneous system of parameters* (h.s.o.p.) for R . Again if we wish to emphasize that $\deg \theta_i \in \mathbb{N}^m$, $1 \leq i \leq d$, then we call $\theta_1, \dots, \theta_d$ an $\mathbb{N}^m$ -homogeneous system of parameters. If $m = 1$ then the Noether normalization lemma guarantees the existence of an h.s.o.p. Moreover, if k is infinite and R is generated by R_1 , then we can choose each θ_i , $1 \leq i \leq d$, to have degree one. However, when $m > 1$ an h.s.o.p. usually will not exist; indeed, a crucial point of this paper concerns the existence of an h.s.o.p. in certain situations when $m > 1$ (Theorem 4.1). At any rate, suppose $\theta_1, \dots, \theta_d$ is an h.s.o.p. for R . Let $S = R/(\theta_1, \dots, \theta_d)$. Since $\theta_1, \dots, \theta_d$ are homogeneous, S inherits from R the structure of an $\mathbb{N}^m$ -graded k -algebra. We now say that R is *Cohen-Macaulay* if

$$F(R, \lambda) = F(S, \lambda) \prod_{i=1}^d (1 - \lambda^{\deg \theta_i})^{-1}. \quad (1)$$

This is not the usual definition of a Cohen-Macaulay ring, but it is equivalent. For a reconciliation with the usual definition in terms of R -sequences, see [24]. It is important to realize that the question of whether or not R is Cohen-Macaulay is independent of the grading chosen for R , though this is not immediately obvious from (1). Thus once we know that R is Cohen-Macaulay, we know that (1) holds for whatever grading we choose for R . From the combinatorial point of view, the importance of Cohen-Macaulay rings R is that the much smaller ring S carries a lot of combinatorial information about R , in particular, the Hilbert function of R .

Now given a complex Δ on $V = \{x_1, \dots, x_n\}$, associate with it a certain $\mathbb{N}$ -graded k -algebra A_Δ as follows. Let $A = k[x_1, \dots, x_n]$, the polynomial ring over k on the vertices of Δ . Let I_Δ be the ideal generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ with $i_1 < i_2 < \cdots < i_s$ and $\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\} \notin \Delta$. Define a grading on $A_\Delta = A/I_\Delta$ by setting $\deg x_i = 1$. We say that Δ is a

Cohen-Macaulay complex (always with respect to the field k) if A_Δ is a Cohen-Macaulay ring. This is the algebraic definition of a Cohen-Macaulay complex.

To give the topological definition, recall that if $F \in \Delta$, then the *link* of F is the complex $\text{lk } F = \{G \in \Delta: F \cap G = \emptyset, F \cup G \in \Delta\}$. In particular, $\text{lk } \emptyset = \Delta$.

1.2 THEOREM. *Let Δ be a δ -complex and k a field. The following three conditions are equivalent.*

- (i) Δ is Cohen-Macaulay (over k).
- (ii) For all $F \in \Delta$, $\tilde{H}_i(\text{lk } F) = 0$ if $i < \dim(\text{lk } F)$. (Here $\tilde{H}$ denotes reduced simplicial homology with coefficient field k .)
- (iii) Let $X = |\Delta|$, the geometric realization of Δ , so that Δ is a triangulation of X . Then $\tilde{H}_i(X) = H_i(X, X - p) = 0$ for all $i < \delta$ and all $p \in X$. (Here $\tilde{H}$ denotes reduced singular homology and H relative singular homology, both over k .)

The equivalence of (i) and (ii) above is a theorem of G. Reisner [16], while the equivalence of (ii) and (iii) is a purely topological result first explicitly proved by J. Munkres [15, Theorem 2.1]. A stronger result was later proved by Hochster [10, Theorem 4.1].

Let us remark that the following results are immediate consequences of Theorem 1.2: (a) every Cohen-Macaulay complex is pure, (b) a Cohen-Macaulay complex of dimension greater than zero is connected, and (c) a graph (= complex of dimension zero or one) is Cohen-Macaulay if and only if it has no edges or is connected.

Let $H(\Delta, m)$, $m \in \mathbb{N}$, denote the Hilbert function of A_Δ . It is easy to see [22, Proposition 3.2] that

$$H(\Delta, m) = \begin{cases} 1, & \text{if } m = 0, \\ \sum_{i=0}^{\delta} f_i \binom{m-1}{i}, & \text{if } m > 0, \end{cases} \quad (2)$$

where $(f_0, f_1, \dots, f_\delta)$ is the f -vector of Δ . An immediate consequence of (2) (using [1, Theorem 11.4]) is the result $\dim A_\Delta = 1 + \dim \Delta = 1 + \delta$. Since $H(\Delta, m)$ is a polynomial in m for $m \geq 1$, it follows that there are integers $1 = h_0, h_1, \dots, h_{\delta+1}$ such that

$$(1 - \lambda)^{1+\delta} F(A_\Delta, \lambda) = h_0 + h_1 \lambda + \dots + h_{1+\delta} \lambda^{1+\delta}.$$

The vector $\mathbf{h} = \mathbf{h}(\Delta) = (h_0, h_1, \dots, h_{1+\delta})$ is called the h -vector of Δ . We wish to state a characterization analogous to Theorem 1.1 of the h -vector of a Cohen-Macaulay complex Δ . To do so, recall that a *multiset* M on a set S is a set with repeated elements belonging to S . More precisely, M is a function $S \rightarrow \mathbb{N}$, where $M(x)$ is regarded as the number of repetitions of $x \in S$. The

cardinality of M is $\text{card } M = \sum_{x \in S} M(x)$. A multiset $M': S \rightarrow \mathbb{N}$ is a *submultiset* of M (denoted $M' \subset M$) if $M'(x) \leq M(x)$ for all $x \in S$. A *multicomplex* is a collection of multisets such that if $M \in \Lambda$ and $M' \subset M$, then $M' \in \Lambda$. The dimension and f -vector of a multicomplex are defined in the obvious way in analogy with complexes, i.e., $f_i = \text{card}\{M \in \Lambda: \text{card } M = i + 1\}$ and $\dim \Lambda = \max\{i: f_i \neq 0\}$. Any vector $(f_0, f_1, \dots, f_\delta)$ which is the f -vector of a multicomplex is called an M -vector. We allow $f_{j+1} = f_{j+2} = \dots = f_\delta = 0$ in an M -vector; if also $f_j \neq 0$ this means that the corresponding multicomplex Λ has dimension j .

In analogy with Theorem 1.1 we have the following result essentially due to Macaulay [13] (explaining our terminology " M -vector"). A common generalization of Theorems 1.1 and 1.3 appears in [3], and an exposition of these results appears in [9].

1.3 THEOREM. *Given positive integers f and i , write*

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \dots > n_j \geq j \geq 1$ (exactly as in Theorem 1.1); and define

$$f^{(i)} = \binom{n_i + 1}{i + 1} + \dots + \binom{n_j + 1}{j + 1},$$

with $0^{(i)} = 0$. Then the vector $\mathbf{f} = (f_0, f_1, \dots, f_\delta)$ of nonnegative integers is an M -vector if and only if $f_{i+1} \leq f_i^{(i+1)}$ for $0 \leq i \leq \delta - 1$. $\square$

We can now give the characterization [23, Theorem 6] of the h -vector of a Cohen-Macaulay complex.

