CENTER-BY-METABELIAN GROUPS OF PRIME EXPONENT

BY

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ABSTRACT. We show that a center-by-metabelian group of prime exponent p is nilpotent of class at most p , and this result is best possible. The proof is based on techniques dealing with varieties of groups.

1. Introduction. A classic result due to Meier-Wunderli [5] states that a metabelian group of prime exponent p is nilpotent of class at most p . Thus a center-by-metabelian group of exponent p is nilpotent of class at most $p + 1$. We show below that this bound can be reduced to p . Because such groups do exist for every prime $p \geq 5$ [4, Satz 6], the bound is best possible.

2. Notation and terminology. Our notation is generally the same as that in [6], to which we also refer the reader for elementary results concerning varieties of groups. We, however, use capital (small) italic letters for groups (elements). The variety generated by a particular group G we denote $\text{var}(G)$, and $\{1\}$ denotes the trivial group.

Commutators are left-normed, with $(x_1, x_2, \dots, x_n)$ an element in G_n , the n th term of the descending central series of G . The n th Engel law is the varietal law (x, ny) , defined inductively:

$$(x, 0y) = x, \quad (x, ny) = ((x, (n-1)y), y).$$

We abbreviate $((x, y), (u, v))$ by $(x, y; u, v)$ and $((x, y; u, v), w)$ by $(x, y; u, v; w)$. Thus G is metabelian if it satisfies the law $(x, y; u, v)$; and center-by-metabelian, if it satisfies the law $(x, y; u, v; w)$. We call $(x, y; u, v)$ a *double commutator*.

H is a *factor group* of G if $H \cong G/N$ for some $N < G$. A is a factor of G if $A \cong H/K$, where $\{1\} < K < H < G$. A finite group is *critical* if it is not contained in the variety generated by its proper factors. A group is *basic* if it is critical and generates a join-irreducible variety. The p -group G is *regular* if for every $a, b \in G$, $(a, b)^p = a^p b^p c^p$, where c is a commutator word in a and b . A variety is regular if every finite group in it is regular.

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3. The main result. Let H be a center-by-metabelian group of prime exponent p . Then $\text{var}(H)$ has exponent p , it is regular and it is center-by-metabelian. To show that H is nilpotent of class at most p , we first show that every basic group in $\text{var}(H)$ has that property. From here on, therefore, let G denote a basic center-by-metabelian group of exponent p , $p \geq 5$, and assume that G is nilpotent of the maximum possible class, $p + 1$. Also, let F be a finitely generated relatively free group that generates $\text{var}(G)$.

LEMMA 3.1. *In $\text{var}(G)$, 2-generator groups have class at most p . Equivalently, 2-variable words of weight $p + 1$ are trivial.*

PROOF. We first note that $F'' \leq Z(F)$ because $\text{var}(G)$ is center-by-metabelian. F/F'' is a finite metabelian p -group such that every finite group in $\text{var}(F/F'')$ is regular. By [8, Theorem 1.4], every 2-generator subgroup of F/F'' has class less than p . In F , therefore, every commutator of weight p in 2 variables is contained in F'' . Since $F'' \leq Z(F)$, every commutator of weight $p + 1$ in 2 variables is trivial. Q.E.D.

PROPOSITION 3.2. *$\text{var}(G)$ satisfies the $(p - 1)$ th Engel law.*

PROOF. Let x and y be elements of a free set of generators of F . Then $(x, (p - 1)y) \in F_{p+1}$ because F has exponent p [3, 18.4.13]. Thus in F , $(x, (p - 1)y)$ can be written as a product of simple commutators in x and y , each of weight $p + 1$. By the lemma, each commutator in this product is trivial. Therefore F and, hence, $\text{var}(G)$ satisfy the law $(x, (p - 1)y) = 1$. Q.E.D.

PROPOSITION 3.3. *Let a, b and y be elements of a free set S of generators of F . Then in F , every simple commutator of the form (a, ry, b, sy) is trivial, where $r + s = p - 1$.*

PROOF. Since F satisfies the $(p - 1)$ th Engel law, both $(a, b, (p - 1)y)$ and $(a, (p - 1)y, b)$ are trivial.

