TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 252, August 1979

CONTINUITY OF THE DENSITY OF A GAS FLOW IN A POROUS MEDIUM[1]

BY

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ABSTRACT. The equation of gas in a porous medium is a degenerate nonlinear parabolic equation. It is known that a unique generalized solution exists. In this paper it is proved that the generalized solution is continuous.

0. Introduction. The density $u(x, t)$ of gas in a porous medium satisfies the equation

$$\partial u/\partial t = \Delta u^m \qquad (m > 1) \tag{0.1}$$

for $x \in R^n$, $t > 0$, and an initial condition

$$u(x, 0) = u_0(x). \tag{0.2}$$

Here $u_0(x) \geqslant 0$ and $u(x, t) \geqslant 0$. The equation (0.1) is a nonlinear parabolic equation, degenerating at the points where $u = 0$. The concept of a solution of (0.1), (0.2) is taken in some weak sense (to be defined precisely in §1). The purpose of this paper is to prove that

$$u(x, t) \text{ is continuous.} \tag{0.3}$$

This result is known for $n = 1$; see [9], [10], [1] and [5].

In §1 we state this result more precisely, giving also a uniform modulus of continuity. In §§2 and 3 we establish preliminary estimates. The proof of (0.3) for $t > 0$ is given in §4 and, for $t = 0$, in §5.

1. The main results. Let $u_0(x)$ be a function defined in R^n and satisfying:

$$0 \leqslant u_0(x) \leqslant N \qquad (N < \infty), \tag{1.1}$$

$$\int_{R^n} (u_0(x))^2 \, dx < \infty, \tag{1.2}$$

$$u_0(x) \text{ is continuous in } R^n \text{, and uniformly Hölder continuous in every compact set where } u_0 > 0. \tag{1.3}$$

Received by the editors August 7, 1978.

AMS (MOS) subject classifications (1970). Primary 35K55.

[1]This work is partially supported by National Science Foundation Grants 74 06375 A01 and MC 575-21416 A01.

We consider the Cauchy problem

$$\partial u/\partial t = \Delta u^m \qquad \text{in } R^n \times (0, \infty), \tag{1.4}$$

$$u(x, 0) = u_0(x) \qquad \text{in } R^n, \tag{1.5}$$

where m is a fixed number, $m > 1$.

By a solution of (1.4), (1.5) we mean a function $u(x, t)$ such that, for any $T < \infty$,

$$\int_0^T \int_{R^n} \left[(u(x, t))^2 + |\nabla_x u^m(x, t)|^2 \right] dx\, dt < \infty \tag{1.6}$$

and

$$\int_0^T \int_{R^n} \left(u \frac{\partial f}{\partial t} - \nabla_x u^m \cdot \nabla_x f \right) dx\, dt + \int_{R^n} u_0(x) f(x)\, dx = 0 \tag{1.7}$$

for any continuously differentiable function f with compact support in $R^n \times [0, T)$.

We recall [11] that under the conditions (1.1), (1.2), there exists a unique solution.

Other concepts of a solution can be given which allow for a different decay condition at $x = \infty$ than in (1.2). The results of this paper are not affected by working with these other concepts of a solution.

The solution $u(x, t)$ can be obtained as a limit of solutions $u_\eta(x, t)$ ($\eta \downarrow 0$) of the equation (1.4) with the initial condition

$$u(x, 0) = u_0(x) + \eta \qquad \text{in } R^n; \tag{1.8}$$

see [11]. Notice that the solution u_η of (1.4), (1.8) is taken in the classical sense, $u_\eta \leqslant u_{\eta'}$ if $\eta \leqslant \eta'$, and

$$\eta \leqslant u_\eta(x, t) \leqslant N + \eta \qquad \text{in } R^n \times (0, \infty). \tag{1.9}$$

We define the parabolic distance between two points (x^1, t^1), (x^2, t^2) by

$$d((x^1, t^1), (x^2, t^2)) = |x^1 - x^2| + |t^1 - t^2|^{1/2}.$$

When we shall speak of a modulus of continuity of a function $v(x, t)$, we shall always mean the distance between two points to be the parabolic distance.

We now introduce two moduli of continuity:

$$\omega_\varepsilon(r) = C|\log r|^{-\varepsilon} \qquad (0 < \varepsilon < 2/n), \tag{1.10}$$

$$\tilde{\omega}(r) = C 2^{-c|\log r|^{1/2}} \tag{1.11}$$

where $C > 0$, $c > 0$.

The main result of the paper is stated in the following theorem.

THEOREM 1.1 (i) *The solution u of* (1.4), (1.5) *is continuous in $R^n \times [0, \infty)$;*

(ii) *For any $\delta_0 > 0$, u^m has a modulus of continuity $\omega_\varepsilon(r)$ in $R^n \times [\delta_0, \infty)$ for any $0 < \varepsilon < 2/n$, if $n \geqslant 3$, and a modulus of continuity $\tilde{\omega}(r)$ in $R^n \times [\delta_0, \infty)$ if $n = 2$.*

The proof of continuity for $t > 0$ and the proof of (ii) are given in §4. It will become obvious from the proof that the condition (1.3) is not required for this part of the theorem.

The proof of continuity for $t = 0$ is given in §5.

§§2 and 3 develop some estimates needed in §4.