1.4 THEOREM. *A vector $(h_0, h_1, \dots, h_{\delta+1})$ is the h -vector of a Cohen-Macaulay complex of dimension δ if and only if $h_0 = 1$ and $(h_1, h_2, \dots, h_{\delta+1})$ is an M -vector. $\square$*

2. Balanced complexes. We wish to introduce a class of Cohen-Macaulay complexes for which Theorem 1.4 can be considerably strengthened and refined. Recall that an *ordered partition* of a finite set V is a sequence $(V_1, \dots, V_m)$ of nonvoid, pairwise disjoint subsets of V satisfying $V_1 \cup \dots \cup V_m = V$.

DEFINITION. A *balanced complex of type $(a_1, \dots, a_m)$* is a pair (Δ, π) satisfying:

- (i) Δ is a pure δ -complex on a vertex set V ,
- (ii) $\pi = (V_1, \dots, V_m)$ is an ordered partition of V , and
- (iii) for every maximal face $F \in \Delta$ and every $i \in [m]$, we have $\text{card}(F \cap V_i) = a_i$. (Hence $a_1 + \dots + a_m = \delta + 1$.)

A balanced complex of type $(1, 1, \dots, 1)$ is called *completely balanced*. Note that a balanced complex of type $(\delta + 1)$ (i.e., with $m = 1$) is really nothing more than a pure δ -complex, since condition (iii) holds automatically. We could have altered our definition somewhat so that Δ need not be pure, but nothing significant is gained by doing so. In particular, we are primarily concerned with Cohen-Macaulay complexes, and these are always pure.

We now give some examples of completely balanced complexes. Let P be a poset (= partially ordered set) on a finite set V , and define $\Delta(P)$ to be the complex on V whose faces are the chains (= linearly ordered subsets) of P . We will use such terminology as “ P is pure” or “ P is Cohen-Macaulay” to mean the corresponding statement for $\Delta(P)$. Thus P is pure if and only if all maximal chains of P have the same length, and $\dim P$ is the length of the longest chain in P . Moreover, a *Cohen-Macaulay poset* P is one for which $\Delta(P)$ is a Cohen-Macaulay complex. Suppose now that P is pure of dimension δ . If $x \in V$, let $\rho(x)$ be the largest integer r for which there is a chain $x_1 < x_2 < \dots < x_r = x$ in P . We call $\rho(x)$ the *rank* of x and ρ the *rank function* of P . (Some authors would call $\rho(x) - 1$ the rank or height of x .) If we set $V_i = \{x \in P: \rho(x) = i\}$, $1 \leq i \leq \delta + 1$, then clearly $\pi = (V_1, V_2, \dots, V_{\delta+1})$ is an ordered partition of V and $(\Delta(P), \pi)$ is completely balanced. We call π the *standard* ordered partition of P . If Δ is any complex, let $Q = Q(\Delta)$ be the poset of nonvoid faces of Δ , ordered by inclusion. Then $\Delta(Q)$ is just the first barycentric subdivision of Δ . Hence any space X which possesses a finite pure triangulation possesses a completely balanced triangulation. There does not seem to be a nice characterization of completely balanced complexes, though a sufficient condition for Δ to be completely balanced is mentioned in [5]. On the other hand, one can characterize complexes Δ of the form $\Delta(P)$ for a finite poset P . Namely, it is necessary and sufficient that Δ satisfy the following:

- (i) any minimal set of vertices which do not form a face of Δ has two elements (i.e., the ideal I_Δ is generated by quadratic monomials), and
- (ii) let Γ be the 1-skelton of Δ . Then Γ must satisfy the well-known conditions of Gilmore and Hoffman [8], Ghouila-Houri [7], or Gallai [6] for being a comparability graph.

3. Numerical invariants of balanced complexes. If (Δ, π) is a balanced complex of type $\mathbf{a} = (a_1, a_2, \dots, a_m)$ and if $\mathbf{b} = (b_1, b_2, \dots, b_m) \in \mathbb{N}^m$, then define $f_{\mathbf{b}} = f_{\mathbf{b}}(\Delta, \pi)$ to be the number of faces $F \in \Delta$ for which $\text{card}(F \cap V_i) = b_i$, $1 \leq i \leq m$. Note that $f_{\mathbf{b}} = 0$ unless $b_i \leq a_i$ for all i (written $\mathbf{b} \leq \mathbf{a}$). Note also that $f_i(\Delta) = \sum f_{\mathbf{b}}(\Delta, \pi)$, where the sum is over all vectors $\mathbf{b} \leq \mathbf{a}$ such that $\sum b_j = i + 1$. Hence the numbers $f_{\mathbf{b}}$ are a refinement of the numbers f_i .

If $S \subset [m]$, let (Δ_S, π_S) be the balanced complex defined as follows:

- (i) $\Delta_S = \{F \in \Delta: F \cap V_i = \emptyset \text{ if } i \notin S\}$,
- (ii) if

$$S = \{c_1, c_2, \dots, c_r\} \text{ with } c_1 < c_2 < \dots < c_r, \quad (3)$$

then $\pi_S = (V_{c_1}, V_{c_2}, \dots, V_{c_r})$.

Hence (Δ_S, π_S) is balanced of type $(a_{c_1}, a_{c_2}, \dots, a_{c_r})$.

For instance, if P is a pure poset of dimension δ and if π is the standard ordered partition of P , then $\Delta(P)_S = \Delta(P_S)$, where P_S is the poset obtained from P by removing all elements whose ranks do not belong to S . In particular, $P_{[\gamma]}$ is the so-called "rank γ upper-truncation of P ". Note that in general if (Δ, π) has type $(a_1, \dots, a_m)$, then $(\Delta_{[m]}, \pi_{[m]}) = (\Delta, \pi)$, and $\Delta_\emptyset = \emptyset$.

The following result is an immediate consequence of the definition of (Δ_S, π_S) .

3.1 PROPOSITION. *Let (Δ, π) be a balanced complex of type $(a_1, a_2, \dots, a_m) = \mathbf{a}$. Let $S \subset [m]$, say $S = \{c_1, c_2, \dots, c_r\}$ with $c_1 < c_2 < \dots < c_r$, so that (Δ_S, π_S) is balanced of type $(a_{c_1}, a_{c_2}, \dots, a_{c_r}) = \mathbf{a}'$. If $(b_{c_1}, \dots, b_{c_r}) = \mathbf{b}' \leq \mathbf{a}'$, define $(b_1, \dots, b_m) = \mathbf{b}$ by letting $b_i = 0$ if i is not one of the c_j . Then $f_{\mathbf{b}}(\Delta_S, \pi_S) = f_{\mathbf{b}}(\Delta, \pi)$. $\square$*

The significance of Proposition 3.1 is the following. If we know the numbers $f_{\mathbf{b}}(\Delta, \pi)$ for all $\mathbf{b}$, then we also know the numbers $f_{\mathbf{b}}(\Delta_S, \pi_S)$ for all $S \subset [m]$ and all $\mathbf{b}'$.