As a metabelian group, F/F'' satisfies the Witt identity. Thus

$$(a, b, y)(b, y, a)(y, a, b) \in F''.$$

Since $F'' \leq Z(F)$, we get

$$((a, b, y)(b, y, a)(y, a, b), (p - 2)y) = 1.$$

Now

$$((a, b, y)(b, y, a)(y, a, b), y) = (a, b, y, y)(b, y, a, y)(y, a, b, y)g_7,$$

where $g_7 \in F_7$.

Proceeding by induction, we obtain

$$\begin{aligned} 1 &= ((a, b, y)(b, y, a)(y, a, b), (p - 2)y) \\ &= (a, b, y, (p - 2)y)(b, y, a, (p - 2)y)(y, a, b, (p - 2)y)g_{p+4}, \end{aligned}$$

where $g_{p+4} \in F_{p+4}$. Since $F_{p+4} = \{1\}$ and $(a, b, (p - 1)y) = 1$, we obtain the following equation:

$$(b, y, a, (p - 2)y) = (y, a, b, (p - 2)y)^{-1} = (a, y, b, (p - 2)y). \quad (\text{A})$$

Since 2-variable commutators of weight $p + 1$ are trivial in F ,

$$(x, y, x, (p - 2)y) = 1,$$

where $x \in S$. Replacing x by ab and expanding, we get

$$(a, y, b, (p - 2)y)(b, y, a, (p - a)y) = 1.$$

Equation (A) then implies that $(a, y, b, (p - 2)y)^2 = 1$. Since $p > 2$, we have $(a, y, b, (p - 2)y) = 1$.

Thus if $r = 0, 1$ or $p - 1$, $(a, ry, b, sy) = 1$ in F , where $r + s = p - 1$. Continuing by induction, assume $1 \leq r < p - 2$ and $(a, ry, b, sy) = 1$. Now

$$((a, ry), y, b)(y, b, (a, ry))(b, (a, ry), y) \in F''$$

by the Witt identity, and since $F'' \leq Z(F)$,

$$(((a, ry)y, b)(y, b, (a, ry))(b, (a, ry), y), (s - 1)y) = 1.$$

Reasoning as above, and expanding, this yields

$$(a, ry, y, b, (s - 1)y)(y, b, (a, ry), (s - 1)y)(b, (a, ry), y, (s - 1)y) = 1. \quad (\text{B})$$

Since $(y, b, (a, ry)) \in F'' \leq Z(F)$, $(y, b, (a, ry), (s - 1)y) = 1$. Also

$$(b, (a, ry), y, (s - 1)y) = ((a, ry), b, y, (s - 1)y)^{-1} = (a, ry, b, sy)^{-1},$$

which is trivial by the induction hypothesis. Thus (B) becomes

$$(a, (r + 1)y, b, (s - 1)y) = 1,$$

and this induction step completes the proof. Q.E.D.

COROLLARY 3.4. *In F , $(a, ry; b, sy) = 1$, where r and s are positive integers such that $r + s = p - 1$. Hence $(a, ry; b, sy)$ is a law in $\text{var}(G)$.*

PROOF. Since $(a, ry; b, sy)$ is a double commutator of weight $p + 1$, it can be written as a product of simple left-normed commutators of weight $p + 1$, each of which contains exactly one a , one b , and $p - 1$ y 's. By the proposition, each of these is trivial. Q.E.D.

LEMMA 3.5. *Let $\{a, b, x, y, y_1, y_2, \dots, y_{p-3}\}$ be a free set of generators of F . Then the following laws hold in F :*

$$(a, b; x, y, y_1, y_2, \dots, y_{p-3}) = (a, b; x, y, y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(p-3)}), \quad (1)$$

where σ is any permutation of $\{1, 2, \dots, p-3\}$.

