TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 255, November 1979

RESULTS ON WEIGHTED NORM INEQUALITIES FOR MULTIPLIERS

BY

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ABSTRACT. Weighted L^p-norm inequalities are derived for multiplier operators on Euclidean space. The multipliers are assumed to satisfy conditions of the Hörmander-Mikhlin type, and the weight functions are generally required to satisfy conditions more restrictive than A_p which depend on the degree of differentiability of the multiplier. For weights which are powers of $|x|$, sharp results are obtained which indicate such restrictions are necessary. The method of proof is based on the function $f^\#$ of C. Fefferman and E. Stein rather than on Littlewood-Paley theory. The method also yields results for singular integral operators.

1. Let $m(x)$ be a bounded function on $\mathbf{R}^n$ and consider the multiplier operator Tf defined initially for functions f in the Schwartz space $\mathcal{S}$ by $(Tf)\hat{}(x) = m(x)\hat{f}(x)$, where $\hat{g}$ is the Fourier transform of g. Denote by s a real number greater than or equal to 1, l a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index of nonnegative integers α_j with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We say $m \in M(s, l)$ if

$$\sup_{R>0}\left(R^{s|\alpha|-n}\int_{R<|x|<2R}|D^\alpha m(x)|^s\,dx\right)^{1/s} < +\infty \quad \text{for all } |\alpha| \leq l. \tag{1.1}$$

The condition (1.1) has been known to be related to multipler theorems for some time. The classic works in this direction are the theorems of Marcinkiewicz (see [**18**]) and Hörmander-Mikhlin (see [**7**]):

THEOREM A. *Let $n = 1$, $1 < p < \infty$, and $m \in M(1, 1)$. Then there exists a constant C, independent of f, such that $\|Tf\|_p \leq C\|f\|_p$.*

THEOREM B. *Let $l > n/2$, $1 < p < \infty$, and $m \in M(2, l)$. Then there exists a constant C, independent of f, such that $\|Tf\|_p \leq C\|f\|_p$.*

Much work has been done to extend these results. Using interpolation methods, Calderón and Torchinsky [**2**] have considered the condition $m \in M(s, l)$ for $s \geqslant 2$ and $l > n/s$. Hirschman [**6**], Krée [**11**], and Triebel [**20**] have extended these results in various directions to weighted L^p spaces for weights which are powers of $|x|$. More recently, Kurtz [**12**] extended Theo-

Received by the editors November 9, 1978.
AMS (MOS) subject classifications (1970). Primary 42A18.
[1]Supported in part by NSF Grant MCS 77-03980.

rems A and B to L^p spaces with more general weights by using the weighted norm inequalities derived in [15] for the function g_λ^*.

The purpose of this paper is two-fold. We consider $s < 2$ and present a method of proof based on the function $f^\#$ of Fefferman and Stein [5] rather than on Littlewood-Paley theory.

We say $f \in L_w^p(\mathbf{R}^n)$, $1 < p < \infty$ and $w(x) > 0$, if

$$\|f\|_{p,w} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < +\infty.$$

The weights w we will consider satisfy an A_r condition; i.e., $w \in A_r$ if there is a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(r-1)}\,dx\right)^{r-1} < C, \qquad 1 < r < \infty,$$

$$\frac{1}{|Q|}\int_Q w(x)\,dx < C \operatorname*{ess\,inf}_Q w, \qquad r = 1,$$

for all cubes $Q \subset \mathbf{R}^n$. When $r = 1$, the condition that $w \in A_1$ means $w^*(x) < Cw(x)$ for almost every x, where g^* is the Hardy-Littlewood maximal function of g. Finally, $w \in A_\infty$ if there exist positive constants C and δ such that for any cube $Q \subset \mathbf{R}^n$ and for any measurable set $E \subset Q$,

$$\frac{m_w(E)}{m_w(Q)} < C\left(\frac{|E|}{|Q|}\right)^\delta,$$

where $m_w(E) = \int_E w(x)\,dx$. Results concerning A_p functions can be found in Muckenhoupt [13] and Coifman and Fefferman [3]. Note, in particular, that $w \in A_p$ implies $w \in A_\infty$.

We use p' to denote the index conjugate to p: $1/p + 1/p' = 1, p > 1$.

The main result of this paper is:

THEOREM 1. *Let* $1 < s < 2$, $n/s < l < n$, *and* $m \in M(s, l)$. *If*

(1) $n/l < p < \infty$ *and* $w \in A_{pl/n}$, *or*

(2) $1 < p < (n/l)'$ *and* $w^{-1/(p-1)} \in A_{p'l/n}$,

then there is a constant C, *independent of* f, *such that*

$$\|Tf\|_{p,w} < C\|f\|_{p,w}.$$

When $l < n$, *we may take* $p = n/l$ *in* (1) *and* $p = (n/l)'$ *in* (2). *If*

(3) $w^{n/l} \in A_1$,

there is a constant C, *independent of* f *and* λ, *such that*

$$m_w(\{x \in \mathbf{R}^n: |Tf(x)| > \lambda\}) < \frac{C}{\lambda}\|f\|_{1,w}, \qquad \lambda > 0.$$

Using interpolation, other conditions on the weight can be found which guarantee that T is a bounded operator. One result which we will prove is:

THEOREM 2. *If* $1 < p < \infty$, $1 < s \leq 2$, $n/s < l \leq n$, $m \in M(s, l)$, *and* $w^{n/l} \in A_p$ *then*

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}$$

for a constant independent of f.

This result does not give the best possible condition on the weight. When $w(x) = |x|^\beta$, we have $w \in A_p$ if $-n < \beta < n(p-1)$. Interpreting Theorem 1 for such w and using interpolation with change of measures, we will show:

THEOREM 3. *Let* $1 < s \leq 2$, $n/s < l \leq n$, *and* $m \in M(s, l)$. *If* $1 < p < \infty$ *and* $\max\{-n, -lp\} < \beta < \min\{n(p-1), lp\}$, *then there is a constant C, independent of f, such that*

$$\|Tf\|_{p,|x|^\beta} \leq C\|f\|_{p,|x|^\beta}.$$

In particular, if $n/l < p < (n/l)'$, *we get* $-n < \beta < n(p-1)$; *we may also take* $p = n/l$ *and* $p = (n/l)'$ *if* $l < n$.

We will show that this result is sharp with the possible exception of the endpoint values of β.

Let $\check{g}$ denote the inverse Fourier transform of g. If we set $K = \check{m}$, then for $f \in \mathcal{S}$, $Tf(x) = (K * f)(x)$. Our proof of Theorem 1 is based on using information about m to get estimates on approximations to K, so it is not surprising that the technique carries over to convolution operators.