We can refine the h -vector of a balanced δ -complex (Δ, π) , where $\pi = (V_1, \dots, V_m)$, just as we did the f -vector, as follows. We make the ring A_Δ into an $\mathbb{N}^m$ -graded k -algebra by defining, for a vertex x of Δ , $\deg x$ to be the i th unit coordinate vector $\mathbf{e}_i \in \mathbb{N}^m$. Equivalently, $\lambda^{\deg x} = \lambda_i$ if $x \in V_i$.

3.2 PROPOSITION. *Let (Δ, π) be a balanced complex of type $\mathbf{a} = (a_1, \dots, a_m)$, where $\pi = (V_1, \dots, V_m)$, and let $H(A_\Delta, \mathbf{b})$ denote the Hilbert function of A_Δ with the above $\mathbb{N}^m$ -grading. Then for all $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{N}^m$,*

$$H(A_\Delta, \mathbf{b}) = \sum_{\mathbf{c}} f_{\mathbf{c}}(\Delta, \pi) \prod_{b_i > 0} \binom{b_i - 1}{c_i - 1},$$

where the sum is over all $(c_1, \dots, c_m) = \mathbf{c} \leq \mathbf{a}$ such that $c_i = 0 \Leftrightarrow b_i = 0$.

Note that Proposition 3.2 reduces to (2) when $m = 1$.

PROOF OF PROPOSITION 3.2. If $M = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a nonzero monomial appearing in A_Δ , then define the *support* of M by $\text{supp } M = \{x_i \in V: \alpha_i > 0\}$. Note $\text{supp } M \in \Delta$. Given $F \in \Delta$ with $c_i = \text{card } F \cap V_i$, the number of monomials M satisfying $\text{supp } M = F$ and $\deg M = \mathbf{b}$ is $\prod_{i: b_i > 0} \binom{b_i - 1}{c_i - 1}$, since there are $\binom{b_i - 1}{c_i - 1}$ monomials of degree b_i in c_i variables with each variable having positive exponent. Summing over all $F \in \Delta$ completes the proof. $\square$

It is an immediate consequence of Proposition 3.2, or can be easily seen directly, that

$$F(A_\Delta, \lambda) = \sum_{F \in \Delta} \prod_{x \in F} \lambda_{\rho(x)} (1 - \lambda_{\rho(x)})^{-1}, \quad (4)$$

where for each $x \in V$, $\rho(x)$ is defined by $x \in V_{\rho(x)}$.

Hence we obtain the following result, which will be of use later.

3.3 PROPOSITION. *Let (Δ, π) be a balanced complex of type $(a_1, \dots, a_m)$. Then $F(A_\Delta, \lambda) \prod_{i=1}^m (1 - \lambda_i)^{a_i}$ (or equivalently $F(A_\Delta, \lambda) \prod_{x \in F} (1 - \lambda_{\rho(x)})$) for any maximal face F of Δ is a polynomial $P(A_\Delta, \lambda)$ in $\lambda_1, \dots, \lambda_m$. Moreover, the degree of $P(A_\Delta, \lambda)$ with respect to λ_i is no more than a_i . In particular, if (Δ, π) is completely balanced then every monomial appearing in $P(A_\Delta, \lambda)$ is squarefree.*

□

Now if $\mathbf{b} \in \mathbb{N}^m$ define $h_{\mathbf{b}} = h_{\mathbf{b}}(\Delta, \pi)$ to be the coefficient of $\lambda^{\mathbf{b}}$ in the polynomial $F(A_\Delta, \lambda) \prod_{i=1}^m (1 - \lambda_i)^{a_i}$. Proposition 3.3 asserts that

$$h_{\mathbf{b}} = 0 \quad \text{unless } \mathbf{b} \leq \mathbf{a}. \quad (5)$$

Clearly $h_i(\Delta) = \sum h_{\mathbf{b}}(\Delta, \pi)$, where $\mathbf{b}$ ranges over all $(b_1, \dots, b_m) \in \mathbb{N}^m$ satisfying $b_1 + \dots + b_m = i$. Hence the numbers $h_{\mathbf{b}}$ are a refinement of the numbers h_i . Note that

$$\begin{aligned} F(A_\Delta, \lambda) \prod_{i=1}^n (1 - \lambda_i)^{a_i} &= \left[\sum_{F \in \Delta} \prod_{x \in F} \lambda_{\rho(x)} (1 - \lambda_{\rho(x)})^{-1} \right] \prod_{i=1}^n (1 - \lambda_i)^{a_i} \\ &= \sum_{\mathbf{c}} \sum_{\substack{F \in \Delta \\ \text{card}(F \cap V_i) = c_i}} \lambda^{\mathbf{c}} \prod_{i=1}^n (1 - \lambda_i)^{a_i - c_i} \\ &= \sum_{\mathbf{c}} f_{\mathbf{c}}(\Delta, \pi) \lambda^{\mathbf{c}} \prod_{i=1}^n (1 - \lambda_i)^{a_i - c_i} \end{aligned}$$

from which we get

$$h_{\mathbf{b}}(\Delta, \pi) = \sum_{\mathbf{c} \leq \mathbf{b}} f_{\mathbf{c}}(\Delta, \pi) \prod_{i=1}^m (-1)^{b_i - c_i} \binom{a_i - c_i}{b_i - c_i}. \quad (6)$$

3.4 EXAMPLE. (a) Let $\Delta = \langle abc, abe, acd, ade, bcd, bef, bfg, cdg, def, dfg \rangle$, so $|\Delta| \approx S^2$. Let $V_1 = \{a, c, e, f, g\}$, $V_2 = \{b, d\}$ and $\pi = (V_1, V_2)$. Then (Δ, π) is balanced of type $(2, 1)$. We have (writing f_{rs} for $f_{(r,s)}$) $f_{00} = 1$, $f_{10} = 5$, $f_{01} = 2$, $f_{20} = 5$, $f_{11} = 10$, $f_{21} = 10$. Also (writing $\lambda = (\mu, \lambda)$),

$$\begin{aligned} F(A_\Delta, \lambda) &= (1 - \mu)^2(1 - \lambda) + 5\mu(1 - \mu)(1 - \lambda) + 2\lambda(1 - \mu)^2 \\ &\quad + 5\mu^2(1 - \lambda) + 10\mu\lambda(1 - \mu) + 10\mu^2\lambda \\ &= 1 + 3\mu + \lambda + \mu^2 + 3\mu\lambda + \mu^2\lambda. \end{aligned}$$

Hence $h_{00} = h_{01} = h_{20} = h_{21} = 1$, $h_{10} = h_{11} = 3$.

(b) Suppose $\Delta = \langle ab, cd \rangle$ with $V_1 = \{a, c\}$, $V_2 = \{b, d\}$, and $\pi = (V_1, V_2)$. Then (Δ, π) is of type $(1, 1)$, i.e., is completely balanced. We have $f_{00} = 1, f_{10} = f_{01} = f_{11} = 2$. Also

$$\begin{aligned} F(A_\Delta, \lambda) &= (1 - \mu)(1 - \lambda) + 2\lambda(1 - \mu) + 2\mu(1 - \lambda) + 2\mu\lambda \\ &= 1 + \mu + \lambda - \mu\lambda. \end{aligned}$$

Hence $h_{00} = h_{10} = h_{01} = 1$, $h_{11} = -1$.