2. Preliminary lemmas. In this section and in §3 we obtain various auxiliary results for the solution $u_\eta(x, t)$ of (1.4), (1.8). For simplicity we shall denote this solution by $u(x, t)$; we also take $0 < \eta < 1$.

All the estimates which we shall obtain, and all the constants will be independent of η. We set $M = N + 1$, so that, by (1.9),

$$0 < u(x, t) < M. \tag{2.1}$$

LEMMA 2.1. *The following inequalities hold*:

$$t\frac{\partial u}{\partial t} \geqslant -\frac{u}{m-1}, \tag{2.2}$$

$$t\frac{\partial u^m}{\partial t} \geqslant -\frac{m}{m-1} u^m. \tag{2.3}$$

This result is due to Aronson and Benilan [4]. Since the proof is short, we briefly give it here. The function $w = t(\partial u/\partial t)$ satisfies

$$\partial w/\partial t = m\Delta(u^{m-1}w) + \Delta u^m.$$

The function $z = -u/(m-1)$ satisfies the same equation, and $z(x, 0) \leqslant 0 = w(x, 0)$ (w is continuous at $t = 0$ if $u_0(x)$ is smooth). By comparison, then, (2.2) follows if $u_0(x)$ is smooth; for general u_0, one uses approximation.

The inequality (2.3) follows immediately from (2.2).

Let δ_0 be a fixed positive number. For any $x^0 \in R^n$, $t^0 > 2\delta_0$, we introduce the sets

$$C_{r,h}(x^0, t^0) = \{(x, t);\ |x - x^0| < r,\ t^0 - h < t < t^0\},$$
$$B_r(x^0) = \{x;\ |x - x^0| < r\},$$

where $h < \delta_0$. Denote the volume of $B_r(x^0)$ by $|B_r(x^0)|$. Set

$$C_0 = m/(m-1)\delta_0 \tag{2.4}$$

and let $h_0 = h_0(\delta_0)$ be any positive number satisfying:

$$e^{2C_0h_0} \leqslant \tfrac{4}{3}, \qquad h_0 < \delta_0. \tag{2.5}$$

LEMMA 2.2. *For any $x^0 \in R^n$, $t^0 > 2\delta_0$, $\lambda > 0$, $r > 0$, $0 < h < h_0(\delta_0)$, the following is true*: *if*

$$\frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} u^m(x, t^0 - h)\, dx \geqslant \lambda \tag{2.6}$$

and if

$$h > M\gamma_n r^2/\lambda, \tag{2.7}$$

then

$$u^m(x^0, t^0) > \tfrac{1}{2}\lambda; \tag{2.8}$$

here γ_n is a positive number depending only on the dimension n.

PROOF. By (2.3), (2.4), $\partial u^m/\partial t \geqslant -(m/(m-1)t)u^m \geqslant -C_0u^m$ if $t^0 - h_0 < t < t^0$ (since $t > 2\delta_0 - h_0 > \delta_0$). Hence

$$u^m(x^0, t^0) \geqslant e^{-C_0(t-t^0+h)}u^m(x^0, t) \geqslant e^{-C_0h_0}u^m(x^0, t). \tag{2.9}$$

Again, by (2.3), (2.4), the function

$$\varphi(t) = \frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} u^m(x, t)\, dx$$

satisfies

$$\varphi'(t) \geqslant -(m/(m-1)t)\varphi(t) \geqslant -C_0\varphi(t)$$

so that

$$\varphi(t) \geqslant e^{-C_0(t-t^0+h)}\varphi(t_0 - h) \geqslant e^{-C_0h_0}\varphi(t_0 - h) \geqslant e^{-C_0h_0}\lambda, \tag{2.10}$$

where (2.6) was used in the last inequality.

Suppose $n \geqslant 3$ and let

$$G(\rho) = \rho^{2-n} - r^{2-n} - ((n-2)/2)r^{-n}(r^2 - \rho^2), \qquad \rho = |x - x^0|. \tag{2.11}$$

Notice that $G(r) = 0$, $G'(r) = 0$ Since $G'(\rho) < 0$ if $\rho < r$, $G(\rho)$ is positive in $B_r(x^0)$. By Green's formula,

$$u^m(x^0, t) = \tfrac{1}{4}\gamma_n\int_{B_r(x^0)} G\Delta u^m\, dx + \frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} u^m(x, t)\, dx \tag{2.12}$$

where γ_n is a positive constant depending only on n.

Suppose now that the assertion (2.8) is not true. Then (2.9) gives

$$u^m(x^0, t) < \tfrac{1}{2}\lambda e^{C_0h_0}.$$

Substituting this and (2.10) into (2.12) and using the first inequality of (2.5), we obtain

$$\tfrac{1}{4}\lambda \leqslant \tfrac{1}{4}\gamma_n\int_{B_r(x^0)} G\Delta u^m\, dx.$$

Integrating this inequality with respect to t, $t^0 - h < t < t^0$, we get

$$\begin{aligned}\lambda h &< \gamma_n\int_{t^0-h}^{t^0}\int_{B_r(x^0)} G(\rho)\Delta u^m(x, t)\, dx\, dt \\ &= \gamma_n\int_{t^0-h}^{t^0}\int_{B_r(x^0)} G(\rho)\frac{\partial u(x, t)}{\partial t}\, dx\, dt \\ &\leqslant \gamma_n\int_{B_r(x^0)} G(\rho)u(x, t^0)\, dx \leqslant \gamma_n M\int_{B_r(x^0)} G(\rho)\, dx.\end{aligned}$$

Since the right-hand side is bounded by $\gamma_n M r^2$, with another constant γ_n, we obtain a contradiction to (2.7). This completes the proof in case $n \geqslant 3$. For $n = 2$ the proof is the same provided we replace $G(\rho)$, defined in (2.11), by $G(\rho) = \log(r/\rho) - \frac{1}{2} r^{-2}(r^2 - \rho^2)$.