$$\begin{aligned} (a, b; x, y, y_1, \dots, y_{p-3})^{-1} &= (b, a; x, y, y_1, \dots, y_{p-3}) \\ &= (a, b; y, x, y_1, \dots, y_{p-3}) = (a^{-1}, b; x, y, y_1, \dots, y_{p-3}) \\ &= \dots = (a, b; x, y, y_1, \dots, y_{p-3}^{-1}) = (x, y, y_1, \dots, y_{p-3}; a, b). \end{aligned} \quad (2)$$

$$\begin{aligned} (a, b; x, y, y_1, y_2, \dots, y_{p-3})(a, b; y, y_1, x, y_2, \dots, y_{p-3}) \\ \times (a, b; y_1, x, y, y_2, \dots, y_{p-3}) = 1. \end{aligned} \quad (3)$$

$$\text{Let } d_1, d_2 \in F'. \text{ Then } (d_1, a, d_2) = (d_1; d_2, a^{-1}). \quad (4)$$

PROOF. The following are well-known properties of metabelian groups:

(i) $(x, y, y_1, y_2, \dots, y_{p-3}) = (x, y, y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(p-3)})$, where σ is any permutation of $\{1, 2, \dots, p-3\}$.

(ii) $(a, b)^{-1} = (b, a)$.

(iii) $(x, y, y_1)(y, y_1, x)(y_1, x, y) = 1$ (the Witt identity).

With these, parts (1), (2) and (3) of the lemma follow easily since F is center-by-metabelian, and all double commutators are of maximal weight. Part (4) is Lemma 6.2 of [2]. Q.E.D.

PROPOSITION 3.6. *In F , commutators of weight 2 commute with commutators of weight $p-1$; i.e., $(F_2, F_{p-1}) = 1$.*

PROOF. Let $\{a, b, x, y, y_1, y_2, \dots, y_{p-2}\}$ be a free set of generators of F . It suffices to show that $(a, b; x, y_1, y_2, \dots, y_{p-2})$ is a law in F .

When working with laws in F , we shall use the notation " $a \mapsto b$ " to mean that each occurrence of a is replaced by b . Then " $a \leftrightarrow b$ " will mean $a \mapsto b$ and $b \mapsto a$ simultaneously.

Also, we shall use the term "separation" to denote the following procedure: Assume $1 = f_1 f_2 \dots f_r$ is a law in F , where $r < p$ and each f_i is a product of nontrivial commutators of maximal weight, each one containing exactly i occurrences of x . Then each f_i is a law in F , $i = 1, 2, \dots, r$. (See, e.g., [1, Corollary 1.1] for a more detailed description of this.)

We now proceed to the proof of the proposition. In the law

$$1 = (a, y; b, (p-2)y),$$

which is valid by (3.4), let $y \mapsto yy_1$. After some expanding, we get

$$\begin{aligned} 1 &= (a, y; b, y, (p-3)yy_1)(a, y_1; b, y, (p-3)yy_1) \\ &\quad \times (a, y; b, y_1, (p-3)yy_1)(a, y_1; b, y_1, (p-3)yy_1). \end{aligned}$$

Using separation, we obtain from this a law written as the product of double commutators, in which exactly one y_1 occurs in each double commutator. We

thus have

$$1 = (a, y; b, y, y_1, (p-4)y)^{p-3} (a, y_1; b, y, (p-3)y) (a, y; b, y_1, (p-3)y).$$

Since F has exponent p , this becomes

$$1 = (a, y; y, b, y_1, (p-4)y)^3 (a, y_1; b, (p-2)y) (a, y; b, y_1, y, (p-4)y). \quad (\text{A})$$

Using 3.5(3) and 3.5(2), we get that

$$\begin{aligned} & (a, y; b, y_1, y, (p-4)y) \\ &= (a, y; y, y_1, b, (p-4)y) (a, y; b, y, y_1, (p-4)y). \end{aligned}$$

Inserting this in equation (A) yields

$$\begin{aligned} 1 &= (a, y; y, b, y_1, (p-4)y)^3 (a, y_1; b, (p-2)y) \\ &\quad \times (a, y; y, y_1, b, (p-4)y) (a, y; b, y, y_1, (p-4)y) \\ &= (a, y; y, b, y_1, (p-4)y)^2 (a, y_1; b, (p-2)y) (a, y; y, y_1, b, (p-4)y) \\ &= (a, y; y, b, (p-4)y, y_1)^2 (a, y_1; b, (p-3)y, y) \\ &\quad \times (a, y; y, y_1, b, (p-4)y) \\ &= (a, y, y_1^{-1}; y, b, (p-4)y)^2 (a, y_1, y^{-1}; b, (p-3)y) \\ &\quad \times (a, y; y, y_1, b, (p-4)y). \end{aligned}$$