In general it is difficult to obtain any intuition for the numbers $h_b(\Delta, \pi)$. There are two special circumstances, however, in which they have additional interpretations. First define the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ of a complex Δ by $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$, where $\chi(\Delta)$ is the usual Euler characteristic. Equivalently, $\tilde{\chi}(\Delta) = -1 + f_0(\Delta) - f_1(\Delta) + \dots$. In particular, $\tilde{\chi}(\emptyset) = -1$. Note that if the reduced homology of Δ satisfies $\tilde{H}_i(\Delta) = 0$ for $i < \delta = \dim \Delta$, then $(-1)^\delta \tilde{\chi}(\Delta) = \dim_k \tilde{H}_\delta(\Delta) \geq 0$.

3.5 PROPOSITION. *Let (Δ, π) be a balanced complex of type $(a_1, \dots, a_m)$. Let $S \subset [m]$, and define $\mathbf{b} = (b_1, \dots, b_m)$ by*

$$b_i = \begin{cases} a_i, & \text{if } i \in S, \\ 0, & \text{if } i \notin S. \end{cases}$$

Let (Δ_S, π_S) be the balanced complex defined by (3). Then $h_b(\Delta, \pi) = (-1)^\delta \tilde{\chi}(\Delta_S)$, where $\delta = b_1 + \dots + b_m - 1 = \dim \Delta_S$.

PROOF. Let $\mathbf{c} = (c_1, \dots, c_m) \leq \mathbf{b}$. For all i we have $\binom{a_i - c_i}{b_i - c_i} = 1$, since $b_i = a_i$ or $b_i = c_i = 0$. Hence from (6),

$$\begin{aligned} h_b(\Delta, \pi) &= \sum_{\mathbf{c} \leq \mathbf{b}} f_{\mathbf{c}}(\Delta, \pi) \prod_{i=1}^m (-1)^{b_i - c_i} \\ &= (-1)^\delta \sum_{i=-1}^{\delta} f_i(\Delta_S) (-1)^i \quad (\text{with } f_{-1}(\Delta) = 1) \\ &= (-1)^\delta \tilde{\chi}(\Delta_S). \quad \square \end{aligned}$$

Note that if (Δ, π) is completely balanced then any $\mathbf{b} \leq \mathbf{a}$ satisfies the hypothesis of Proposition 3.5. Hence in the completely balanced case every $h_b(\Delta, \pi)$ may be interpreted as a reduced Euler characteristic (up to sign).

For our second alternative interpretation of the numbers $h_b(\Delta, \pi)$, we need to define the notion of “shellability.” Our definition is slightly more general than that sometimes given, e.g., [4]. If Δ is a pure δ -complex, then a *shelling* of Δ is an ordering $F_1, F_2, \dots, F_r$ of the δ -faces of Δ (so $r = f_\delta(\Delta)$) such that for $1 \leq i \leq r - 1$, $(F_1 \cup F_2 \cup \dots \cup F_i) \cap F_{i+1}$ is a nonvoid union of $(\delta - 1)$ -faces of F_{i+1} . In exactly the same manner as McMullen’s interpretation [14, p.

182] of $h_i(\Delta)$ when Δ is shellable (McMullen uses $g_{i-1}^{(d)}$ for our h_i), we obtain the following result.

3.6 PROPOSITION. *Let (Δ, π) be a balanced complex where $\pi = (V_1, \dots, V_m)$, and let $F_1, F_2, \dots, F_r$ be a shelling of Δ . For each $i \in [r]$, define G_i to be the unique minimal face of F_i which is not contained in $F_1 \cup F_2 \cup \dots \cup F_{i-1}$. (In particular, $G_1 = \emptyset$.) Define $\mathbf{b}(i) = (b_1(i), \dots, b_m(i))$ by $b_j(i) = \text{card } G_i \cap V_j$. Then $h_{\mathbf{b}}(\Delta, \pi)$ is equal to the number of integers $i \in [m]$ for which $\mathbf{b} = \mathbf{b}(i)$.*

PROOF. Let $F_i(\lambda)$ be the Poincaré series for the pure subcomplex of Δ whose maximal faces are $F_1, F_2, \dots, F_i$. Then by the definition of G_i and $\mathbf{b}(i)$,

$$F_{i+1}(\lambda) = F_i(\lambda) + \frac{\lambda^{\mathbf{b}(i)}}{\prod_{j=1}^m (1 - \lambda_j)^{a_j}},$$

so that

$$F(A_{\Delta}, \lambda) \prod_{i=1}^m (1 - \lambda_i)^{a_i} = \sum_{i=1}^m \lambda^{\mathbf{b}(i)}.$$

The proof now follows from the definition of $h_{\mathbf{b}}(\Delta, \pi)$. $\square$

3.7 EXAMPLE. Let (Δ, π) be the balanced complex of Example 3.4(a). Then (writing abc for $\{a, b, c\}$, etc.) $abc, acd, ade, abe, cdg, dfg, def, bef, bfg, bcd$ is a shelling of Δ . We have $G_1 = \emptyset$, $G_2 = d$, $G_3 = e$, $G_4 = be$, $G_5 = g$, $G_6 = f$, $G_7 = ef$, $G_8 = bf$, $G_9 = bg$, $G_{10} = bcd$. Since $V_1 = \{a, c, e, f, g\}$ and $V_2 = \{b, d\}$, we have $\mathbf{b}(1) = (0, 0)$, $\mathbf{b}(2) = (0, 1)$, $\mathbf{b}(3) = (1, 0)$, $\mathbf{b}(4) = (1, 1)$, $\mathbf{b}(5) = (1, 0)$, $\mathbf{b}(6) = (1, 0)$, $\mathbf{b}(7) = (2, 0)$, $\mathbf{b}(8) = (1, 1)$, $\mathbf{b}(9) = (1, 1)$, $\mathbf{b}(10) = (2, 1)$, and we obtain the same values for $h_{\mathbf{b}}(\Delta, \pi)$ as before.

Proposition 3.6 shows that $h_{\mathbf{b}}(\Delta, \pi) \geq 0$ when Δ is shellable. However, it is easy to see that all shellable complexes are Cohen-Macaulay, so that the inequality $h_{\mathbf{b}}(\Delta, \pi) \geq 0$ is subsumed and generalized by Theorem 4.4 below. For a survey of some aspects of the subject of shellability, see [4]. Examples of shellable complexes include: (i) the boundary complex of a simplicial convex polytope (but not necessarily a triangulation of a sphere), (ii) connected graphs and triangulations of 2-cells, (iii) the independent set complex and broken circuit complex [23, §7] of a finite matroid, and (iv) the complex $\Delta(P)$ where P is an admissible lattice in the sense of [21]. Example (iii) is due to Scott Provan and (iv) to Anders Björner.

There is a somewhat weaker condition than shellability which implies that $h_{\mathbf{b}}(\Delta, \pi) \geq 0$. If Δ is a complex and if $G \subset F$ are faces of Δ , then define the interval $[G, F]$ by $[G, F] = \{F' \mid G \subset F' \subset F\}$. An *upper partition* of a pure δ -complex Δ is a collection $[G_1, F_1], \dots, [G_r, F_r]$ of intervals of Δ satisfying:

- (i) $[G_i, F_i] \cap [G_j, F_j] = \emptyset$ if $i \neq j$,
- (ii) $\Delta = [G_1, F_1] \cup \dots \cup [G_r, F_r]$,

(iii) $\dim F_i = \delta$ for all $i \in [r]$ (so $r = f_\delta(\Delta)$).