Denote by $\Sigma = \Sigma_{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$, $x' = x/|x| \in \Sigma$ $(x \neq 0)$, and ρ any rotation of Σ with magnitude $|\rho| = \sup_{x \in \Sigma}|\rho x - x|$. Let $1 \leq r \leq \infty$ and $\Omega \in L^r(\Sigma)$ be positively homogeneous of degree zero. We say that Ω satisfies the L^r-Dini condition if

$$\int_0^1 \omega_r(\delta)\frac{d\delta}{\delta} < +\infty,$$

where

$$\omega_r(\delta) = \sup_{|\rho| < \delta}\left(\int_\Sigma |\Omega(\rho x) - \Omega(x)|^r \, d\sigma_x\right)^{1/r}.$$

Set $K(x) = \Omega(x')/|x|^n$, with $\int_\Sigma \Omega(x) d\sigma_x = 0$, and $Tf(x) = (K * f)(x)$ in the usual principal-value sense. If Ω satisfies the L^r-Dini condition then it also satisfies the L^1-Dini condition, which by [1] implies T is a bounded operator on L^p, $1 < p < \infty$. Recently, Kaneko and Yano [10] have shown that if Ω satisfies the L^∞-Dini condition then T maps L^p_w into itself for $1 < p < \infty$ and $w \in A_p$. We have extended this to:

THEOREM 4. *Let* $1 < r \leq \infty$, $\Omega \in L^r(\Sigma)$, *and* $\int_\Sigma \Omega(x)\, d\sigma_x = 0$. *Suppose* Ω *satisfies the* L^r*-Dini condition. If*

(1) $r' < p < \infty$ *and* $w \in A_{p/r'}$, *or*

(2) $1 < p < r$ *and* $w^{-1/(p-1)} \in A_{p'/r'}$
then there is a constant C, *independent of* f, *such that*

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}.$$

When $r < \infty$, *we may take* $p = r'$ *in* (1) *and* $p = r$ *in* (2). *If*
(3) $w^{r'} \in A_1$, *then*

$$m_w(\{x \in \mathbf{R}^n: |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda}\|f\|_{1,w}, \qquad \lambda > 0,$$

where C *is independent of* f *and* λ.

Theorem 4 is a direct analogue of Theorem 1. (We could also have stated a version of Theorem 3. See also **[14]**.) In fact, when $r \geq 2$, r' plays the same role as n/l. For example, notice the similarity between $m \in M(s, n)$, $1 < s \leq 2$, and Ω satisfying the L^∞-Dini condition. Our technique, however, does not allow for either r or s to be equal to 1.

§2 contains the basic lemma and a collection of results used in the proof of Theorem 1. This theorem and Theorems 2 and 3 are proved in §3. The proof of Theorem 4 is found in §4. The paper concludes with a counterexample showing Theorem 3 is best possible except for the question of endpoint equalities for β. The basic lemma and the counterexample are generalizations to $n > 1$ of results in **[16]**, and we gratefully acknowledge many helpful discussions with W.-S. Young and B. Muckenhoupt.

2. Following **[7]**, we select an approximation to the identity

$$\sum_{j=-\infty}^{+\infty} \phi(2^{-j}x) = 1, \qquad x \neq 0,$$

where φ is an infinitely differentiable, nonnegative function supported in $\frac{1}{2} < |x| < 2$. Let $m_j(x) = m(x)\varphi(2^{-j}x)$, so that

$$m(x) = \sum_{j=-\infty}^{+\infty} m_j(x), \qquad x \neq 0.$$

Notice that $m_j(x)$ is supported in $2^{j-1} < |x| < 2^{j+1}$ and that for such x, $m_k(x) = 0$ unless $k = j - 1, j$, or $j + 1$. It follows easily that if $m \in M(s, l)$ and $|\alpha| \leq l$, then

$$\left(\int_{\mathbf{R}^n} |D^\alpha m_j(x)|^s \, dx\right)^{1/s} \leq C(2^j)^{n/s - |\alpha|},$$

with C independent of j.

We also have that $m_j \in L^1 \cap L^\infty$. Define $k_j(x)$ by $k_j(x) = \check{m}_j(x)$, and let

$$m^N(x) = \sum_{j=-N}^{N} m_j(x), \qquad K_N(x) = (m^N)\check{}(x) = \sum_{j=-N}^{N} k_j(x).$$

It follows that $\|m^N\|_\infty \leq C$, uniformly in N, and that $m^N(x) \to m(x)$, $x \neq 0$, as $N \to \infty$. Now define $T_N f$ by $T_N f = (m^N \hat{f})\check{}$, so that $T_N f = f * K_N$ for $f \in L^2$, say. The following lemma shows how conditions on m can be interpreted as conditions on K_N.

LEMMA 1. *Let* $1 < s \leq 2$, $m \in M(s, l)$ *for a positive integer* l, *and let* K_N *be defined as above. If* d *is an integer such that* $0 < d \leq l$, $1 < t \leq s$, $n/t < d < n/t + 1$, *and* $1 \leq p \leq t'$, *then*

$$\left(\int_{R<|x|<2R} |K_N(x-y) - K_N(x)|^p\, dx\right)^{1/p} \leq CR^{-d+n/p-n/t'}|y|^{d-n/t}$$

for all $|y| < \dfrac{R}{2}$,

with C *independent of* N, R, *and* y.

PROOF. Since $K_N(x) = \sum_{j=-N}^{N} k_j(x)$,

$$\left(\int_{R<|x|<2R} |K_N(x-y) - K_N(x)|^p\, dx\right)^{1/p} \leq \sum_j \left(\int_{R<|x|<2R} |k_j(x-y) - k_j(x)|^p\, dx\right)^{1/p}. \tag{2.1}$$

Also, $|y| < R/2$ and $R < |x| < 2R$ imply $R/2 < |x-y| < 5R/2$, so that

$$\begin{aligned}\left(\int_{R<|x|<2R} |k_j(x-y) - k_j(x)|^p\, dx\right)^{1/p} &\leq \left(\int_{R<|x|<2R} |k_j(x-y)|^p\, dx\right)^{1/p} + \left(\int_{R<|x|<2R} |k_j(x)|^p\, dx\right)^{1/p}\\ &\leq 2\left(\int_{R/2<|x|<5R/2} |k_j(x)|^p\, dx\right)^{1/p}.\end{aligned}$$

Therefore, we need to estimate

$$\left(\int_{R/2<|x|<5R/2} |k_j(x)|^p\, dx\right)^{1/p} \quad \text{and} \quad \left(\int_{R<|x|<2R} |k_j(x-y) - k_j(x)|^p\, dx\right)^{1/p}.$$

Let d be an integer such that $0 < d \leq l$ and $1 < t \leq s$ such that $p \leq t'$. It is easy to see that $m \in M(t, d)$. Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then

$$\begin{aligned}\left(\int_{R/2<|x|<5R/2} |k_j(x)|^p\, dx\right)^{1/p} &\leq CR^{-d}\left(\int_{R/2<|x|<5R/2} \big|\,|x|^d k_j(x)\big|^p\, dx\right)^{1/p}\\ &\leq CR^{-d} \sum_{|\alpha|=d} \left(\int_{R/2<|x|<5R/2} |x^\alpha k_j(x)|^p\, dx\right)^{1/p}.\end{aligned}$$