In the next lemma we take $h_0 = h_0(\delta_0)$ to satisfy, in addition to (2.5), the inequality

$$h_0 < 2n(m-1)\delta_0 c/M, \qquad c > 0, \tag{2.13}$$

and let C_1 be any constant satisfying:

$$C_1 - 1 \geqslant 2^{n+1}, \qquad C_1 - 1 > \gamma_n M 2^{n+2}. \tag{2.14}$$

LEMMA 2.3. *Let $t^0 > \delta_0$, $0 < h < h_0(\delta_0)$, $\lambda = c\sigma(h)$ where $\sigma(h) > 0$, $\sigma(h) \to 0$ if $h \to 0$. Let x^0 be any point in R^n such that*

$$|x^0| < h^{1/2}(\sigma(h))^{1/2} \equiv r. \tag{2.15}$$

If

$$u^m(x^0, t^0) > C_1\lambda, \tag{2.16}$$

then

$$u^m(0, t^0 + h) > \lambda. \tag{2.17}$$

PROOF. Let $v(x) = u^m(x, t^0) + c_1|x - x^0|^2$, $c_1 > 0$. Then by Lemma 2.1 and (2.1),

$$\Delta v = \partial u/\partial t + 2nc_1 \geqslant -M/(m-1)\delta_0 + 2nc_1 = 0$$

provided

$$c_1 = M/2n(m-1)\delta_0. \tag{2.18}$$

Thus, v is subharmonic. This implies that, for any $r > 0$,

$$\frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} v \geqslant v(x^0) = u^m(x^0, t^0) > C_1\lambda,$$

where (2.16) was used. Since the left-hand side is bounded above by

$$cr^2 + \frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} u^m,$$

we obtain, if r is taken as in (2.15),

$$I \equiv \frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} u^m \geqslant C_1\lambda - c_1 r^2 = C_1\lambda - c_1 h\sigma(h).$$

Recalling the definition of λ, c_1 and the restriction (2.13), we find that $I \geqslant (C_1 - 1)\lambda$. If we now assume that $|x^0| \leqslant r$ then $B_{2r}(0) \supset B_r(x^0)$, and we conclude that

$$\frac{1}{|B_{2r}(0)|}\int_{B_{2r}(0)} u^m \geqslant (C_1 - 1)\lambda\frac{1}{2^n}.$$

We now wish to apply Lemma 2.2 with λ, r replaced by $\lambda' = (C_1 - 1)\lambda 2^{-n}$, $r' = 2r$, and with $t^0 - h$ replaced by t^0. The condition (2.7) is satisfied by virtue of the second inequality in (2.14). We conclude that $u^m(0, t^0 + h) > \frac{1}{2}\lambda' > \lambda$, where the last inequality is a consequence of the first inequality in (2.14).

We conclude this section with the following lemma.

Lemma 2.4. *Let $\delta_0 > 0$ and let*

$$c_* = 1/mM^{m-1}. \tag{2.19}$$

Then, for all $t > \delta_0$,

$$\Delta u^m > -M/(m-1)\delta_0, \tag{2.20}$$

$$\Delta u^m - c_* \partial u^m/\partial t > -M/(m-1)\delta_0. \tag{2.21}$$

Proof. By Lemma 2.1,

$$\Delta u^m = \frac{\partial u}{\partial t} > -\frac{u}{(m-1)t} > -\frac{M}{(m-1)\delta_0},$$

and (2.20) follows. To prove (2.21), we can write

$$\Delta u^m - c_*\frac{\partial u^m}{\partial t} = \frac{\partial u}{\partial t} - c_* m u^{m-1}\frac{\partial u}{\partial t} = \left(1 - c_* m u^{m-1}\right)\frac{\partial u}{\partial t}$$
$$> -\left(1 - c_* m u^{m-1}\right)\frac{u}{(m-1)t},$$

since $1 - c_* m u^{m-1} > 0$ by (2.19). Since the right-hand side is $> -u/(m-1)t > -M/(m-1)\delta_0$, (2.21) follows.

3. A priori estimates. The main result of this section is stated in Lemma 3.3. Let ε be any positive number such that $\varepsilon < 2/n$.

Fix a point (x^0, t^0) in $R^n \times (2\delta_0, \infty)$ and define

$$R_k = \{(x, t): |x - x^0| < 2^{-k}, t^0 - 2^{-2k} < t < t^0\},$$

where k is a positive integer. In Lemma 3.1 we shall be interested only in $k > k^*$. Here k^* is a positive number sufficiently large, to be determined in the proof of Lemma 3.1. It depends only on ε, δ_0, N and it also satisfies the inequality $2^{-2k^*} < h_0$ where $h_0 = h_0(\delta_0)$ is the number satisfying all the restrictions imposed in Lemmas 2.2, 2.3.