A complex Δ possessing an upper partition is said to be *partitionable*. It is easily seen that every shellable complex is partitionable. The converse is false. In fact, if $\Delta = \langle ab, cd, ce, de \rangle$ then Δ is partitionable but not even Cohen-Macaulay. An upper partition of Δ is given by $[\emptyset, ab]$, $[c, cd]$, $[d, de]$, $[e, ce]$. We do not know if every Cohen-Macaulay complex is partitionable. Now assume (Δ, π) is balanced with $\pi = (V_1, \dots, V_m)$, and suppose $[G_1, F_1], \dots, [G_r, F_r]$ is an upper partition of Δ . Then as a slight generalization of Proposition 3.6, it is easily shown that if $\mathbf{b} = (b_1, \dots, b_m)$, then $h_{\mathbf{b}}(\Delta, \pi)$ is equal to the number of $j \in [r]$ for which $\text{card } G_j \cap V_i = b_i$ for all $i \in [m]$. Partitionable complexes were independently considered by Provan [25, Appendix 4].

4. Balanced Cohen-Macaulay complexes. A balanced complex (Δ, π) for which Δ is a Cohen-Macaulay complex is called a *balanced Cohen-Macaulay complex*. Our main aim is to give restrictions on the numbers $h_{\mathbf{b}}$ associated with a balanced Cohen-Macaulay complex which strengthens and refines Theorem 1.4. Our results are based on the following fundamental algebraic property of balanced complexes.

4.1 THEOREM. *Let (Δ, π) be a balanced complex of type $(a_1, \dots, a_m)$, where $\pi = (V_1, \dots, V_m)$. Let Δ_i stand for the complex $\Delta_{(i)}$ of (3), so that Δ_i is just the restriction of Δ to V_i . Set $A_i = A_{\Delta_i}$, and give this ring an $\mathbb{N}$ -grading by defining $\deg x = 1$ for $x \in V_i$. Now give A_{Δ} the $\mathbb{N}^m$ -grading $\deg x = \mathbf{e}_i$ if $x \in V_i$ (as defined preceding Proposition 3.2). Suppose Ψ_i is an $\mathbb{N}$ -homogeneous system of parameters for A_i . Then $\Psi = \Psi_1 \cup \dots \cup \Psi_m$ is an $\mathbb{N}^m$ -homogeneous system of parameters for A_{Δ} .*

PROOF. If $\theta \in \Psi_i$ has degree $\alpha \in \mathbb{N}$ in A_i , then θ is $\mathbb{N}^m$ -homogeneous in A_{Δ} , with $\deg \theta = \alpha \mathbf{e}_i \in \mathbb{N}^m$. Since $a_1 + a_2 + \dots + a_m = \dim A_{\Delta}$, it remains only to show $\dim_k A_{\Delta}/(\Psi) < \infty$, where (Ψ) denotes the ideal generated by all $\theta \in \Psi$. Now A_{Δ} is a quotient ring of $A_1 \otimes_k A_2 \otimes_k \dots \otimes_k A_m$, so

$$\dim_k A_{\Delta}/(\Psi) \leq \prod_{i=1}^m \dim_k A_i/(\Psi_i) < \infty.$$

This completes the proof. $\square$

4.2 COROLLARY. *Let (Δ, π) be completely balanced with $\pi = (V_1, \dots, V_m)$. Let $\theta_i = \sum_{x \in V_i} x$. Then $\theta_1, \dots, \theta_m$ is an $\mathbb{N}^m$ -homogeneous system of parameters for A_{Δ} . Indeed, $\deg \theta_i = \mathbf{e}_i$.*

PROOF. We have $A_i = k[V_i]/I_i$, where I_i is generated by all products xx' such that $x, x' \in V_i$ and $x \neq x'$. From this it follows that the single element

θ_i is a system of parameters for A_i , and the proof follows from Theorem 4.1. $\square$

REMARK. When (Δ, π) is completely balanced, Corollary 4.2 gives an h.s.o.p. for A_Δ consisting of linear forms (i.e., of N-degree one). There is a more general result which gives a necessary and sufficient condition for a set of linear forms to be an N-h.s.o.p. for *any* A_Δ . Let $d = \dim A_\Delta = 1 + \dim \Delta$, and let

$$\theta_i = \sum_{j=1}^n \alpha_{ij} x_j, \quad \alpha_{ij} \in k, 1 \leq i \leq d,$$

be a set of d linear forms in A_Δ . Then $\theta_1, \dots, \theta_d$ is an N-h.s.o.p. if and only if for every $F \in \Delta$ (equivalently, for every maximal $F \in \Delta$) the $d \times (\text{card } F)$ matrix (α_{ij}) , where $1 \leq i \leq d$ and $x_j \in F$, has rank equal to $\text{card } F$. We omit the proof.

Theorem 4.1 allows an easy proof of the next result.

4.3 THEOREM. *Let (Δ, π) be a balanced Cohen-Macaulay complex of type $(a_1, \dots, a_m)$, and let $S \subset [m]$. Then (Δ_S, π_S) is a balanced Cohen-Macaulay complex of type $(a_{c_1}, a_{c_2}, \dots, a_{c_r})$, where $S = \{c_1, c_2, \dots, c_r\}$ and $c_1 < c_2 < \dots < c_r$.*

PROOF. We only need to prove that Δ_S is Cohen-Macaulay. By Theorem 1.2, the desired result is of a purely topological nature. An almost equivalent result was first proved by J. Munkres [15, Theorem 6.4] using topological methods, and his proof straightforwardly extends to Theorem 4.3. However, it may be of interest to give a simple alternative proof based directly on the definition (1) of a Cohen-Macaulay ring.

Let A_Δ have the usual $\mathbb{N}^m$ -grading defined by $\deg x = \mathbf{e}_i$ if $x \in V_i$. Let $\Psi = \psi_1 \cup \dots \cup \psi_m$ be a homogeneous system of parameters of the type described by Theorem 4.1. Then by (1),

$$F(A_\Delta, \lambda) \prod_{\theta \in \Psi} (1 - \lambda^{\deg \theta}) = F(A_\Delta / (\Psi), \lambda). \quad (7)$$