Using the fact that $\check{m}_j = k_j$, Hölder's inequality, and the Hausdorff-Young theorem, we have for $|\alpha| = d$ that

$$\left(\int_{R/2<|x|<5R/2} |x^\alpha k_j(x)|^p \, dx\right)^{1/p} = \left(\int_{R/2<|x|<5R/2} |(D^\alpha m_j)\check{}(x)|^p \, dx\right)^{1/p}$$

$$= R^{n/p}\left(R^{-n}\int_{R/2<|x|<5R/2} |(D^\alpha m_j)\check{}(x)|^p \, dx\right)^{1/p}$$

$$< CR^{n/p}\left(R^{-n}\int_{R/2<|x|<5R/2} |(D^\alpha m_j)\check{}(x)|^{t'} \, dx\right)^{1/t'}$$

$$< CR^{n/p-n/t'}\left(\int_{\mathbf{R}^n} |D^\alpha m_j(x)|^t \, dx\right)^{1/t}$$

$$< CR^{n/p-n/t'}(2^j)^{n/t-d}.$$

Combining these estimates gives

$$\left(\int_{R/2<|x|<5R/2} |k_j(x)|^p \, dx\right)^{1/p} < CR^{-d+n/p-n/t'}(2^j)^{n/t-d}. \tag{2.2}$$

For the integral of the difference of the k_j's we have

$$\left(\int_{R<|x|<2R} |k_j(x-y) - k_j(x)|^p \, dx\right)^{1/p}$$

$$= \left(\int_{R<|x|<2R} |\{m_j(x)(e^{ix\cdot y} - 1)\}\check{}|^p \, dx\right)^{1/p}$$

$$< CR^{-d}\left(\int_{R<|x|<2R} \left| \, |x|^d \{m_j(x)(e^{ix\cdot y} - 1)\}\check{}\right|^p \, dx\right)^{1/p}$$

$$< CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n}\int_{R<|x|<2R} |x^\alpha \{m_j(x)(e^{ix\cdot y} - 1)\}\check{}|^p \, dx\right)^{1/p}$$

$$< CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n}\int_{R<|x|<2R} |\{D^\alpha[m_j(x)(e^{ix\cdot y} - 1)]\}\check{}|^p \, dx\right)^{1/p}$$

$$< CR^{n/p-d} \sum_{|\alpha|=d} \left(R^{-n}\int_{R<|x|<2R} |\{D^\alpha[m_j(x)(e^{ix\cdot y} - 1)]\}\check{}|^{t'} \, dx\right)^{1/t'}$$

$$< CR^{n/p-d-n/t'} \sum_{|\alpha|=d} \left(\int_{\mathbf{R}^n} |D^\alpha[m_j(x)(e^{ix\cdot y} - 1)]|^t \, dx\right)^{1/t}$$

$$< CR^{n/p-d-n/t'} \sum_{|\beta|+|\gamma|=d} \left(\int_{\mathbf{R}^n} |D^\beta m_j(x) \cdot D^\gamma(e^{ix\cdot y} - 1)|^t \, dx\right)^{1/t}.$$

Consider first $|\gamma| = 0$, $|\beta| = d$. Since $|e^{ix\cdot y} - 1| < |x|\,|y|$,

$$\left(\int_{\mathbf{R}^n} |(D^\beta m_j(x))(e^{ix\cdot y} - 1)|^t \, dx\right)^{1/t} < \left(\int_{\mathbf{R}^n} |\, |x|\, |y| D^\beta m_j(x)|^t \, dx\right)^{1/t}$$

$$< C2^j|y|(2^j)^{n/t-d} = C|y|(2^j)^{n/t-d+1}.$$

If $|\gamma| > 0$, $|D^\gamma(e^{ix\cdot y} - 1)| < |y|^{|\gamma|}$ and $|\beta| = d - |\gamma|$, so that

$$\left(\int_{\mathbf{R}^n} |D^\beta m_j(x) \cdot D^\gamma(e^{ix\cdot y} - 1)|^t \, dx\right)^{1/t} < \left(\int_{\mathbf{R}^n} |\, |y|^{|\gamma|} D^\beta m_j(x)|^t \, dx\right)^{1/t}$$

$$< C|y|^{|\gamma|}(2^j)^{n/t-|\beta|} = C|y|^{|\gamma|}(2^j)^{n/t-d+|\gamma|}.$$

Adding these estimates, we obtain

$$\left(\int_{R<|x|<2R} |k_j(x - y) - k_j(x)|^p \, dx\right)^{1/p} < CR^{n/p-d-n/t'} \sum_{m=1}^{d} |y|^m (2^j)^{n/t-d+m}. \tag{2.3}$$

But, if $2^j < |y|^{-1}$ $(|y| < 2^{-j})$,

$$|y|^m (2^j)^{n/t-d+m} < |y|(2^j)^{n/t-d+1},$$

so for these values of j, the estimate (2.3) becomes $CR^{n/p-d-n/t'}|y|(2^j)^{n/t-d+1}$.

Using (2.2) and (2.3) in (2.1), we get

$$\left(\int_{R<|x|<2R} |K_N(x - y) - K_N(x)|^p \, dx\right)^{1/p}$$

$$< C \sum_{2^j<|y|^{-1}} R^{n/p-d-n/t'}|y|(2^j)^{n/t-d+1} + C \sum_{2^j>|y|^{-1}} R^{n/p-d-n/t'}(2^j)^{n/td}$$

$$< CR^{n/p-d-n/t'}|y|^{d-n/t}$$

as long as $n/t < d < n/t + 1$. This completes the proof of Lemma 1.

Although we will not use it, we would like to point out that if $l > \max\{n/p', n/s\}$, then

$$\left(\int_{R<|x|<2R} |K_N(x)|^p \, dx\right)^{1/p} < CR^{n/p-n}.$$

This follows from (2.2) with $d = l$ and the estimate

$$\left(\int_{R<|x|<2R} |k_j(x)|^p \, dx\right)^{1/p} < C2^{jn}R^{n/p},$$

which is a consequence of $|k_j(x)| = |\check{m}_j(x)| < \|m_j\|_1 < C2^{jn}$.

REMARK 1. We may replace the domain of integration in Lemma 1 by $\{x \in \mathbf{R}^n \colon R < |x|\}$; that is, under the conditions of Lemma 1,

$$\left(\int_{R<|x|}|K_N(x-y)-K_N(x)|^p\,dx\right)^{1/p} \le CR^{-d+n/p-n/t'}|y|^{d-n/t}.$$

For, if t, d, and y satisfy the conditions of Lemma 1,

$$\begin{aligned}\left(\int_{R<|x|}|K_N(x-y)-K_N(x)|^p\,dx\right)^{1/p} &\le \sum_{j=0}^{\infty}\left(\int_{2^jR<|x|<2^{j+1}R}|K_N(x-y)-K_N(x)|^p\,dx\right)^{1/p}\\ &\le \sum_{j=0}^{\infty}C(2^jR)^{-d+n/p-n/t'}|y|^{d-n/t}\\ &= CR^{-d+n/p-n/t'}|y|^{d-n/t}\sum_{j=0}^{\infty}(2^j)^{-d+n/p-n/t'}\\ &= CR^{-d+n/p-n/t'}|y|^{d-n/t},\end{aligned}$$

since $-d+n/p-n/t'<0$ for $n/t<d$.