Define

$$\mu_k = \sup_{R_k} u^m, \quad M_k = \max\{\mu_k, 4C_1 k^{-\varepsilon}\},$$

where C_1 is the constant appearing in Lemma 2.3.

Lemma 3.1. *Suppose*

$$u^m(x^0, t^0) < k_0^{-\varepsilon} \tag{3.1}$$

for some $k_0 > k^*$. *Then, for any* $k^* < k \leq k_0$,

$$\mu_{k+1} \leq M_k(1 - Ck^{-\varepsilon n/2}) \tag{3.2}$$

where C is a constant depending only on ε, δ_0, N.

PROOF. We wish to apply Lemma 2.3 with $\sigma(h) = |\log h|^{-\varepsilon}$, $h = 2^{-2k}$, $c = (2\log 2)^\varepsilon$ (so that $\lambda = k^{-\varepsilon}$). The lemma asserts that if $u^m(x, t^0 - 2^{-2k}) > C_1\lambda$ for some x such that $|x - x^0| \leq 2^{-k}k^{-\varepsilon/2}$, then $u(x^0, t^0) > \lambda$. Since the last inequality contradicts (3.1) (since $k \leq k_0$), it follows that

$$u^m(x, t^0 - 2^{-2k}) \leq C_1 k^{-\varepsilon} \quad \text{if } |x - x^0| < 2^{-k}k^{-\varepsilon/2} \tag{3.3}$$

for all $k^* < k \leq k_0$.

Let

$$v(x) = u^m(x, t^0 - h) + c_1|x - x^*|^2 \quad \text{where } |x^* - x^0| < \tfrac{1}{3}\, 2^{-k}; \tag{3.4}$$

c_1 is chosen as in (2.18), so that $\Delta v \geq 0$.

Denote by B the ball with center x^* and radius $\frac{2}{3}\, 2^{-k}$, and denote by B' the ball with center x^0 and radius $2^{-k}k^{-\varepsilon/2}$. Then B' is contained in B (if $k > k^*$).

Since v is subharmonic,

$$u^m(x^*, t^0 - h) = v(x^*, t^0 - h) \leq \frac{1}{|B|} \int_B v(x)\, dx$$

$$\leq c_1\left(\tfrac{2}{3}\, 2^{-k}\right)^2 + \frac{1}{|B|} \int_B u^m(x, t^0 - h)\, dx. \tag{3.5}$$

Using the inequality (3.3) in B', and the inequality $u^m(x, t^0 - h) \leq M_k$ in $B - B'$, we can estimate the last term in (3.5) by $M_k(1 - k^{-\varepsilon n/2}) + C_1 k^{-\varepsilon n/2}k^{-\varepsilon}$. Substituting this estimate in (3.5) we get

$$u^m(x^*, t^0 - h) \leq M_k(1 - k^{-\varepsilon n/2}) + 2C_1 k^{-\varepsilon n/2}k^{-\varepsilon}.$$

Recalling the $M_k \geq 4C_1k^{-\varepsilon}$, we conclude (if $k > k^*$) that

$$u^m(x^*, t^0 - h) \leq M_k\left(1 - \tfrac{1}{2}\, k^{-\varepsilon n/2}\right) \quad \text{if } |x^* - x^0| < \tfrac{1}{3}\, 2^{-k}. \tag{3.6}$$

Let $x' = (x - x^0)/2^k$, $t' = (t - t^0)/2^{2k}$, $w(x', t') = u^m(x, t)$.

By Lemma 2.4,

$$\left(\Delta - c_*\partial/\partial t'\right)w = 2^{-2k}\left(\Delta - c_*\partial/\partial t\right)u^m \geq -\tilde{C}2^{-2k} \tag{3.7}$$

where $\tilde{C} = M/(\delta_0(m - 1))$.

Consider the function

$$z = 1 - w/M_k \tag{3.8}$$

in the cylinder $|x'| < 1$, $-1 < t' < 0$. Clearly $z \geq 0$ in this cylinder. Also, by (3.6),

$$z(x', -1) \geq \tfrac{1}{2}\, k^{-\varepsilon n/2} \quad \text{if } |x'| < \tfrac{1}{3}. \tag{3.9}$$

By (3.7) we further have

$$\left(\Delta - c_*\partial/\partial t'\right)z \leqslant \tilde{C}2^{-2k}/M_k. \tag{3.10}$$

Denote by $G(x', \xi', t')$ the Green function of $\Delta - c_*(\partial/\partial t')$ in the cylinder $|x'| < 1$, $-1 < t' < 0$. Representing z in terms of Green's function, and using (3.9), (3.10), we get

$$z(x', t') \geqslant -\int_{-1}^{t'} \int_{|\xi'|<1} \frac{\tilde{C}2^{-2k}}{M_k} G(x', \xi', t' - s)\, d\xi'\, ds$$
$$+ \tfrac{1}{2} k^{-\varepsilon n/2} \int_{|\xi'|<1/3} G(x', \xi', t' + 1)\, d\xi'. \tag{3.11}$$

We restrict (x', t') to the set $|x'| < \frac{1}{2}$, $-\frac{1}{2} < t' < 0$. Then the last integral is larger than a positive constant $4c$, and

$$\int_{-1}^{t'} \int_{|\xi'|<1} G(x', \xi', s)\, d\xi'\, ds \leqslant C,$$

where c and C depend only on c_*, n. It follows that

$$z(x', t') \geqslant 2ck^{-\varepsilon n/2} - C\tilde{C}2^{-2k}/M_k \geqslant ck^{-\varepsilon n/2}, \tag{3.12}$$

where the inequality $M_k \geqslant 4C_1 k^{-\varepsilon}$ was used; here again we take $k > k^*$ with k^* sufficiently large.