Since Δ_S consists of those faces $F \in \Delta$ for which $x \in V_{c_1} \cup \dots \cup V_{c_r}$ whenever $x \in F$, it follows that $F(A_{\Delta_S}, \lambda)$ is obtained from $F(A_\Delta, \lambda)$ by setting $\lambda_i = 0$ if $i \notin S$ and then substituting λ_i for λ_{c_i} . Let $\Psi_S = \bigcup_{i \in S} \Psi_i$. By Theorem 4.1, Ψ_S is an h.s.o.p. for A_{Δ_S} . If $\deg$ (resp. $\deg_S$) denotes degree in A_Δ (resp. A_{Δ_S}), it follows that $\prod_{\theta \in \Psi_S} (1 - \lambda^{\deg_S \theta})$ is obtained from $\prod_{\theta \in \Psi} (1 - \lambda^{\deg \theta})$ by the same substitution as above. Now note that $A_{\Delta_S} / (\Psi_S) = A_\Delta / (\Psi, X)$, where X consists of all $x \in V$ such that $x \notin V_i$ for any $i \in S$. Since $A_\Delta / (\Psi)$ is $\mathbb{N}^m$ -graded, a k -basis for $A_\Delta / (\Psi, X)$ consists of those monomials in $A_\Delta / (\Psi)$ whose support lies in $V_{c_1} \cup \dots \cup V_{c_r}$. The remaining monomials in $A_\Delta / (\Psi)$ are zero modulo X . Hence $F(A_\Delta / (\Psi, X), \lambda)$ is

obtained from $F(A_{\Delta}/(\Psi), \lambda)$ by making once again the same substitution $\lambda_i \rightarrow 0$ if $i \notin S$ and then $\lambda_{c_i} \rightarrow \lambda_i$. Hence when we make this substitution in (7) we obtain

$$F(A_{\Delta_S}, \lambda) \prod_{\theta \in \Psi_S} (1 - \lambda^{\deg_S \theta}) = F(A_{\Delta_S}/(\Psi_S), \lambda).$$

By (1), it follows that A_{Δ_S} is Cohen-Macaulay. $\square$

We are now in a position to discuss the h -vectors of balanced Cohen-Macaulay complexes.

4.4 THEOREM. *Let (Δ, π) be a balanced Cohen-Macaulay complex of type $\mathbf{a} = (a_1, \dots, a_m)$, with $\pi = (V_1, \dots, V_m)$. Let $v_i = \text{card } V_i$, and let T be a set with $\sum_{i=1}^m (v_i - a_i)$ elements, say $T = \{y_{ij}: 1 \leq i \leq m, 1 \leq j \leq v_i - a_i\}$. Then there exists a multicomplex Λ on T with the following property: For every $(b_1, \dots, b_m) \in \mathbb{N}^m$, the number of $M \in \Lambda$ satisfying*

$$\sum_{j=1}^{v_i - a_i} M(y_{ij}) = b_i, \quad \text{for all } i \in [m], \quad (8)$$

is equal to $h_b(\Delta, \pi)$. Hence by (5),

$$\sum_{j=1}^{v_i - a_i} M(y_{ij}) \leq a_i, \quad \text{for all } i \in [m]. \quad (9)$$

Before proving this result, we first discuss its significance. According to Theorem 1.4, the h -vector of a Cohen-Macaulay complex Δ is the f -vector of some multicomplex Λ . Theorem 4.4 asserts that Λ must have certain special properties when (Δ, π) is balanced of type $(a_1, \dots, a_m)$. It follows from (9) that $M(y_{ij}) \leq a_i$ for all $i \in [m]$. Thus each $y_{ij} \in T$ has a restriction as to its multiplicity in any $M \in \Lambda$. In general, given a vector $\mathbf{c} = (c_1, c_2, \dots, c_r)$ where each c_i is a positive integer or ∞ , there is a characterization analogous to Theorems 1.1 and 1.3 for the f -vector of a multicomplex Λ on a set $S = \{y_1, \dots, y_r\}$ such that $M(y_i) \leq c_i$ for all $i \in [r]$. This characterization is essentially due to Clements and Lindström [3], although an explicit numerical statement first appeared in [2] and is restated succinctly in [9]. Note that Theorem 1.1 corresponds to the case $\mathbf{c} = (1, 1, \dots, 1)$ and Theorem 1.3 to the case $\mathbf{c} = (\infty, \infty, \dots, \infty)$. At any rate, Theorem 4.4 shows that Λ must satisfy the characterization with $v_i - a_i$ of the c_j 's equal to a_i . But Theorem 4.4 actually asserts a much stronger result, viz., the elements of certain subsets of T cannot have their *combined* multiplicities greater than a_i . Moreover, Theorem 4.4 places a restriction not merely on the ordinary h -vector of Δ , but on the "refined" numbers $h_b(\Delta, \pi)$. Unfortunately the condition which Theorem 4.4 places on $h_b(\Delta, \pi)$ is not strong enough to characterize the number $h_b(\Delta, \pi)$ when (Δ, π) is balanced of some fixed type $(a_1, \dots, a_m)$.

unless $m = 1$ (Theorem 1.4), so for $m > 1$ we do not have as complete a result as Theorem 1.4.

The completely balanced case of Theorem 4.4 is of special interest and deserves a separate statement. When (Δ, π) is completely balanced, (9) implies that each $M(y_i) \leq 1$. In other words, Λ is actually a complex, not just a multicomplex, and we obtain:

4.5 COROLLARY. *Let (Δ, π) be a completely balanced Cohen-Macaulay complex. Then the h -vector of Δ is the f -vector of some complex Λ and therefore satisfies Theorem 1.1. (In other words, $\mathbf{h}(\Delta)$ is not merely an M -vector, but also a K -vector.) Even more strongly, there is an ordered partition $(T_1, \dots, T_m)$ of the vertex set T of Λ such that $\text{card } T_i \cap F \leq 1$ for all $F \in \Lambda$, $i \in [m]$. Equivalently, the 1-skeleton of Λ can be m -colored in the usual graph-theoretical sense. Moreover, Λ can be chosen so that for any 0-1 vector $\mathbf{b} = (b_1, \dots, b_m)$, there are exactly $h_{\mathbf{b}}(\Delta, \pi)$ faces $F \in \Lambda$ satisfying: $T_i \cap F = \emptyset$ if and only if $b_i = 0$.*

□

As an example to show that the conditions on Λ given by Corollary 4.5 are not sufficient to characterize the h -vector of a completely balanced Cohen-Macaulay complex, let Λ be the complex on $T = \{x_1, \dots, x_7\}$ with maximal faces $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\{x_5\}$, $\{x_6\}$, $\{x_7\}$. Let $T_1 = \{x_1, x_3, x_5, x_6, x_7\}$, $T_2 = \{x_2, x_4\}$. Then (Δ, π) would satisfy $h_{00} = 1$, $h_{10} = 5$, $h_{01} = 2$, $h_{11} = 2$. In particular, (Δ, π) would be of type $(1, 1)$ (i.e., a bipartite graph) with 6 vertices and 10 edges, and no such graph exists.

We remark in passing one additional property of completely balanced Cohen-Macaulay complexes (Δ, π) . Namely, π is uniquely determined up to order. In other words, if (Δ, σ) is also completely balanced, then the entries V_i of π are a permutation of those of σ . Thus the 1-skeleton of Δ is a so-called "uniquely m -colorable graph." The proof is omitted. The corresponding statement for arbitrary completely balanced complexes is false, as shown by the example $\Delta = \langle ab, cd \rangle$.