Using 3.5(2) we rewrite this as

$$1 = (a, y, y_1; b, y, (p-4)y)^2 (y_1, a, y; b, (p-3)y) (a, y; y, y_1, b, (p-4)y). \quad (\text{B})$$

Using the Witt identity, we have

$$(y_1, a, y; b, (p-3)y) = (y, a, y_1; b, (p-3)y) (y_1, y, a; b, (p-3)y),$$

from which equation (B) becomes

$$1 = (a, y, y_1; b, (p-3)y) (y_1, y, a; b, (p-3)y) (a, y; y, y_1, b, (p-4)y).$$

Taking inverses and rearranging yields

$$1 = (a, y; y_1, y, b, (p-4)y) (y, a, y_1; b, (p-3)y) (y, y_1, a; b, (p-3)y). \quad (\text{C})$$

Now from (3.4), $1 = (a, y; b, (p-3)y)$. Letting $y \mapsto xy$, we get, after some expanding,

$$\begin{aligned} 1 &= (a, x; b, x, (p-3)xy) (a, y; b, y, (p-3)xy) \\ &\quad \times (a, x; b, y, (p-3)xy) (a, y; b, x, (p-3)xy). \end{aligned}$$

Using separation, we obtain from this a law in which each double commutator contains exactly one x . We get

$$\begin{aligned} 1 &= (a, y; b, y, x, (p-4)y)^{p-3} (a, x; b, y, (p-3)y) (a, y; b, x, (p-3)y) \\ &= (a, y; b, y, (p-4)y, x)^{-3} (a, x; b, (p-3)y, y) (a, y; b, x, (p-3)y) \\ &= (a, y, x^{-1}; b, (p-3)y)^{-3} (a, x, y^{-1}; b, (p-3)y) (a, y; b, x, (p-3)y), \end{aligned}$$

or

$$1 = (a, y, x; b, (p-3)y)^3 (x, a, y; b, (p-3)y) (a, y; b, x, (p-3)y). \quad (\text{D})$$

Using the Witt identity again, we have

$$(x, a, y; b, (p-3)y) = (y, a, x; b, (p-3)y) (x, y, a; b, (p-3)y).$$

Substituting this into (D) yields

$$1 = (a, y, x; b, (p-3)y)^2 (x, y, a; b, (p-3)y) (a, y; b, x, (p-3)y).$$

Letting $x \leftrightarrow b$, we get

$$1 = (a, y, b; x, (p-3)y)^2 (b, y, a; x, (p-3)y) (a, y; x, b, (p-3)y).$$

Multiplying these last two laws yields

$$\begin{aligned} 1 &= (a, y, x; b, (p-3)y)^2 (a, y, b; x, (p-3)y)^2 \\ &\quad \times (x, y, a; b, (p-3)y) (b, y, a; x, (p-3)y). \end{aligned} \quad (\text{E})$$

We now note the following:

$$\begin{aligned} (b, y, a; x, (p-3)y) &= (b, y; x, (p-3)y, a^{-1}) \\ &= (b, y; x, y, a^{-1}, (p-4)y) \\ &= (b, y, y^{-1}; x, y, a^{-1}, (p-5)y) = \cdots \\ &= (b, y, (p-4)y^{-1}; x, y, a^{-1}) \\ &= (b, y, (p-4)y; x, y, a^{-1})^{(-1)^{p-4}} \\ &= (b, (p-3)y; x, y, a). \end{aligned}$$

Using this in equation (E) yields

$$1 = [(a, y, x; b, (p-3)y) (a, y, b; x, (p-3)y)]^2.$$

Because p is odd, we get

$$1 = (a, y, x; b, (p-3)y) (a, y, b; x, (p-3)y). \quad (\text{I})$$

From equation (C), with $a \leftrightarrow b$ and $y_1 \mapsto x$, we obtain

$$\begin{aligned} 1 &= (b, y; x, y, a, (p-4)y)(y, b, x; a, (p-3)y)(y, x, b; a(p-3)y) \\ &= (b, y, a^{-1}; x, (p-3)y)(y, b, x; a, (p-3)y)(y, x, b; a, (p-3)y). \end{aligned}$$