Recalling now (3.8), the assertion (3.2) follows from (3.12).

LEMMA 3.2. *Under the assumptions of Lemma* 3.1,

$$\mu_k \leqslant \hat{C}k^{-\varepsilon} \quad \textit{for all } k < k_0 \tag{3.13}$$

where $\hat{C}$ is a constant depending only on ε, δ_0, N.

PROOF. We choose $\hat{C}$ so large that (3.13) holds for all $1 \leqslant k \leqslant k^* + 1$ and $\hat{C} > 1$. Next we proceed by induction on k. Suppose (3.13) holds for some k, $k^* < k < k_0$; we shall prove it for $k + 1$. In view of the definition of M_k, $M_k \leqslant \hat{C}k^{-\varepsilon}$. Substituting this into (3.2) we get

$$\mu_{k+1} \leqslant \hat{C}k^{-\varepsilon}(1 - Ck^{-\varepsilon n/2}) \leqslant \hat{C}(k + 1)^{-\varepsilon}$$

provided

$$1 - \frac{C}{k^{\varepsilon n/2}} \leqslant \left(\frac{k + 1}{k}\right)^{-\varepsilon},$$

which is certainly the case if k^* is sufficiently large (depending on ε, C) since $\varepsilon n/2 < 1$.

LEMMA 3.3. *For any $0 < \varepsilon < 2/n$, $\delta_0 > 0$ there exists a constant $C^* > 1 + N$, depending only on ε, δ_0, N, such that for any (x, t), (x^0, t^0) in $R^n \times (\delta_0, \infty)$,*

$$u^m(x, t) \leqslant C^* \max\left\{|\log(|x - x^0| + |t - t^0|^{1/2})|^{-\varepsilon}, u^m(x^0, t^0)\right\}. \tag{3.14}$$

PROOF. We may assume that $u^m(x^0, t^0) < 1$.

Let k_0 be a positive integer such that

$$(k_0 + 1)^{-\varepsilon} \leq u^m(x^0, t^0) \leq (k_0)^{-\varepsilon}. \tag{3.15}$$

Consider the first case where $k_0 > k^*$. By Lemma 3.2,

$$u^m(x, t) \leq \hat{C}k^{-\varepsilon} \quad \text{if } |x - x^0| \leq 2^{-k}, t^0 - 2^{-2k} \leq t \leq t^0, \tag{3.16}$$

provided $k < k_0$. This gives

$$u^m(x, t) \leq C|\log(|x - x^0| + |t - t^0|^{1/2})|^{-\varepsilon}$$

as long as (x, t) does not satisfy: $|x - x^0| \leq 2^{-k_0}, t^0 - 2^{-2k_0} \leq t \leq t^0$. If, on the other hand $|x - x^0| \leq 2^{-k_0}, t^0 - 2^{-2k_0} \leq t \leq t^0$, then (3.16), with $k = k^0$, gives

$$u^m(x, t) \leq \hat{C}(k_0)^{-\varepsilon} \leq Cu^m(x^0, t^0),$$

where (3.15) has been used. Thus we have proved (3.14) if $t \leq t^0$ and $k_0 > k^*$. If $k_0 \leq k^*$ then $u^m(x^0, t^0) \geq (k^* + 1)^{-\varepsilon}$ and (3.14) follows by choosing $C^* > M(k^* + 1)^{\varepsilon}$. We have thus completed the proof of (3.14) in case $t \leq t^0$. In particular it follows that

$$u^m(x, t^0) \leq C^*|\log|x - x^0||^{-\varepsilon} + C^*u^m(x^0, t^0). \tag{3.17}$$

Let v be the solution of

$$\Delta v - c_* \frac{\partial v}{\partial t} = -\frac{M}{2(m-1)\delta_0} \qquad \text{in } R^n \times (t^0, \infty),$$

$$v(x, t^0) = u^m(x, t^0) \qquad \text{in } R^n.$$

Since u^m satisfies (2.21), we can compare v with u^m and conclude that

$$u^m(x, t) \leq v(x, t). \tag{3.18}$$

Representing v in terms of the fundamental solution of $\Delta - c_*(\partial/\partial t)$ and then using (3.17), we find that

$$v(x, t) \leq C^*|\log(|x - x^0| + |t - t^0|^{1/2})|^{-\varepsilon} + C^*u^m(x^0, t^0) \quad \text{if } t > t^0,$$

with another constant C^*. Using (3.18), the inequality (3.14) then follows (with yet another constant C^*) for $t > t^0$.

4. Proof of Theorem 1.1 for $t > 0$. We begin by deriving another version of Lemma 3.3.

LEMMA 4.1. *For any $0 < \varepsilon < 2/n$, $\delta_0 > 0$, if $(x^i, t^i) \in R^n \times (2\delta_0, \infty)$ for $i = 0, 1$ and if*

$$|\log(|x^1 - x^0| + |t^1 - t^0|^{1/2})|^{-\varepsilon} < (1/C^*)u^m(x^i, t^i) \tag{4.1}$$

for $i = 0$ or $i = 1$, then

$$1/C^* < u^m(x^1, t^1)/u^m(x^0, t^0) < C^*. \tag{4.2}$$

Here C^* is the same constant as in Lemma 3.3.