REMARK 2. The Hörmander-Mikhlin theorem follows easily from Lemma 1. To see this, let $m\in M(s,l)$, $1<s\le 2$ and $l>n/s$. Choose $t\le s$ so that $n/t<l<n/t+1$. By Remark 1 with $p=1$ and $R=2|y|$, we have

$$\int_{|x|>2|y|}|K_N(x-y)-K_N(x)|dx \le C(2|y|)^{-l+n-n/t'}|y|^{l-n/t}=C.$$

Thus, the kernels K_N satisfy, uniformly in N, the Hörmander condition

$$\int_{|x|>2|y|}|K(x-y)-K(x)|dx\le C \quad \text{for all } y\neq 0,$$

so that $T_Nf=K_N*f$ is bounded on L^p, uniformly in N, for $1<p<\infty$.

For $f\in\mathcal{S}$, we have $Tf=(m\hat{f})\check{}$. It follows that

$$\|Tf-T_Nf\|_\infty\le\|(m-m^N)\hat{f}\|_1\to 0$$

since m^N converges pointwise and boundedly to m. Then, applying Fatou's lemma, we get

$$\|Tf\|_p\le C\|f\|_p,$$

for $f\in\mathcal{S}$, where C is the uniform bound for the T_N on L^p. The result extends to all of L^p by continuity.

Part (1) of Theorem 1 is proved using Lemma 1 and the following three known results.

LEMMA 2. *Set $f_r^*(x)=((f^r)^*)^{1/r}(x)$. If $0<r<p<\infty$ and $w\in A_{p/r}$, then*

$$\|f_r^*\|_{p,w}\le C\|f\|_{p,w}$$

with C independent of f.

This is an immediate corollary of results in [13].

LEMMA 3. *Let*

$$f^{\#}(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y) - \mathrm{av}_Q f| \, dy,$$

where $\mathrm{av}_Q f = |Q|^{-1} \int_Q f(z)\, dz$. *Let* $0 < p < \infty$ *and* $w \in A_\infty$. *Then*

$$\|f^*\|_{p,w} \leq C \|f^{\#}\|_{p,w}$$

with C independent of f.

This is proved in [4]. The following result is a special case of interpolation with change of measures. It is proved in [17] and [19].

LEMMA 4. *Let* $1 < r \leq q < \infty$ *and let* w_0 *and* w_1 *be two positive weights. If T is a bounded linear operator from* $L^r_{w_0}$ *into itself and* $L^q_{w_1}$ *into itself, then T is bounded from* L^p_w *into itself for* $r \leq p \leq q$ *and* $w = w_0^t w_1^{1-t}$, *provided* $t = (q-p)/(q-r)$ *for* $r \neq q$ *and* $0 \leq t \leq 1$ *for* $r = q$.

We would like to point out that $w^{n/l} \in A_p$, $n/l \geq 1$, if and only if $w \in A_p$ and satisfies the reverse Hölder's inequalities

$$\frac{1}{|Q|} \int_Q w^{n/l}(x)\, dx \leq C \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right)^{n/l}$$

and

$$\frac{1}{|Q|} \int_Q \left(w(x)^{-1/(p-1)} \right)^{n/l} dx \leq C \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{n/l};$$

when $p = 1$, we only need the first inequality. For $p > 1$, if $w \in A_p$ and satisfies the above inequalities, then

$$\left(\frac{1}{|Q|} \int_Q w^{n/l}(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q (w^{n/l}(x))^{-1/(p-1)}\, dx \right)^{p-1}$$
$$\leq C \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right)^{n/l} \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{(p-1)n/l}$$

so that $w^{n/l} \in A_p$. For $p = 1$, if $w \in A_1$ and satisfies the first inequality above, then

$$\frac{1}{|Q|} \int_Q w^{n/l}(x) dx \leq C \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{n/l}$$
$$\leq C \left(\operatorname*{ess\,inf}_Q w \right)^{n/l} = C \operatorname*{ess\,inf}_Q w^{n/l},$$

so that $w^{n/l} \in A_1$. For the other implication, note first that $w^{n/l} \in A_p$ implies $w \in A_p$ since $n/l \geq 1$. If $p > 1$, by the A_p condition,

$$\left(\frac{1}{|Q|}\int_Q w^{n/l}(x)dx\right)^{l/n} \leq C\left(\frac{1}{|Q|}\int_Q w(x)^{-(n/l)(1/(p-1))}\,dx\right)^{-(l/n)(p-1)}.$$

Thus, the first reverse Hölder's inequality will follow if we show

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-(n/l)(1/(p-1))}\,dx\right)^{-(l/n)(p-1)} \leq \frac{1}{|Q|}\int_Q w(x)\,dx,$$

or equivalently

$$1 \leq \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-n/l(p-1)}\,dx\right)^{l(p-1)/n}.$$

But, if $s > 1$, using Hölder's inequality, we have

$$\begin{aligned} 1 = \frac{1}{|Q|}\int_Q dx &= \frac{1}{|Q|}\int_Q w^{1/s}(x)w^{-1/s}(x)\,dx \\ &\leq \left(\frac{1}{|Q|}\int_Q w(x)dx\right)^{1/s}\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(s-1)}\,dx\right)^{(s-1)/s}. \end{aligned}$$

Setting $s - 1 = l(p-1)/n$, or $s = 1 + l(p-1)/n > 1$, we get the desired inequality. Since $w^{n/l} \in A_p$ implies $(w^{-1/(p-1)})^{n/l} \in A_{p'}$, we also obtain the other reverse Hölder's inequality from the argument above. Finally, when $p = 1$, by the A_1 condition

$$\frac{1}{|Q|}\int_Q w^{n/l}(x)dx \leq c\operatorname*{ess\,inf}_Q w^{n/l} = c(\operatorname{ess\,inf} w)^{n/l} \leq c\left(\frac{1}{|Q|}\int_Q w(x)dx\right)^{n/l}.$$

Notice that the above is true if we replace n/l by any $t \geq 1$.

3. We begin the proof of Theorem 1 by noting that (2) is a consequence of (1) by duality. To see this, suppose $1 < p \leq (n/l)'$ and $w^{-1/(p-1)} \in A_{p'l/n}$. Then, for $f \in \mathcal{S}$,

$$\|Tf\|_{p,w} = \left(\int_{\mathbf{R}^n}|Tf(x)|^p\,w(x)dx\right)^{1/p} = \sup\left|\int_{\mathbf{R}^n} Tf(x)g(x)dx\right|,$$

where the supremum is taken over all functions $g \in \mathcal{S}$ such that $\|g\|_{p',w^{-1/(p-1)}} = 1$.