Taking inverses yields

$$1 = (b, y, a; x, (p-3)y)(b, y, x; a, (p-3)y)(x, y, b; a, (p-3)y). \quad (J)$$

Now from (I), with $a \leftrightarrow b$, we get

$$1 = (b, y, x; a, (p-3)y)(b, y, a; x, (p-3)y).$$

Using this, we eliminate the first two factors in (J), getting

$$1 = (x, y, b; a, (p-3)y). \quad (K)$$

Now using the Witt identity one last time, we have

$$1 = (x, y, b; a, (p-3)y)(y, b, x; (p-3)y)(b, x, y; a, (p-3)y).$$

Therefore, by using (K) and then taking inverses, we get

$$1 = (b, y, x; a, (p-3)y)(x, b, y; a, (p-3)y).$$

From this, and from (K) with $b \leftrightarrow x$, we obtain

$$1 = (x, b, y; a, (p-3)y) = (x, b; a, (p-3)y, y^{-1}).$$

Letting $a \leftrightarrow x$ and taking inverses yields

$$1 = (a, b; x, (p-2)y).$$

Thus, $F/Z(F')$ satisfies the law $(x, (p-2)y)$. Since $F'' \leq Z(F')$ and $F_p \leq Z(F')$, $F/Z(F')$ is a metabelian group of class at most $p-1$. By a well-known result [1, Corollary 2.1], any metabelian group of small class satisfying the law $(x, (p-2)y)$ also satisfies the law $(x, y_1, y_2, \dots, y_{p-2})$. Therefore $F/Z(F')$ satisfies this law, and $(a, b; x, y_1, y_2, \dots, y_{p-2})$ is a law in F . Q.E.D.

THEOREM 3.7. *Let G be a basic center-by-metabelian group of prime exponent p . Then G is nilpotent of class at most p .*

PROOF. Let F be a relatively free group that generates $\text{var}(G)$, having $\{a, b, x, y_1, y_2, \dots, y_{p-2}\}$ as a free set of generators. It is known that F is nilpotent of class at most $p+1$; so assume F has class exactly $p+1$. From the above proposition, we have that $1 = (a, b; x, y_1, y_2, \dots, y_{p-2})$. By 3.5(4), we get $1 = (a, b, y_{p-2}^{-1}; x, y_1, y_2, \dots, y_{p-3})$, and taking inverses yields $1 = (a, b, y_{p-2}; x, y_1, y_2, \dots, y_{p-3})$.

Proceeding by induction, we see that every double commutator of weight $p+1$ is trivial. Because F/F'' is a metabelian group of exponent p , $F_{p+1} \leq F''$ [5]. Thus the simple commutator $(a, b, x, y_1, y_2, \dots, y_{p-2})$ can be written as a product of double commutators, each of weight $p+1$ and containing all

the generators of F . Since these are all trivial, so is F_{p+1} . Therefore, F has class at most p ; and, hence, so does G . Q.E.D.

THEOREM 3.8. *Let H be a center-by-metabelian group of prime exponent p . Then H is nilpotent of class at most p .*

PROOF. Since groups of exponent 2 are abelian, and groups of exponent 3 are nilpotent of class at most 3 [6, Theorem 34.32], H is metabelian if $p < 5$. Thus we may assume that $p \geq 5$ and H is not metabelian.

Since H is center-by-metabelian but not metabelian, $Z(H) > \{1\}$ and $H'' \leq Z(H)$. Therefore H and, hence, $\text{var}(H)$ are solvable. Because a solvable group of exponent p is locally finite [7, Theorem 7.16], $\text{var}(H)$ is a locally finite variety. Hence $\text{var}(H)$ is generated by its basic groups, and it remains only to show that those are nilpotent of class at most p .

Let G be a basic group in $\text{var}(H)$. If G is abelian, it has class 1; and if G is metabelian, it has class at most p since every metabelian group of exponent p does.

If G is not metabelian, it must be a basic center-by-metabelian group of exponent p . Hence it has class at most p by 3.7. Therefore all the basic groups in $\text{var}(H)$ have class at most p . Hence H itself is nilpotent of class at most p . Q.E.D.

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