PROOF. It suffices to prove (4.2) when (4.1) holds for $i = 0$, that is,

$$\left|\log(|x^1 - x^0| + |t' - t^0|^{1/2})\right|^{-\varepsilon} < (1/C^*)u^m(x^0, t^0). \tag{4.3}$$

The inequality $u^m(x^1, t^1) \leqslant C^* u^m(x^0, t^0)$ is of course a consequence of (3.14) and (4.1) (for $i = 0$). To prove that $u^m(x^1, t^1) \geqslant u^m(x^0, t^0)/C^*$, we proceed by assuming that

$$u^m(x^1, t^1) < (1/C^*)u(x^0, t^0) \tag{4.4}$$

and deriving a contradiction.

We write (3.14) with (x, t) and (x^0, t^0) replaced, respectively, by (x^0, t^0) and (x^1, t^1) and then use the relations (4.3), (4.4). We obtain $u^m(x^0, t^0) < u^m(x^0, t^0)$, which is impossible.

Take now any point (x^0, t^0) with $t_0 > 2\delta_0$, and let k_0 be a positive integer such that

$$M^m(k_0 + 1)^{-\varepsilon} \leqslant u^m(x^0, t^0) \leqslant M^m k_0^{-\varepsilon}. \tag{4.5}$$

Define

$$\Sigma_0 = \left\{(x, t): |x - x^0| + |t - t^0|^{1/2} < 2^{-\mu c k_0}, t > \delta_0\right\} \tag{4.6}$$

where c and μ are positive numbers to be determined below (independently of k_0), and $\mu > 2$.

If $(x, t) \notin \Sigma_0$, $t > \delta_0$, then $|\log(|x - x^0| + |t - t^0|^{1/2})|^{-\varepsilon} > (k_0 \mu c \log 2)^{-\varepsilon}$. Recalling (4.5) and using Lemma 3.3, we then obtain $u^m(x, t) \leqslant C|\log(|x - x^0| + |t - t^0|^{1/2})|^{-\varepsilon}$ where C is a positive constant depending on δ_0, ε, N, μ, c. Since the same inequality holds also for $u^m(x^0, t^0)$, we obtain

$$\left|u^m(x, t) - u^m(x^0, t^0)\right| \leqslant 2C\left|\log(|x - x^0| + |t - t^0|^{1/2})\right|^{-\varepsilon}. \tag{4.7}$$

We shall now evaluate the left-hand side for $(x, t) \in \Sigma_0$. By (4.5) and Lemma 4.1,

$$1/C^* \leqslant u^m(\bar{x}, \bar{t})/u^m(x^0, t^0) \leqslant C^* \tag{4.8}$$

provided

$$|\log(|\bar{x} - x^0| + |\bar{t} - t^0|^{1/2})|^{-\varepsilon} < (N^m/C^*)(k_0 + 1)^{-\varepsilon},$$

that is, provided $|\bar{x} - x^0| + |\bar{t} - t^0|^{1/2} < 2^{-\bar{c}k_0}$ where $\bar{c}$ is a positive constant depending on ε, δ_0, N. We now choose, in (4.6), $c = \bar{c}$. It follows that if

$$|\bar{x} - x^0| \leqslant 2^{-ck_0}, \qquad |\bar{t} - t^0| \leqslant 2^{-2ck_0} \tag{4.9}$$

then (4.8) holds.

Introduce variables

$$x' = (x - x^0)/2^{-ck_0}\lambda, \qquad t' = (t - t^0)/2^{-2ck_0},$$

where $\lambda = k_0^{-(m-1)\varepsilon/2m}$ and let $v(x', t') = u(x, t)$. Then

$$\partial v/\partial t' = \nabla(a(x', t')\nabla v') \tag{4.10}$$

where $a = mu^{m-1}k_0^{\varepsilon(m-1)/m}$.

If

$$|x'| \leqslant k_0^{\varepsilon(m-1)/m}, \qquad |t'| \leqslant 1, \tag{4.11}$$

then

$$|x - x^0| \leqslant 2^{-ck_0}\lambda^2 k_0^{\varepsilon(m-1)/m} = 2^{-ck_0}, \qquad |t - t^0| \leqslant 2^{-2ck_0},$$

so that (4.9) holds with $(\bar{x}, \bar{t}) = (x, t)$; consequently also (4.8) is thus satisfied. Then, $C_1 < a(x', t') < C_2$ where C_1, C_2 are positive constants depending only on ε, δ_0, N. We can now apply the Nash estimate [8] to v and conclude that for some α, $0 < \alpha < 1$,

$$\begin{aligned} |u(x, t) - u(x^0, t^0)| &= |v(x', t') - v(0, 0)| \leqslant C(|x'|^\alpha + |t'|^{\alpha/2}) \\ &= C\frac{|x - x^0|^\alpha}{(2^{-ck_0}\lambda)^\alpha} + C\frac{|t - t^0|^{\alpha/2}}{(2^{-2ck_0})^\alpha} \end{aligned} \tag{4.12}$$

where C is a generic constant depending only on ε, δ_0, N.