Let $\overline{T}$ be the operator with multiplier $\overline{m}$, the complex conjugate of m. Then $\overline{m}$ satisfies the same estimates as m and we have

$$\begin{aligned} \|Tf\|_{p,w} = \sup\left|\int_{\mathbf{R}^n} f(x)\overline{T}g(x)dx\right| &\leq \sup\|f\|_{p,w}\|\overline{T}g\|_{p',w^{-1/(p-1)}} \\ &\leq C\|f\|_{p,w}\sup\|g\|_{p',w^{-1/(p-1)}} = C\|f\|_{p,w} \end{aligned}$$

by (1), since $p' \geq n/l$ and $w^{-1/(p-1)} \in A_{p'l/n}$.

Turning to the proof of (1), fix $p > n/l$ and $w \in A_{pl/n}$. Choose an $r \leq s$ such that n/r is not an integer, $n/l < r < p$ and $w \in A_{p/r}$. There is an integer $d < l$ for which $n/r < d < n/r + 1$. We will show

$$(T_N f)^{\#}(x) < C f_r^*(x) \tag{3.1}$$

with a C independent of f and N.

Fix $x \in \mathbf{R}^n$ and let Q be a cube centered at x with diameter δ. Write

$$f(y) = f_0(y) + \sum_{j=1}^{\infty} f_j(y),$$

where

$$f_0(y) = f(y)\chi(\{y \in \mathbf{R}^n: |x - y| < 2\delta\})$$

and

$$f_j(y) = f(y)\chi(\{y \in \mathbf{R}^n: 2^j\delta < |x - y| < 2^{j+1}\delta\}), \qquad j = 1, 2, \ldots.$$

For $y \in Q$,

$$(K_N * f)(y) = (K_N * f_0)(y) + \sum_{j=1}^{\infty} (K_N * f_j)(y).$$

By Hölder's inequality and Remark 2, for any $q > 1$ we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(K_N * f_0)(y)|\,dy &< \left(\frac{1}{|Q|}\int_Q |(K_N * f_0)(y)|^q\,dy\right)^{1/q}\\ &< C\frac{\|f_0\|_q}{|Q|^{1/q}} < C f_q^*(x),\end{aligned}$$

with C independent of N. For any j,

$$\begin{aligned}(K_N * f_j)(y) &= (K_N * f_j)(x) + \int \{K_N(y - z) - K_N(x - z)\} f_j(z)\,dz\\ &\equiv c_j + \varepsilon_j,\end{aligned}$$

say. Note that c_j is independent of y and

$$\begin{aligned}|\varepsilon_j| &< \int_{2^j\delta < |x-z| < 2^{j+1}\delta} |K_N(y - z) - K_N(x - z)|\,|f(z)|\,dz\\ &< \left(\int_{2^j\delta < |x-z| < 2^{j+1}\delta} |K_N(y - z) - K_N(x - z)|^{r'}\,dz\right)^{1/r'}\\ &\quad\cdot\left(\int_{|x-z| < 2^{j+1}\delta} |f(z)|^r\,dz\right)^{1/r}.\end{aligned}$$

Applying Lemma 1 with $p = r'$ and $t = r$ and noting that $|x - y| < \delta$, we obtain

$$\begin{aligned}|\varepsilon_j| &< C|x - y|^{d-n/r}(2^j\delta)^{-d}(2^{j+1}\delta)^{n/r}\left\{(2^{j+1}\delta)^{-n}\int_{|x-z|<2^{j+1}\delta}|f(z)|^r\,dz\right\}^{1/r}\\ &< C(2^j)^{n/r-d} f_r^*(x).\end{aligned}$$

Therefore,

$$\frac{1}{|Q|}\int_Q\left|(K_N*f)(y)-\sum_{j=1}^{\infty}c_j\right|dy=\frac{1}{|Q|}\int_Q\left|\sum_{j=0}^{\infty}(K_N*f_j)(y)-\sum_{j=1}^{\infty}c_j\right|dy$$

$$\leq\frac{1}{|Q|}\int_Q|(K_N*f_0)(y)|dy+\sum_{j=1}^{\infty}\frac{1}{|Q|}\int_Q|(K_N*f_j)(y)-c_j|\,dy$$

$$\leq Cf_r^*(x)+C\sum_{j=1}^{\infty}(2^j)^{n/r-d}f_r^*(x)=Cf_r^*(x),$$

since $n/r-d<0$. The fact that this estimate is true for any cube centered at x implies (3.1). Now, using Lemmas 2 and 3, since $w\in A_{p/r}$, we obtain

$$\|(K_N*f)\|_{p,w}\leq\|(K_N*f)^*\|_{p,w}\leq C\|(K_N*f)^\#\|_{p,w}\leq C\|f_r^*\|_{p,w}\leq C\|f\|_{p,w},$$

uniformly in N. Arguing as in Remark 2, we have

$$\|Tf\|_{p,w}=\|(K*f)\|_{p,w}\leq C\|f\|_{p,w}.$$

When $l<n$ and $p=n/l$, the above proof fails. However, using Lemma 4 and the fact that $w\in A_1$ implies there is a $b>1$ such that $w^b\in A_1$, we will prove the result. So, fix such a b. Then $w^b\in A_{ql/n}$ for any $q>n/l$. Setting $w_0(x)=1$ and $w_1(x)=w^b(x)$, we need to find q and r so that $r<n/l<q$ and $w(x)=(w^b(x))^{(n/l-r)/(q-r)}$. Thus we need $b((n/l-r)/(q-r))=1$ or $b(n/l-r)=q-r$. Then, choosing r, $1<r<n/l$, and solving for q, which is necessarily greater than n/l since $b>1$, completes the proof.

The proof of Theorem 1 will be finished once we show the weak-type (1, 1) result. This will be done using standard techniques which are included for completeness. Fix a nonnegative f in $L^1\cap L^1_w$ and $\lambda>0$. Applying the Calderón-Zygmund decomposition to f, we get a sequence of disjoint cubes $\{Q_k\}$ and functions g and b, $f(x)=g(x)+b(x)$, satisfying

(i) $|Q_k|\leq(C/\lambda)\int_{Q_k}f(y)dy$,

(ii) $\|g\|_{2,w}^2\leq\lambda\|f\|_{1,w}$,

(iii) $b(y)=f(y)-|Q_k|^{-1}\int_{Q_k}f(z)dz$ for $y\in Q_k$, $\operatorname{supp} b\subset\bigcup Q_k$ and $\int_{Q_k}b(y)dy=0$.