PROOF OF THEOREM 4.4. If K is an extension field of k , then the ring $A_{\Delta} \otimes_k K$ has the same Hilbert function as A_{Δ} and is Cohen-Macaulay if and only if A_{Δ} is Cohen-Macaulay. Hence we may assume k is infinite. By Theorem 4.1 there is an N^m -homogeneous system of parameters $\theta_1, \dots, \theta_d$ ($d = a_1 + \dots + a_m = 1 + \dim \Delta$) for A_{Δ} , and our assumption that k is infinite implies we can choose them so that exactly a_i of them have degree $e_i \in N^m$. Let $B_{\Delta} = A_{\Delta}/(\theta_1, \dots, \theta_d)$. Hence by (1) and our assumption that (Δ, π) is Cohen-Macaulay, we have

$$F(A_{\Delta}, \lambda) \prod_{i=1}^m (1 - \lambda_i)^{a_i} = F(B_{\Delta}, \lambda).$$

Thus by the definition of $h_b(\Delta, \pi)$, we have

$$F(B_\Delta, \lambda) = \sum_b h_b(\Delta, \pi) \lambda^b. \quad (10)$$

Now suppose $V_i = \{x_{i1}, \dots, x_{i\nu_i}\}$. Let y_{ij} denote the image in B_Δ of x_{ij} . Since the a_i parameters of degree $\mathbf{e}_i$ are linearly independent, it follows that B_Δ is generated as a k -algebra by the elements y_{ij} , $1 \leq i \leq m$, $1 \leq j \leq \nu_i - a_i$. Hence B_Δ has a k -basis consisting of monomials in these y_{ij} . A simple argument due to Macaulay [13] and also given in [24, Theorem 2.1] shows that we can pick this k -basis to be a multicomplex Λ on the set of y_{ij} . By (10), it follows that the number of $M \in \Lambda$ satisfying (8) is $h_b(\Delta, \pi)$, completing the proof. $\square$

5. Posets and Möbius functions. In the special case that $\Delta = \Delta(P)$ for some poset P , our previous results are closely related to certain well-known concepts associated with P . In this section we will sketch this relation.

Let P be a (finite) pure poset with rank function ρ , i.e., $\rho(x)$ is the cardinality of a saturated chain of P with maximum element x . Let $\hat{P}$ denote the poset obtained by adjoining a minimum element $\hat{0}$ and maximum element $\hat{1}$ to P , i.e., $\hat{0} < x < \hat{1}$ for all $x \in P$. Let μ denote the Möbius function of $\hat{P}$, as defined in [17]. Thus, μ is a function from $\{(x, y) \in \hat{P} \times \hat{P} : x \leq y\}$ to $\mathbb{Z}$ satisfying

$$\begin{aligned} \mu(x, x) &= 1 \quad \text{for all } x \in \hat{P}, \\ \sum_{x < y < z} \mu(x, y) &= 0 \quad \text{for all fixed pairs } x < z \text{ in } \hat{P}. \end{aligned}$$

We also write $\mu(x)$ for $\mu(\hat{0}, x)$ and $\mu(P)$ for $\mu(\hat{0}, \hat{1})$.

If $\dim P = \delta$ (i.e., every maximal chain of P has cardinality $\delta + 1$) and $S \subset [\delta + 1]$, define $\alpha(P, S)$ to be the number of chains $x_1 < x_2 < \dots < x_s$ in P such that $\{\rho(x_1), \dots, \rho(x_s)\} = S$. Thus $\alpha(P, \emptyset) = 1$, $\alpha(P, \{i\})$ is the number of elements in P of rank i , and $\alpha(P, [\delta + 1])$ is the number of maximal chains of P . Equivalently, $\alpha(P, S)$ is the number of maximal chains of the poset

$$P_S = \{x \in P : \rho(x) \in S\},$$

or equivalently, the number of maximal chains of the poset $\hat{P}_S$ consisting of P_S with $\hat{0}$ and $\hat{1}$ adjoined. Now for $S \subset [\delta + 1]$ define

$$\beta(P, S) = \sum_{T \subset S} (-1)^{\text{card}(S-T)} \alpha(P, T).$$

Equivalently, $\alpha(P, S) = \sum_{T \subset S} \beta(P, T)$. It is an immediate consequence of “Philip Hall’s theorem” [17, Proposition 6] that

$$\beta(P, S) = (-1)^{1 + \text{card } S} \mu(P_S). \quad (11)$$

The numbers $\alpha(P, S)$ and $\beta(P, S)$ were studied for various classes of posets in [18], [19], [21], where they were shown to have many interesting properties. For instance, if $\hat{P}$ is a semimodular lattice then $\beta(P, S) \geq 0$ for all $S \subset [\delta + 1]$. It comes as no surprise that this result may be regarded as a consequence of the fact that semimodular lattices are Cohen-Macaulay, and Cohen-Macaulay posets seem like the right context for obtaining such results.

Many well-known numerical invariants of (pure) posets $\hat{P}$ can be expressed in terms of the basic numbers $\beta(P, S)$. For instance, the "Whitney number of the second kind" $W_i(\hat{P})$ is defined by $W_i(\hat{P}) = \text{card}\{x \in P: \rho(x) = i\}$ and is clearly given by

$$W_i(\hat{P}) = \alpha(P, \{i\}) = \beta(P, \{i\}) + 1.$$

The "Whitney number of the first kind" $w_i(\hat{P})$ is defined by $w_i(\hat{P}) = \sum_{\rho(x)=i} \mu(x)$, and it is not hard to see that

$$(-1)^i w_i(\hat{P}) = \beta(P, [i-1]) + \beta(P, [i]).$$

Another commonly studied invariant is

$$(-1)^\delta \sum_{\rho(x)=i} \mu(x) \mu(x, \hat{1}) = \beta(P, [\delta+1] - \{i\}) + \beta(P, [\delta+1]).$$

A related invariant of $\hat{P}$ is the *zeta polynomial* [20, §3]. If $m \in \mathbb{N}$, define $Z(P, m)$ to be the number of chains $\hat{0} = x_0 \leq x_1 \leq \dots \leq x_m = \hat{1}$ in $\hat{P}$. Thus $Z(P, 0) = 0$, $Z(P, 1) = 1$, and $Z(P, 2) = \text{card } \hat{P}$. It is easily seen that $Z(P, m)$ is a polynomial function of m of degree $\delta + 2$. It follows that there are constants $e_0, \dots, e_{\delta+1}$ and $h_0, \dots, h_{\delta+1}$ such that $Z(P, m) = \sum_{i=1}^{\delta+2} e_{i-1} \binom{m}{i}$ and

$$(1 - \lambda)^{\delta+3} \sum_{m=0}^{\infty} Z(P, m) \lambda^m = \lambda(h_0 + h_1 \lambda + \dots + h_{\delta+1} \lambda^{\delta+1}). \quad (12)$$

It is not hard to see that

$$e_i = \sum_{\substack{S \subset [\delta+1] \\ \text{card } S = i}} \alpha(P, S) \quad \text{and} \quad h_i = \sum_{\substack{S \subset [\delta+1] \\ \text{card } S = i}} \beta(P, S).$$

We now discuss the relationship between the numbers $\beta(P, S)$ and the complex $\Delta(P)$. We have already noted in §2 that when P is pure of dimension δ , there is a standard ordered partition $\pi = (V_1, \dots, V_{\delta+1})$ defined by $V_i = \{x \in P: \rho(x) = i\}$ which makes $(\Delta(P), \pi)$ completely balanced. Now Philip Hall's theorem is equivalent to the formula

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y)), \quad x < y,$$

where $(x, y) = \{z \in P: x < z < y\}$ (see [17, p. 346]). In particular, $\mu(P_S) = \tilde{\chi}(P_S)$. It is then an immediate consequence of Proposition 3.5 and (11) (or

otherwise) that $h_b(\Delta(P), \pi) = \beta(P, S)$, where $\mathbf{b} = (b_1, \dots, b_{\delta+1})$, $b_i = 1$ if $i \in S$, $b_i = 0$ if $i \notin S$. Hence we see that if P is a Cohen-Macaulay poset, then the numbers $\beta(P, S) = h_b(\Delta(P), \pi)$ satisfy the stringent requirements of Corollary 4.5. In particular, we obtain new restrictions on the Whitney numbers of the first and second kind of a Cohen-Macaulay poset.