Now, if we take $\mu > 4/\alpha$ then, for $(x, t) \in \Sigma_0$,

$$|x - x^0|^{\alpha/2} \leqslant (2^{-\mu c k_0})^{\alpha/2} \leqslant (2^{-ck_0}\lambda)^\alpha,$$

$$|t - t^0|^{\alpha/4} \leqslant (2^{-2\mu c k_0})^{\alpha/4} \leqslant (2^{-2ck_0})^\alpha.$$

Substituting this into (4.12), we get

$$|u(x, t) - u(x^0, t^0)| \leqslant C(|x - x^0|^{\alpha/2} + |t - t^0|^{\alpha/4}).$$

Combining this with (4.7) we find that (4.7) (with a different C) is valid for all (x, t), (x^0, t^0) in $R^n \times (2\delta_0, \infty)$.

We now recall that the function u which we have been considering so far is actually the solution u_η of (1.4), (1.8), and $u_\eta(x, t)\downarrow u(x, t)$ as $\eta\downarrow 0$. Hence, by taking $\eta\downarrow 0$ in (4.7) (for u_η) we obtain the same inequality for u. This completes the proof of Theorem 1.1(ii) in case $n \geqslant 3$.

If $n = 2$ we can improve the modulus of continuity. We take in the proof of Lemma 3.1 $\sigma(h) = 2^{-|\log h|^{1/2}}$, $h = 2^{-2k}$, $\lambda = 2^{-k^{1/2}}$ and apply Lemma 2.3 (with $r = 2^{-k}2^{-k^{1/2}}$). The function z defined in (3.8) then satisfies (3.10) and

$$z(x', -1) > 1 - c2^{-k^{1/2}}/\mu_k > \tfrac{1}{2} \quad \text{if } |x'| < 2^{-k^{1/2}},$$

where $M_k = \max\{\mu_k, 2^{-C'k^{1/2}}\}$ and c, C' are positive constants. Let $\zeta(x') = z(x', -1) - (C2^{-2k}/M_k)|x'|^2$. Then $(\Delta - c_*\partial/\partial t')\zeta \leqslant 0$. Also

$$\zeta > \tfrac{1}{3} \quad \text{if } |x'| < 2^{-k^{1/2}},$$

$$\zeta \geqslant -C2^{-2k}/M_k \quad \text{if } |x'| = 1.$$

We compare ζ with the function

$$\eta(x') = \frac{1}{4}\,\frac{\log|x'|}{\log 2^{-k^{1/2}}} - \frac{C2^{-2k}}{M_k}.$$

By the maximum principle, $\zeta > \eta$ if $2^{-k^{1/2}} < |x'| < 1$. Hence, if $|x'| < \frac{1}{2}$,

$$z(x', -1) > \frac{1}{2}\frac{\log(1/2)}{\log 2^{-k^{1/2}}} - \frac{C2^{-2k}}{M_k} + \frac{C2^{-2k}}{M_k}|x'|^2$$
$$> \frac{C}{k^{1/2}} - C2^{-k} > \frac{C}{k^{1/2}},$$

where C is a generic constant depending on δ_0, N.

We can now proceed as in (3.11) and obtain

$$z(x', t') > C/k^{1/2} \qquad \text{if } |x'| < \tfrac{1}{2}, -\tfrac{1}{2} < t' < 0.$$

Therefore $\mu_{k+1} < M_k(1 - C/k^{1/2})$ for any $k > k^*$.

Proceeding analogously to the proof of Lemma 3.2, we establish by induction on k that

$$\mu_k < C2^{-C'k^{1/2}} \qquad (k^* < k < k_0). \tag{4.13}$$

provided C' is chosen sufficiently small.

With (4.13) at hand, we can now proceed as in the case $n > 3$, replacing everywhere the modulus of continuity $C|\log r|^{-\epsilon}$ by $C2^{-C'|\log r|^{1/2}}$.

5. Continuity at $t = 0$. We first prove continuity at a point $(y, 0)$ where $u_0(y) > 0$. Consider the function

$$w(x, t) = \frac{1}{At + 1}\left(1 - \frac{|x|^2}{(At + 1)^{\alpha}}\right)^{1/(m-1)} \qquad (\alpha > 0, A > 0). \tag{5.1}$$

By direct calculation we find that in the region where $|x|^2 < At + 1$, $0 < t < \delta$: $\Delta w^m - w_t > 0$ provided $B \equiv A^{m+\alpha-1}$ and α satisfy $B > 2mn/(m-1)$, $\alpha B < 4m/(m-1)$, and provided δ is sufficiently small. Define

$$w_{c,L}(x, t) = cw(Lx, c^{m-1}L^2t), \qquad c > 0, L > 0. \tag{5.2}$$

Then (cf. [6])

$$\Delta(w_{c,L})^m - \partial w_{c,L}/\partial t > 0 \quad \text{if } |x|^2 < Ac^{m-1}t + L^{-2}, t < \delta_0, \tag{5.3}$$

where $\delta_0 = \delta/(c^{m-1}L^2)$.