Since $T_Nf=T_Ng+T_Nb$,

$$m_w(\{x\in\mathbf{R}^n\colon|T_Nf(x)|>2\lambda\})$$

$$\leq m_w(\{x\in\mathbf{R}^n\colon|T_Ng(x)|>\lambda\})+m_w(\{x\in\mathbf{R}^n\colon|T_Nb(x)|>\lambda\}).$$

We can apply (1) of Theorem 1 to the first term on the right because $w\in A_1$. Then, using (ii), we get

$$m_w(\{x\in\mathbf{R}^n\colon|T_Ng(x)|>\lambda\})\leq\frac{C}{\lambda^2}\|g\|_{2,w}^2\leq\frac{C}{\lambda}\|f\|_{1,w}.$$

Let Q_k^* be Q_k expanded concentrically twice. Then using (i) and the fact that

$w \in A_1$, we have

$$m_w\Big(\bigcup Q_k^*\Big) < \sum m_w(Q_k^*) < C\sum m_w(Q_k) < C\sum \frac{1}{\lambda}\int_{Q_k} f(y)\frac{m_w(Q_k)}{|Q_k|}\,dy$$
$$< \frac{C}{\lambda}\sum \int_{Q_k} f(y)w(y)\,dy < \frac{C}{\lambda}\|f\|_{1,w}.$$

Thus, we have only to show

$$m_w\Big(\Big\{x \notin \bigcup Q_k^*\colon |T_N b(x)| > \lambda\Big\}\Big) < \frac{C}{\lambda}\|f\|_{1,w}. \tag{3.2}$$

Let y_k and δ_k be the center and diameter of Q_k. Then

$$\int_{x\notin \cup Q_k^*}|T_N b(x)|w(x)dx = \int_{x\notin\cup Q_k^*}\left|\int_{\mathbf{R}^n} K_N(x-y)b(y)dy\right| w(x)dx$$
$$= \int_{x\notin\cup Q_k^*}\left|\sum_k \int_{Q_k} K_N(x-y)b(y)dy\right| w(x)dx$$
$$= \int_{x\notin\cup Q_k^*}\left|\sum_k \int_{Q_k}\{K_N(x-y) - K_N(x-y_k)\}b(y)dy\right| w(x)dx$$
$$< \sum_k \int_{Q_k}\left(\int_{x\notin Q_k^*}|K_N(x-y) - K_N(x-y_k)|w(x)dx\right)|b(y)|dy.$$

If we can show, for any $y \in Q_k$, that the inner integral is bounded by a constant independent of k and N times $\operatorname{ess\,inf}_{Q_k} w$, then our result will follow, as we now show. For, by (iii),

$$m_w\Big(\Big\{x\notin \bigcup Q_k^*\colon |T_N b(x)|>\lambda\Big\}\Big) < \frac{1}{\lambda}\int_{x\notin\cup Q_k^*}|T_N b(x)|w(x)dx$$
$$< \frac{C}{\lambda}\sum\int_{Q_k}|b(x)|\operatorname*{ess\,inf}_{Q_k} w\,dx < \frac{C}{\lambda}\sum\int_{Q_k}|b(x)|w(x)\,dx$$
$$< \frac{C}{\lambda}\sum\int_{Q_k} f(x)w(x)dx + \frac{C}{\lambda}\sum\int_{Q_k}\left(\frac{1}{|Q_k|}\int_{Q_k} f(z)dz\right)w(x)\,dx$$
$$< \frac{C}{\lambda}\|f\|_{1,w} + \frac{C}{\lambda}\sum\int_{Q_k} f(z)\frac{m_w(Q_k)}{|Q_k|}\,dz$$
$$< \frac{C}{\lambda}\|f\|_{1,w} + \frac{C}{\lambda}\sum\int_{Q_k} f(z)w(z)dx < \frac{2C}{\lambda}\|f\|_{1,w}.$$

Therefore,

$$m_w(\{x\in\mathbf{R}^n\colon |T_N f(x)|>\lambda\}) < \frac{C}{\lambda}\|f\|_{1,w} \tag{3.3}$$

with a constant independent of N, f, and λ. If $f \in \mathbb{S}$, $f = f^+ - f^-$ where f^+ and f^- and nonnegative and in $L^1 \cap L^1_w$, so that (3.3) holds for $f \in \mathbb{S}$. Then

$$
\begin{aligned}
m_w(\{x \in \mathbf{R}^n \colon |Tf(x)| > \lambda\})
&\leq m_w(\{x \in \mathbf{R}^n \colon |T_N f(x)| + |Tf(x) - T_N f(x)| > \lambda\}) \\
&\leq m_w\left(\left\{x \in \mathbf{R}^n \colon |T_N f(x)| > \frac{\lambda}{2}\right\}\right) \\
&\quad + m_w\left(\left\{x \in \mathbf{R}^n \colon |Tf(x) - T_N f(x)| > \frac{\lambda}{2}\right\}\right).
\end{aligned}
$$

Since $T_N f$ converges uniformly to Tf for $f \in \mathcal{S}$, choosing N large enough the second term on the right is zero. By (3.3),

$$m_w(\{x \in \mathbf{R}^n \colon |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda}\|f\|_{1,w} \quad \text{for } f \in \mathcal{S},$$

which extends to L^1_w.

To complete the proof of Theorem 1 we need to show

$$\int_{x \notin Q_k^*} |K_N(x-y) - K_N(x-y_k)| w(x)\, dx \leq C \operatorname*{ess\,inf}_{Q_k} w \quad \text{if } y \in Q_k,$$

with C independent of k and N. Choose $r \leq s$ so that $n/r < l < n/r + 1$ and $w^r \in A_1$. Then, using Lemma 1 with $p = r'$ and $t = r$ and noting that $x \notin Q_k^*$ implies $|x - y_k| > 2\delta_k$, we have for $y \in Q_k$ that

$$
\begin{aligned}
&\int_{|x-y_k|>2\delta_k} |K_N(x-y) - K_N(x-y_k)| w(x)dx \\
&= \sum_{j=1}^{\infty} \int_{2^j\delta_k < |x-y_k| \leq 2^{j+1}\delta_k} |K_N(x-y) - K_N(x-y_k)| w(x)\, dx \\
&\leq \sum_{j=1}^{\infty} \left(\int_{2^j\delta_k < |x-y_k| \leq 2^{j+1}\delta_k} |K_N(x-y) - K_N(x-y_k)|^{r'} dx\right)^{1/r'} \\
&\quad \cdot \left(\int_{|x-y_k| \leq 2^{j+1}\delta_k} w^r(x)dx\right)^{1/r} \\
&\leq C \sum_{j=1}^{\infty} (2^j\delta_k)^{-l}(\delta_k)^{l-n/r}(2^{j+1}\delta_k)^{n/r}\left\{(2^{j+1}\delta_k)^{-n}\int_{|x-y_k|<2^{j+1}\delta_k} w^r(x)dx\right\}^{1/r}.
\end{aligned}
$$

Thus, since $w^r \in A_1$,

$$
\begin{aligned}
&\int_{|x-y_k|>2\delta_k} |K_N(x-y) - K_N(x-y_k)| w(x)dx \\
&\qquad \leq C \sum_{j=1}^{\infty} (2^j)^{n/r-l} \operatorname*{ess\,inf}_{|x-y_k|<2^{j+1}\delta_k} w(x) \\
&\qquad \leq C \operatorname*{ess\,inf}_{|x-y_k|<\delta_k} w(x) \sum_{j=1}^{\infty} (2^j)^{n/r-l} \leq C \operatorname*{ess\,inf}_{Q_k} w
\end{aligned}
$$

with C independent of k and N. This completes the proof of Theorem 1.