Now note that if (x, y) is an open interval of P , then (x, y) is the link of a face of $\Delta(P)$ of the form $x_1 < x_2 < \dots < x_r = x < y = y_s < y_{s-1} < \dots < y_1$, where $\rho(x_i) = i$ and $\rho(y_j) = \delta + 2 - j$. Hence by Theorem 1.2, (x, y) is also a Cohen-Macaulay poset. The fact that $\beta(P, S) \geq 0$ for any Cohen-Macaulay poset P then implies the following: Let P be a Cohen-Macaulay poset, and let (x, y) be an open interval of length l in $\hat{P}$. Then $(-1)^l \mu(x, y) \geq 0$, where μ is the Möbius function of $\hat{P}$. Indeed, $(-1)^l \mu(x, y)$ is the l th Betti number (with respect to the field k) of the complex $\Delta(x, y)$. In the terminology of poset theory, the Möbius function of $\hat{P}$ "alternates in sign." Since by Theorem 4.3 each P_S is Cohen-Macaulay when P is, it follows that the Möbius function of each P_S also alternates in sign [23, §8].

It should also be pointed out that when P is any pure poset with the standard ordered partition π , then the numbers $\alpha(P, S)$ are identical to $f_b(\Delta(P), \pi)$, with $b_i = 1$ if $i \in S$, $b_i = 0$ if $i \notin S$. The zeta polynomial $Z(P, m)$ is just the function $H(\Delta, m - 1)$ of (2), and the vector $(h_0, h_1, \dots, h_{\delta+1})$ of (12) is just the h -vector of $\Delta(P)$.

There are two main classes of Cohen-Macaulay posets known: (i) semimodular lattices, or more generally, semimodular posets. (A poset P is *semimodular* if for every closed interval I of $\hat{P}$, and for every x, y in I such that x and y cover some element u of I , there is a $v \in I$ which covers both x and y . If $\hat{P}$ is a lattice, then it suffices to consider only the case $I = \hat{P}$.) (ii) The lattice of faces of a regular cell complex (e.g., a simplicial complex or the boundary complex (not necessarily simplicial) of a convex polytope) whose underlying space X satisfies Theorem 1.2(iii). In addition, if P and Q are Cohen-Macaulay, then so is their ordinal sum $P \oplus Q$. ($P \oplus Q$ is the partial order on the disjoint union of P and Q defined by $x \leq y$ in $P \oplus Q$ if (i) $x \leq y$ in P , or (ii) $x \leq y$ in Q , or (iii) $x \in P$ and $y \in Q$.) Indeed, $\Delta(P \oplus Q)$ is just the *join* $\Delta(P) * \Delta(Q)$, and $A_{\Delta(P \oplus Q)} = A_{\Delta(P)} \otimes_k A_{\Delta(Q)}$.

When $\hat{P}$ is an admissible lattice there is a combinatorial interpretation of the numbers $\beta(P, S)$ which implies they are nonnegative [21]. To give the reader the flavor of this result, we mention the interpretation of $\beta(P, S)$ when $\hat{P}$ is the lattice of subspaces of an n -dimensional vector space over $GF(q)$. In this case,

$$\beta(P, S) = \sum_{\pi} q^{i(\pi)}, \quad (13)$$

where the sum is over all permutations $\pi = (a_1, \dots, a_n)$ of $[n]$ satisfying $S = \{i: a_i > a_{i+1}\}$, and where $i(\pi) = \text{card}\{(i, j): i < j \text{ and } a_i > a_j\}$. This suggests that admissible lattices are Cohen-Macaulay, and indeed this has been shown by Anders Björner (to be published). It would be of considerable interest to obtain results analogous to (13) for other classes of Cohen-Macaulay posets, such as the semimodular posets which are not lattices, or the lattice of faces of a convex polytope.

REFERENCES

1. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass. 1969.
2. G. F. Clements, *More on the generalized Macaulay theorem*, Discrete Math. **1** (1971), 247–255.
3. G. Clements and B. Lindström, *A generalization of a combinatorial theorem of Macaulay*, J. Combinatorial Theory **7** (1969), 230–238.
4. G. Danaraj and V. Klee, *Which spheres are shellable?*, Ann. Discrete Math. **2** (1978), 33–52.
5. R. Edwards, *An amusing reformulation of the Four Color Problem*, Notices Amer. Math. Soc. **24** (1977), A-257–A-258. Abstract #77T-G16.
6. T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66.
7. A. Ghouïla-Houri, *Caractérisation des graphes nonorientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C. R. Acad. Sci. Paris Sér. A-B **254** (1962), 1370–1371.
8. P. Gilmore and A. J. Hoffman, *A characterization of comparability graphs and interval graphs*, Canad. J. Math. **16** (1967), 539–548.
9. C. Greene and D. J. Kleitman, *Proof techniques in the theory of finite sets*, MAA Studies in Combinatorics (G.-C. Rota, editor), Math. Assoc. Amer., Washington, D. C. (to appear).
10. M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Proc. Second Oklahoma Ring Theory Conf. (March, 1976), Dekker, New York, 1977.
11. G. Katona, *A theorem of finite sets*, Proc. Colloq. Theory of Graphs (Tihany, Hungary, 1966), Academic Press, New York; Akad. Kiadó, Budapest, 1968, pp. 187–207.
12. J. B. Kruskal, *The number of simplices in a complex*, Mathematical Optimization Techniques, Univ. of California Press, Berkeley, Calif., 1963, pp. 251–278.
13. F. S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. London Math. Soc. **26** (1927), 531–555.
14. P. McMullen, *The maximum number of faces of a convex polytope*, Mathematika **17** (1970), 179–184.
15. J. Munkres, *Topological results in combinatorics* (preprint).
16. G. A. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math. **21** (1976), 30–49.
17. G.-C. Rota, *On the foundations of combinatorial theory. I, Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.
18. R. Stanley, *Ordered structures and partitions*, Mem. Amer. Math. Soc. No. 119, 1972.
19. ———, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217.
20. ———, *Combinatorial reciprocity theorems*, Advances in Math. **14** (1974), 194–253.
21. ———, *Finite lattices and Jordan-Hölder sets*, Algebra Universalis **4** (1974), 361–371.
22. ———, *The Upper Bound Conjecture and Cohen-Macaulay rings*, Studies in Appl. Math. **54** (1975), 135–142.
23. ———, *Cohen-Macaulay complexes*, Higher Combinatorics (M. Aigner, editor), NATO Advanced Study Institute Series, Reidel, Dordrecht and Boston, 1977, pp. 51–52.
24. ———, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978), 57–83.

25. J. S. Provan, *Decompositions, shellings, and diameters of simplicial complexes and convex polyhedra*, Thesis, Cornell University, 1977.

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