Consider the function

$$v(x, t) = \begin{cases} w_{c,L}(x, t) & \text{if } |x|^2 < Ac^{m-1}t + L^{-2}, \\ 0 & \text{if } |x|^2 > Ac^{m-1}t + L^{-2}. \end{cases} \tag{5.4}$$

Since v^m vanishes on $|x|^2 = Ac^{m-1}t + L^{-2}$ to an order larger than 1, it is clear that v is a subsolution of (1.4). Further,

$$v(x, 0) = 0 \qquad \text{if } |x| > 1/L,$$
$$v(x, 0) < c \qquad \text{if } |x| < 1/L. \tag{5.5}$$

Now, since $u_0(y) > 0$, we have $u_0(x) > c$ if $|x - y| < 1/L$ for some $c > 0$, $L > 0$. We can therefore compare the solution $u_\eta(x, t)$ with $\tilde{v}(x, t) \equiv v(x - y, t)$, and conclude that

$$u_\eta(x, t) \geqslant \tilde{v}(x, t) \quad \text{if } 0 \leqslant t \leqslant \delta_0.$$

Consequently $u_\eta(x, t) > c/2$ if $|x - y| < 1/2L$, $0 < t < \delta'$ (δ' small enough).

We can now apply the Nash estimate [8] and deduce a uniform Hölder continuity on the $u_\eta(x, t)$ for

$$|x - y| \leqslant 1/3L, \qquad 0 \leqslant t \leqslant \tfrac{1}{2}\delta', \tag{5.6}$$

with exponent and coefficient which are independent of η. Taking $\eta \to 0$, we conclude that $u(x, t)$ is also Hölder continuous in the set (5.6).

It remains to prove continuity at a point $(y, 0)$ for which $u_0(y) = 0$. For any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$u_0(x) < \varepsilon \qquad \text{if } |x - y| < \delta. \tag{5.7}$$

Consider the parabolic problem

$$\begin{aligned} \partial w/\partial t &= \Delta w^m \qquad \text{if } |x - y| < \delta, t > 0,\\ w(x, 0) &= 2\varepsilon \qquad \text{if } |x - y| < \delta,\\ w(x, t) &= N + \varepsilon \qquad \text{if } |x - y| = \delta, t > 0. \end{aligned}$$

This problem has a classical solution. We shall compare this solution with the function $u_\eta(x, t)$ for $\eta < \varepsilon$. In view of (5.7),

$$u_\eta(x, 0) < w(x, 0) \quad \text{if } |x - y| < \delta.$$

Since also $u_\eta(x, t) < w(x, t)$ when $|x - y| = \delta$, $t > 0$, we conclude that $u_\eta(x, t) < w(x, t)$. Taking $\eta \to 0$ and noting that $w(x, t)$ is continuous at $(y, 0)$, we get

$$\overline{\lim_{(x, t)\to(y, 0)}}\ u(x, t) \leqslant \lim_{(x, t)\to(y, 0)} w(x, t) = 2\varepsilon.$$

Since ε is arbitrary, $u(x, t) \to 0 = u_0(y)$ if $(x, t) \to (y, 0)$. This completes the proof.

Remark 1. Consider the case $n = 1$. Then $u^m_{xx} \geqslant -c$ and, since u^m is bounded, u^m_x must then be bounded. Using the relation $u^m_t \geqslant -cu^m$ we then deduce that if $t_2 \geqslant t$ then

$$u^m(x_2, t_2) > u^m(x_1, t_1) - C(|x_2 - x_1| + |t_2 - t_1|). \tag{5.8}$$

If $v_{xx} - c_* v_t = -C$ ($c_* > 0$, $C > 0$) and $v(x, t_1) = u^m(x, t_1)$, then we get, upon using (5.8) with $t_2 = t_1$, that

$$|v(x, t) - v(x_1, t_1)| \leqslant C(|x - x_1|^\theta + |t - t_1|^{\theta/2}) \qquad (t > t_1)$$

for any $\theta < \frac{1}{2}$. Since $u^m_{xx} - c_* u^m_t \geqslant -C$ for some $c_* > 0$, $C > 0$, we conclude that $u^m \leqslant v$, so that

$$u^m(x_2, t_2) \le u^m(x_1, t_1) + C(|x_2 - x_1|^\theta + |t_2 - t_1|^{\theta/2}).$$

Together with (5.8) we thus obtain a modulus of continuity Ar^θ for u^m, for any $\theta < 1$. Actually, in this case of $n = 1$, a better modulus of continuity is known (Aronson [1], Gilding [5]): $|(u^{m-1})_x| \le C$, and

$$|u(x_2, t_2) - u(x_1, t_1)| \le C(|x_2 - x_1|^\nu + |t_2 - t_1|^{\nu/2})$$

where $\nu = \min(1, 1/(m-1))$.

REMARK 2. Theorem 1.1 implies that the sets

$$\Omega = \{(x, t) \in R^n \times (0, \infty); u(x, t) > 0\},$$
$$\Omega(t) = \{x \in R^n; u(x, t) > 0\}$$

are open subsets of $R^n \times (0, \infty)$ and R^n, respectively. The relation (2.9) (for u_η, $\eta \to 0$) implies

$$u^m(x, t) \ge C(t, t_0)u^m(x, t_0)$$

where $C(t, t_0) > 0$. It easily follows that

$$\Omega(t) \text{ is increasing with } t; \tag{5.9}$$

it is not necessarily strictly increasing (see [2], [3], [7]).

REMARK 3. The results of this paper extend to the more general equation

$$u_t = \Delta\varphi(u)$$

where $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi'(u) > 0$. The analog of Lemma 1.1, for this case, is proved in [4].

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