We will derive Theorem 2 from Theorem 1 by using Lemma 4 and a characterization of A_p functions proved by P. Jones [9]. He has shown that if $w \in A_p$ then there are A_1 weights u and v such that $w = uv^{1-p}$.

Fix p, $1 < p < \infty$, and w so that $w^{n/l} \in A_p$. We have $w^{n/l} = uv^{1-p}$, $u,v \in A_1$, or $w = u^{l/n}v^{l(1-p)/n}$. Next, write this as

$$w = u^{l/n}v^{l(1-p)/n} = (u^\alpha v^\beta)^t(u^\gamma v^\delta)^{1-t} = w_0^t w_1^{1-t}.$$

For this to make sense, we need

$$\alpha t + \gamma(1-t) = \frac{l}{n}, \tag{3.4}$$

$$\beta t + \delta(1-t) = \frac{l}{n}(1-p). \tag{3.5}$$

Then, in order to use Lemma 4 for weights which satisfy Theorem 1, we require

$$w_0^{-1/(r-1)} \in A_{rl/n}, \qquad 1 < r < \min\left\{\left(\frac{n}{l}\right)', p\right\}, \tag{3.6}$$

$$w_1 \in A_{ql/n}, \qquad q > \max\left\{\frac{n}{l}, p\right\}, \tag{3.7}$$

$$t = \frac{q-p}{q-r}. \tag{3.8}$$

Recall that $u \in A_1$ (similarly $v \in A_1$) implies

$$\frac{1}{|Q|}\int_Q u(y)\,dy < Cu(x) \quad \text{for almost all } x \in Q.$$

Therefore, if $\alpha > 0$ and $\beta < 0$, letting $s = r'l/n$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q w_0(x)^{-1/(r-1)}\,dx\right)\left(\frac{1}{|Q|}\int_Q w_0(x)^{(1/(r-1))(1/(s-1))}\,dx\right)^{s-1} \\
&\quad = \left(\frac{1}{|Q|}\int_Q u(x)^{-\alpha/(r-1)}v(x)^{-\beta/(r-1)}\,dx\right) \\
&\qquad \cdot \left(\frac{1}{|Q|}\int_Q u(x)^{(\alpha/(r-1))(1/(s-1))}\,v(x)^{(\beta/(r-1))(1/(s-1))}\,dx\right)^{s-1} \\
&\quad < C\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)^{-\alpha/(r-1)}\left(\frac{1}{|Q|}\int_Q v(x)^{-\beta/(r-1)}\,dx\right) \\
&\qquad \cdot \left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)^{\beta/(r-1)}\left(\frac{1}{|Q|}\int_Q u(x)^{(\alpha/(r-1))(1/(s-1))}\,dx\right)^{s-1} \\
&\quad = C,
\end{aligned}$$

if

$$\alpha = (r-1)\left(\frac{r'l}{n} - 1\right) = \frac{rl}{n} - r + 1 \quad \text{and} \quad \beta = -(r-1);$$

that is $w_0^{-1/(r-1)} \in A_{r'l/n}$ for these values of α and β. Similarly, we can show $w_1 \in A_{ql/n}$ if $\gamma = 1$ and $\delta = -((ql/n) - 1)$. Using these values of α and γ, we have (3.4) if $t = 1/r$. Next, solving (3.5) for q, we get $q = r'(p-1)$. This value of q also satisfies (3.8). Therefore, if we choose $r < \min\{(n/l)', p\}$ so close to 1 that $q = r'(p-1) > \max\{n/l, p\}$, we can satisfy (3.4)–(3.8), proving Theorem 2.

Before proving Theorem 3, notice that $-n \geqslant -lp$ if $n/l \leqslant p$, and $n(p-1) \leqslant lp$ if $p \leqslant (n/l)'$. Therefore, for $l < n$ the conclusion of Theorem 3 can be divided into three cases:

$$1 < p < \frac{n}{l} \quad \text{and} \quad -lp < \beta < n(p-1), \tag{3.9}$$

$$\frac{n}{l} \leqslant p \leqslant \left(\frac{n}{l}\right)' \quad \text{and} \quad -n < \beta < n(p-1), \tag{3.10}$$

$$\left(\frac{n}{l}\right)' < p < \infty \quad \text{and} \quad -n < \beta < lp. \tag{3.11}$$

Since (3.11) is the dual of (3.9), we need only concern ourselves with (3.9) and (3.10).

Next, let us interpret Theorem 1 when $w(x)$ is a power of $|x|$. Because $|x|^\beta \in A_p$ if and only if $-n < \beta < n(p-1)$, we have $(l < n)$ that T is bounded on $L^p_{|x|^\beta}$ if

$$\frac{n}{l} \leqslant p < \infty \quad \text{and} \quad -n < \beta < pl - n, \tag{3.12}$$

$$1 < p \leqslant \left(\frac{n}{l}\right)' \quad \text{and} \quad -n + p(n-l) < \beta < n(p-1). \tag{3.13}$$

However, combining (3.12) and (3.13), we have (3.10) and are left with only proving (3.9).

Let $q = n/l$ and $r < n/l$; then also $r < (n/l)'$. By (3.13) and (3.10), T is bounded on $L^r_{|x|^{\beta_0}}$ and $L^q_{|x|^{\beta_1}}$ for $-n + r(n-l) < \beta_0 < n(r-1)$ and $-n < \beta_1 < n(q-1)$. Using Lemma 4, if $r < p < q$ we see that T is bounded on $L^p_{|x|^\beta}$ for

$$\beta = \beta_0\left(\frac{q-p}{q-r}\right) + \beta_1\left(\frac{p-r}{q-r}\right).$$

Thus β satisfies

$$\{-n + r(n-l)\}\left(\frac{q-p}{q-r}\right) - n\left(\frac{p-r}{q-r}\right)$$
$$< \beta < n(r-1)\left(\frac{q-p}{q-r}\right) + n(q-1)\left(\frac{p-r}{q-r}\right).$$

Simplifying and using the fact that $q = n/l$, we get

$$\frac{n^2(r-1)}{n-lr} + \frac{plr(l-n)}{n-lr} < \beta < n(p-1). \tag{3.14}$$

But, as $r \to 1$, the left-hand side of (3.14) approaches $-lp$. So, taking r sufficiently close to 1 allows us to choose any β satisfying $-lp < \beta < n(p-1)$.

When $l = n$, the restriction in Theorem 3 is $-n < \beta < n(p-1)$ for $1 < p < \infty$. But, when $l = n$ in Theorem 1, we require $w \in A_p$, and $|x|^\beta \in A_p$ if $-n < \beta < n(p-1)$.

4. The proof of Theorem 4 is based on an analogue of Lemma 1.

LEMMA 5. *Let $\Omega \in L^r(\Sigma)$ and satisfy the L^r-Dini condition. Set $K(x) = \Omega(x')/|x|^n$. There exists a constant $\alpha_0 > 0$ such that if $|y| < \alpha_0 R$, then*

$$\left(\int_{R<|x|<2R} |K(x-y) - K(x)|^r\, dx\right)^{1/r} \leq CR^{n/r-n}\left\{\frac{|y|}{R} + \int_{|y|/2R<\delta<|y|/R} \omega_r(\delta)\frac{d\delta}{\delta}\right\}.$$

PROOF. We may choose $\alpha_0 < \frac{1}{2}$; then, since $|x| > R$, $|x-y|$ is equivalent to $|x|$. Therefore,

$$|K(x-y) - K(x)| = \left|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n}\right| \leq C\left\{|\Omega(x)|\frac{|y|}{|x|^{n+1}} + \frac{|\Omega(x-y) - \Omega(x)|}{|x|^n}\right\}.$$

It follows that

$$\left(\int_{R<|x|<2R} |K(x-y) - K(x)|^r\, dx\right)^{1/r} \leq C\left(\int_{R<|x|<2R} |\Omega(x)|^r \frac{|y|^r}{|x|^{(n+1)r}}\, dx\right)^{1/r} + C\left(\int_{R<|x|<2R} \frac{|\Omega(x-y) - \Omega(x)|^r}{|x|^{nr}}\, dx\right)^{1/r}. \tag{4.1}$$

The first term on the right side of (4.1) is bounded by

$$C\|\Omega\|_{L^r(\Sigma)}|y|R^{-(n+1)}R^{n/r} = CR^{n/r-n}\left(\frac{|y|}{R}\right).$$

Changing to polar coordinates, we see the second term equals

$$C\left(\int_R^{2R} t^{-nr+n-1}\left(\int_\Sigma |\Omega(tx'-y)-\Omega(tx')|^r\,d\sigma_{x'}\right)dt\right)^{1/r}$$

$$\leq CR^{n/r-n}\left(\int_R^{2R}\left(\int_\Sigma \left|\Omega\left(\frac{x'-\alpha}{|x'-\alpha|}\right)-\Omega(x')\right|^r d\sigma_{x'}\right)\frac{dt}{t}\right)^{1/r},$$

where $\alpha = y/t$. Arguing as in Calderón, Weiss, and Zygmund [1, pp. 65–72], we see the inner integral is bounded by

$$C\sup_{|\rho|\leq|\alpha|}\int_\Sigma |\Omega(\rho x')-\Omega(x')|^r\,d\sigma_{x'} = C\omega_r^r\left(\frac{|y|}{t}\right)$$

as long as $|\alpha| = |y|/t < \alpha_0$. Thus, the second term is bounded by

$$CR^{n/r-n}\left(\int_R^{2R}\omega_r^r\left(\frac{|y|}{t}\right)\frac{dt}{t}\right)^{1/r} = CR^{n/r-n}\left(\int_{|y|/2R}^{|y|/R}\omega_r^r(\delta)\frac{d\delta}{\delta}\right)^{1/r}$$

$$\leq CR^{n/r-n}\left(\int_{|y|/2R<\delta<|y|/R}\omega_r(\delta)\frac{d\delta}{\delta}\right),$$

since ω_r is essentially constant on intervals of the form $(a, 2a)$, $a > 0$. Lemma 5 is now proved.

Notice that when $R = 2^j|y|$, with a j such that $1/\alpha_0 < 2^j$, we get

$$\left(\int_{2^j|y|<|x|<2^{j+1}|y|}|K(x-y)-K(x)|^r\,dx\right)^{1/r}$$

$$\leq C(2^j|y|)^{n/r-n}\left\{\frac{1}{2^j}+\int_{2^{-(j+1)}}^{2^{-j}}\omega_r(\delta)\frac{d\delta}{\delta}\right\}.$$

Theorem 4 is proved in exactly the same manner as Theorem 1. Using Lemma 5, we show

$$(K * f)^\#(x) \leq Cf_{r'}^*(x),$$

which proves the result for $p > r'$. The only change necessary is in the decomposition $f = f_0 + \Sigma f_j$. For Theorem 4,

$$f_0(y) = f(y)\chi\left(\left\{y \in \mathbf{R}^n\colon |x-y| \leq \frac{1}{\alpha_0}\delta\right\}\right)$$

and the sum of f_j's is over $j \geq \log_2(1/\alpha_0)$. We get the case $p = r'$ by interpolation, and $1 < p \leq r$ follows by duality. In the weak-type (1, 1) proof, we may have to replace the weak-type (2, 2) result for the good function by a weak-type (r', r') result.

5. We conclude by showing that Theorem 3 is best possible, except for endpoint equalities for β. We prove the result for $p > (n/l)'$; the case $p < n/l$ follows by duality. For $n/l \leq p \leq (n/l)'$, the Riesz transforms and

an argument like that in [**8**] show the range of β is best possible.

Let $1 < s \leq 2$, $n/s < l < n$, $(n/l)' < p$ and $\beta > lp$. Define a multiplier m by

$$m(x) = e^{ix\cdot\eta}(1 + |x|^2)^{-l/2}$$

for a fixed η of length 1. Note that $\check{m}(x) = G_l(x - \eta)$ (the Bessel kernel of order l) and that $|D^\alpha m(x)| \leq C_\alpha/(1 + |x|)^l$, so $m \in M(s, l)$, $1 \leq s \leq \infty$. Moreover, $G_l \geq 0$ and there exist $c, \mu > 0$ such that $G_l(x) > c|x|^{l-n}$ if $|x| < \mu$ (see Stein [**19**, p. 132] for details).

Set

$$f(x) = |x|^{-((n+\beta)/p)}|\log|x||^{-\delta}\chi(\{x \in \mathbf{R}^n: |x| < \mu\}).$$

If $\delta p > 1$, $f \in L^p_{|x|^\beta}(\mathbf{R}^n)$. Since $Tf(x) = (G_l(\cdot - \eta) * f)(x)$,

$$\begin{aligned} Tf(x) &= \int_{|y|<\mu} |y|^{-((n+\beta)/p)}|\log|y||^{-\delta}G_l(x - y - \eta)\,dy \\ &= \int_{|x-\eta-z|<\mu} |x - \eta - z|^{-((n+\beta)/p)}|\log|x - \eta - z||^{-\delta}G_l(z)\,dz \end{aligned}$$

by setting $z = x - \eta - y$. Now, if we restrict the integration to $|z| < \frac{1}{2}|x - \eta|$, $|x - \eta - z|$ is equivalent to $|x - \eta|$ and, if $|x - \eta| < \mu/2$,

$$\begin{aligned} Tf(x) &\geq C|x - \eta|^{-((n+\beta)/p)}|\log|x - \eta||^{-\delta}\int_{|z|<\frac{1}{2}|x-\eta|} \frac{dz}{|z|^{n-l}} \\ &= C|x - \eta|^{l-((n+\beta)/p)}|\log|x - \eta||^{-\delta}. \end{aligned}$$

Therefore, $Tf \notin L^p_{|x|^\beta}(\mathbf{R}^n)$ if $\{l - ((n + \beta)/p)\}p < -n$; i.e., if $\beta > lp$.

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